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LOGICS WITH SEPARATING SETS OF MEASURES

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ABSTRACT. A totally bounded uniformity induced by a set of measures M on a logic (orthomodular lattice) L is studied. Some relations between the properties of the uniformity, resp. the compatible topology and the structure of L and M are shown.

1. Introduction

In [12], a uniform topology τ_m induced by a measure m on a logic (= orthomodular lattice) L has been introduced. If m is a valuation, the topology τ_m coincides with the topology induced by the pseudometric ϱ_m, where $\varrho_m(a, b) = m(a \triangle b)$, and $\triangle$ is the symmetric difference in L.

The uniform topology induced by ϱ_m, where m is a valuation or an outer valuation on a logic L, has been thoroughly studied (see e.g. [13]). Another kind of a uniform topology induced by a measure (not necessarily a valuation or an outer valuation) has been introduced and studied in [10].

A generalization of the topology τ_m introduced in [12] is the topology τ_M induced by a set M of measures on a logic L. In [11], the topology τ_M has been compared with the order topology τ_o on L.

Topologies on partially ordered sets and lattices have been studied in [2], [4], [5], [6], [7].

In the present paper, we study two kinds of totally bounded uniformities induced by states on a logic L. We find conditions under which L is a uniform logic (in the sense of [13]).

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2. Definitions and preliminary results

Let $(L, 0, 1, ', \vee, \wedge)$ be a *logic*, i.e. an orthomodular lattice (see [1], [8], [13], [14] for detail). Two elements $a, b \in L$ are *orthogonal* (written $a \perp b$) if $a \leqslant b'$. If $a \leqslant b$ we shall write $b - a$ instead of $b \wedge a'$. The symbol $a \triangle b$ will denote the *symmetric difference*, i.e. $a \triangle b = (a - a \wedge b) \vee (b - a \wedge b) = (a \vee b) - (a \wedge b)$.

A (finite) *measure* on L is a map $m\colon L \to [0, \infty)$ such that $m(a \vee b) = m(a) + m(b)$ for any $a, b \in L$ such that $a \perp b$. A measure m on L is a *valuation* if $m(a \vee b) + m(a \wedge b) = m(a) + m(b)$ for any $a, b \in L$, or equivalently, if it is subadditive, i.e. if $m(a \vee b) \leqslant m(a) + m(b)$ for any $a, b \in L$. A measure m on L is *faithful* if $m(a) = 0$ implies $a = 0$, $a \in L$.

Definition 2.1. *Let M be a set of measures on a logic L. We say that M is*

(i) *weakly separating (for L) if $a, b \in L$, $a \neq b \Rightarrow \exists m \in M\ \exists x \in L$ such that either $m(a \vee x) \neq m(b \vee x)$ or $m(a \wedge x) \neq m(b \wedge x)$,*

(ii) *separating (for L) if $a \in L$, $a \neq 0 \Rightarrow \exists m \in M$ such that $m(a) \neq 0$. If $M = \{m\}$, we say that m is (weakly) separating if $\{m\}$ has this property.*

It is clear that m is separating iff m is faithful.

Lemma 2.2. *Let M be a set of measures on L.*

(i) *If M is separating, then M is weakly separating.*

(ii) *If all the measures in M are valuations, then M is separating iff M is weakly separating.*

Proof. (i) Let $a, b \in L$, $a \neq b$. Then either $(a \vee b) - b \neq 0$ or $b - (a \wedge b) \neq 0$. Hence there is $m \in M$ such that either $m(a \vee b) \neq m(b) = m(b \vee b)$ or $m(a \wedge b) = m(b) = m(b \wedge b)$, i.e. M is weakly separating.

(ii) Let $a \in L$, $a \neq 0$. Let M be a weakly separating set of valuations, and suppose that $m(a) = 0$ for all $m \in M$. Then $m(a \vee x) + m(a \wedge x) = m(a) + m(x)$ implies that $m(a \vee x) = m(x) = m(0 \vee x)$, and $m(a \wedge x) = 0 = m(0 \wedge x)$ for any $x \in L$ and any $m \in M$. This implies that $a = 0$, a contradiction.

The following example shows that the notions "separating" and "weakly separating" are in general not equal.

Example 2.3. Let us consider the logic L, which is a horizontal sum of the Boolean algebra 2^3 and the Boolean algebra 2^2 (see fig. 1). Define a measure on L as folows:

$m(a) = m(b) = m(c) = 1/3$,
$m(a') = m(b') = m(c') = 2/3$,
$m(0) = m(d) = 0$,
$m(1) = m(d') = 1$.

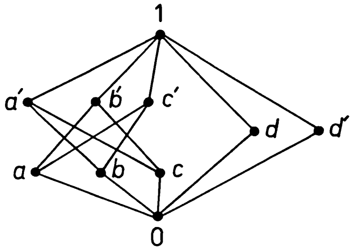

Fig. 1

It can be checked that $\{m\}$ is weakly separating but not faithful.

3. Measures and topologies on a logic

Let L be a logic and let m be a (nontrivial) measure on L. For every $x \in L$ define

$$\varrho_{m,x\vee}(a, b) = |m(a \vee x) - m(b \vee x)|, \quad a, b \in L$$

$$\varrho_{m,x\wedge}(a, b) = |m(a \wedge x) - m(b \wedge x)|, \quad a, b \in L.$$

Let $D(m) = \{\varrho_{m,x\vee} \mid x \in L\} \cup \{\varrho_{m,x\wedge} \mid x \in L\}$. Denote by $\mathfrak{U}_{D(m)}$ the *uniformity on L induced by the family of pseudo-metrics* $D(m)$ and let τ_m denote the *topology on L compatible with* $\mathfrak{U}_{D(m)}$ (see [3], [9]).

If m is a valuation, put $\varrho_m(a, b) = m(a \triangle b)$ and denote by $\mathfrak{U}_{\varrho_m}$ the *uniformity induced by the pseudo-metric* ϱ_m, and let τ_{ϱ_m} denote the *topology compatible with* $\mathfrak{U}_{\varrho_m}$.

The following theorem shows us the interrelations between $\mathfrak{U}_{D(m)}$ and $\mathfrak{U}_{\varrho_m}$, resp. τ_m and τ_{ϱ_m} for a valuation m (see also [12]).

Theorem 3.1. *Let m be a measure on a logic L.*

(i) *The map* $\varrho_m\colon L \times L \to [0, \infty)$ *defined by* $\varrho_m(a, b) = m(a \triangle b)$ *is a pseudometric iff m is a valuation* (*or, equivalently, iff m is subadditive*).

(ii) *If m is a valuation, then* $\mathfrak{U}_{D(m)} \subset \mathfrak{U}_{\varrho_m}$ *and* $\tau_m = \tau_{\varrho_m}$.

Proof. (i) If m is subadditive, then $a \triangle b \leqslant a \triangle c \vee b \triangle c$ for any $a, b, c \in L$ implies that $\varrho_m(a, b) \leqslant \varrho_m(a, c) + \varrho_m(b, c)$. Now it is easy to see that ϱ_m is a pseudometric. On the other hand, if ϱ_m is a pseudometric, then for all $a, b \in L$

$$\varrho_m(a, b) \leqslant \varrho_m(a, a \wedge b) + \varrho_m(a \wedge b, b)$$

and

$$\varrho_m(a, b) \leqslant \varrho_m(a, a \vee b) + \varrho_m(a \vee b, b).$$

From these two inequalities we easily obtain that $m(a) + m(b) = m(a \vee b) + m(a \wedge b)$, i.e. m is a valuation.

(ii) For every measure m on L it holds that

$$\varrho_{m,x\vee}(a, b) = |m(a \vee x) - m(b \vee x)| \leqslant m((a \vee x) \triangle (b \vee x)),$$

$$\varrho_{m,x\wedge}(a, b) = |m(a \wedge x) - m(b \wedge x)| \leqslant m((a \wedge x) \triangle (b \wedge x)).$$

In addition, if m is a valuation, then

$$m((a \vee x) \triangle (b \vee x)) + m((a \wedge x) \triangle (b \wedge x)) \leqslant m(a \triangle b) \quad ([2], \text{ p. } 301).$$

Therefore

$$\varrho_{m,x\vee}(a, b) + \varrho_{m,x\wedge}(a, b) \leqslant \varrho_m(a, b)$$

for all $a, b, x \in L$. This entails that for every $\varepsilon > 0$,

$$W(\varrho_m, \varepsilon) \subset W(\varrho_{m,x\vee}, \varepsilon) \cap W(\varrho_{m,x\wedge}, \varepsilon),$$

where

$$W(\varrho, \varepsilon) = \{(a, b) \in L \times L \mid \varrho(a, b) < \varepsilon\} \quad \text{for any} \quad \varrho \in \{\varrho_m, \varrho_{m,x\vee}, \varrho_{m,x\wedge}\}.$$

Hence every element of the base of $\mathfrak{U}_{D(m)}$ belongs to $\mathfrak{U}_{\varrho_m}$, and hence $\mathfrak{U}_{D(m)} \subset \mathfrak{U}_{\varrho_m}$. This implies that $\tau_m \subset \tau_{\varrho_m}$. Now let $\{a_\alpha\}_\alpha$ be a net in L such that $a_\alpha \xrightarrow{\tau_m} a$. This means that $m(a_\alpha \vee x) \to m(a \vee x)$ and $m(a_\alpha \wedge x) \to m(a \wedge x)$ for every $x \in L$. In particular, $m(a_\alpha \vee a) \to m(a)$, $m(a_\alpha \wedge a) \to m(a)$, i.e. $\varrho_m(a_\alpha, a) = m(a_\alpha \triangle a) \to 0$. Thus $\tau_m = \tau_{\varrho_m}$.

The following example shows that for a valuation m on L in general $\mathfrak{U}_{D(m)} \neq \mathfrak{U}_{\varrho_m}$.

Example 3.2. Let H be a finite-dimensional Hilbert space (real or complex) and let $L = L(H)$ be the Hilbert space logic, i.e., the lattice of all closed linear subspaces of H. It is known that there exists a faithful valuation m on L, and the metric ϱ_m induces the discrete topology on L ([13], p. 62). Then $\mathfrak{U}_{D(m)} \neq \mathfrak{U}_{\varrho_m}$, since $\mathfrak{U}_{D(m)}$ is totally bounded (see Th. 3.4 below), and if it were metrizable, it would be separable ([3], p. 153), i.e. L would contain a countable dense subset with respect to the topology $\tau_m = \tau_{\varrho_m}$. As τ_{ϱ_m} is discrete, the latter conditi‹ n would imply that L itself should be countable.

From the above example we see that $\mathfrak{U}_{D(m)}$ need not be metrizable (or pseudometrizable) even if m is a valuation.

Let M be a set of measures on L. Denote by $\mathfrak{U}_{D(M)}$ the *uniformity induced by the family of pseudo-metrics* $D(M)$, where $D(M) = \bigcup_{m \in M} D(m)$, and let τ_M be the *topology compatible with* $\mathfrak{U}_{D(M)}$. Clearly, $\tau_M \supset \tau_m$ for every $m \in M$.

We recall that the interval topology τ_i on L is a topology with the subbase *consisting of the set-theoretical complements of all intervals* $\langle a, b\rangle \subset L$, *where* $a \leqslant b$ (*in particular* $\{a\} = \langle a, a\rangle$).

In the next theorem we collect some basic properties of $\mathfrak{U}_{D(M)}$ and τ_M.

Theorem 3.3. *Let L be a logic and let M be a set of measures on L. Then*

(i) $(L, \mathfrak{U}_{D(M)})$ *is totally bounded and hence the completion of* $(L, \mathfrak{U}_{D(M)})$ *is compact.*

(ii) *The topology* τ_M *is* T_2 *iff M is weakly separating.*

(iii) *If M is separating, then* $\tau_M \supset \tau_i$.

(iv) *If M is separating and* $(L, \mathfrak{U}_{D(M)})$ *is a complete uniform space, then L is a complete logic (i.e. L is a complete lattice).*

Proof. (i) For every $x \in L$ and $m \in M$ define

$$f_{m,x\vee}(a) = m(a \vee x),\ f_{m,x\wedge}(a) = m(a \wedge x), \quad a \in L.$$

Let $\Phi(M) = \{f_{m,x\vee} \mid m \in M, x \in L\} \cup \{f_{m,x\wedge} \mid m \in M, x \in L\}$. Then the uniformity $\mathfrak{U}_{D(M)}$ is generated by the family $\Phi(M)$ of bounded functions on L, and hence $\mathfrak{U}_{D(M)}$ is totally bounded (see [3], 4.2.13, p. 168).

(ii) Observe that a net $\{a_\alpha\}_\alpha \subset L$ τ_M-converges to a iff $m(a_\alpha \vee x) \to m(a \vee x)$ and $m(a_\alpha \wedge x) \to m(a \wedge x)$ for every $x \in L$ and every $m \in M$. Hence every net has at most one limit iff M is weakly separating.

(iii) Let M be separating. Let $\{a_\alpha\}_\alpha \subset \langle b, c\rangle$, where $b \leqslant c$ $(b, c \in L)$, and let $a_\alpha \xrightarrow{\tau_M} a$. Then $m(a) = \lim m(a_\alpha \wedge c) = m(a \wedge c)$, $m(b) = \lim m(a_\alpha \wedge b) = m(a \wedge b)$ for every $m \in M$. As M is separating, this implies that $a = a \wedge c$, $b = a \wedge b$, i.e. $a \in \langle b, c\rangle$. Thus intervals are closed sets in τ_M, hence $\tau_i \subset \tau_M$.

(iv) If $(L, \mathfrak{U}_{D(M)})$ is complete, it is compact. If M is separating, $\tau_M \supset \tau_i$. This implies that τ_i is also compact and by [6], L is a complete lattice.

For the convenience of readers we recall some necesary definitions.

A *net* $\{a_\alpha\}_\alpha \subset L$ *(o)-converges* to a $(a_\alpha \xrightarrow{(o)} a)$ if there are nets $\{b_\alpha\}_\alpha$, $\{c_\alpha\}_\alpha$ such that $b_\alpha \leqslant a_\alpha \leqslant c_\alpha$ and $b_\alpha \uparrow a$, $c_\alpha \downarrow a$.

The *order topology* τ_o on L is the strongest (= finest) topology such that (o)-convergence implies the topological convergence. The symbol $a_\alpha \downarrow a$ means that the net $\{a_\alpha\}_\alpha$ is nonincreasing and $\wedge\, a_\alpha = a$. The symbol $a_\alpha \uparrow a$ is defined dually.

A *logic* L is *(o)-continuous* if $a_\alpha \uparrow a$ implies $a_\alpha \wedge x \uparrow a \wedge x$ for every $x \in L$ (dually, $a_\alpha \downarrow a$ implies $a_\alpha \vee x \downarrow a \vee x$ for every $x \in L$).

A logic L is *atomic* if every element in L contains an atom. If L is atomic, then every element in L is the supremum of all atoms it contains.

A logic L is *separable* if any set of mutually orthogonal nonzero elements is at most countable.

A *measure* m on L is *(o)-continuous* if $a_\alpha \downarrow a$ implies that $m(a_\alpha) \to m(a)$. We note that (o)-continuity coincides with the *complete additivity* of m.

We shall need the following statement that is of interest itself.

Lemma 3.4. *Let m be a measure on a logic L. If a is an atom in L such that $m(a) \neq 0$, then the intervals $\langle 0, a' \rangle$ and $\langle a, 1 \rangle$ are clopen sets in τ_m.*

Proof. Let a be an atom in L and let $m(a) \neq 0$. Let $\{c_\alpha\}_\alpha \subset \langle a, 1 \rangle$ and let $c_\alpha \xrightarrow{\tau_m} c$. Then $m(c_\alpha \wedge a) \to m(c \wedge a)$ implies that $m(c \wedge a) \neq 0$, i.e. $c \geqslant a$. Thus $\langle a, 1 \rangle$ is closed. By duality, $\langle 0, a' \rangle$ is closed. Hence $A = \langle 0, a' \rangle \cup \langle a, 1 \rangle$ is a closed set. Let $\{c_\alpha\}_\alpha$ be a net in L such that $c_\alpha \xrightarrow{\tau_m} c$, $c \in A$ and $\{c_\alpha\}_\alpha \cap A = \emptyset$. Then $0 = m(c_\alpha \wedge a) \to m(c \wedge a)$, $0 = m(c'_\alpha \wedge a) \to m(c' \wedge a)$, which contradicts $c \in A$. Hence A is a clopen set and since $\langle 0, a' \rangle \cap \langle a, 1 \rangle = \emptyset$, $\langle a, 1 \rangle$ and $\langle 0, a' \rangle$ are also clopen sets.

We note that in any (o)-continuous logic L, $\langle a, 1 \rangle$, $\langle 0, a' \rangle$ are clopen sets in the order topology τ_o for any atom $a \in L$.

Corollary 3.5. *If M is a separating set of measures on L, then for every atom $a \in L$ the intervals $\langle a, 1 \rangle$, $\langle 0, a^\perp \rangle$ are clopen sets in τ_M.*

Proof. As M is separating for L, to every atom $a \in L$ there is $m \in M$ such that $m(a) \neq 0$ and $\tau_m \subset \tau_M$.

Theorem 3.6. *Let L be an (o)-continuous atomic logic. Let M be a separating set of (o)-continuous measures on L. Then*

(i) *For every $x \in L$, the filter $\mathfrak{U}(x)$ of the neighbourhoods of x in τ has a base consisting of intervals which are clopen sets in τ_M.*

(ii) $\tau_M = \tau_o$.

Proof. (i) We note that if $A \subset L$, $A = \emptyset$, then $\vee A = 0$, $\wedge A = 1$. Let $x \in L$. As L is atomic, there are sets of atoms $\{a_\alpha \mid \alpha \in A\}$, $\{b_\alpha \mid \alpha \in B\}$ such that $x = \bigvee_{\alpha \in A} a_\alpha$, $x' = \bigvee_{\alpha \in B} b_\alpha$. Put $C = \{\gamma \subset A \cup B \mid \gamma \text{ is finite}\}$. C is a directed set with respect to the set inclusion. For every $\gamma \in C$ put $x_\gamma = \bigvee_{k \in \gamma \cap A} a_k$, $y_\gamma = \bigwedge_{k \in \gamma \cap B} b'_k$.

Then $x_\gamma \uparrow x$, $y_\gamma \downarrow x'$. By Lemma 3.4, $\langle x_\gamma, 1 \rangle = \left\langle \bigvee_{k \in \gamma \cap A} a_k, 1 \right\rangle = \bigcap_{k \in \gamma \cap A} \langle a_k, 1 \rangle$ and $\langle 0, y_\gamma \rangle = \left\langle 0, \bigcap_{k \in \gamma \cap B} b'_k \right\rangle = \bigcap_{k \in \gamma \cap B} \langle 0, b'_k \rangle$ are clopen sets in τ_M. Let $O(x)$ be any open neighbourhood of x in τ_o. Suppose that for every $\gamma \in C$ there is $z_\gamma \in \langle x_\gamma, y_\gamma \rangle$ such that $z_\gamma \notin O(x)$. As $x_\gamma \leqslant z_\gamma \leqslant y_\gamma$, $x_\gamma \uparrow x$, $y_\gamma \downarrow x$, we obtain $z_\gamma \xrightarrow{(o)} x$ and since $L \backslash O(x)$ is closed in τ_o, we obtain $x \in L \backslash O(x)$, a contradiction. Therefore

$\{\langle x_\gamma, y_\gamma\rangle\}_\gamma$ is a base of $\mathfrak{U}(x)$. Hence $\mathfrak{U}(x)$ has a base consisting of clopen intervals in τ_M for every $x \in L$.

(ii) Suppose that $a_\alpha \xrightarrow{o} a$. Then by the (o)-continuity of L and m, $a_\alpha \xrightarrow{\tau_m} a$ for every $m \in M$. We obtain $\tau_M \subset \tau_o$. Let $G \in \tau_o$. In view of (i) there exists to any $x \in G$ a neighbourhood $\mathscr{V}(x)$ which is clopen in τ_M such that $x \in \mathscr{V}(x) \subset G$. Thus $\tau_o \subset \tau_M$. We conclude that $\tau_M = \tau_o$.

A complete logic L with a T_2-uniformity $\mathfrak{U}$ on L is called a uniform logic if

(i) *the map $a \to a'$ is uniformly continuous,*

(ii) *the map $(a, b) \to a \vee b$ is uniformly continuous,*

(iii) *$a_\alpha \downarrow a \Rightarrow a_\alpha \xrightarrow{\tau_\mathfrak{U}} a$, where $\tau_\mathfrak{U}$ is the topology compatible with $\mathfrak{U}$.*

We note that there can be at most one uniformity $\mathfrak{U}$ on L such that $(L, \mathfrak{U})$ is a uniform logic (see [13], p. 56).

Theorem 3.7. *Let L be a complete (o)-continuous logic such that the interval topology τ_i on L is T_2. Let M be a separating set of (o)-continuous measures on L. Then*

(i) *$\tau_o = \tau_M = \tau_i$ is a compact completely regular T_2-topology. In addition, τ_o-convergence coincides with (o)-convergence.*

(ii) *L is a uniform logic with the uniformity $\mathfrak{U}_{D(M)}$.*

(iii) *L is separable iff τ_o is metrizable and in this case L contains a τ_o-dense countable subset.*

Proof. (i) The facts that L is complete and τ_i is T_2 imply that (a) L is atomic ([13], p. 75), (b) τ_i is compact ([6]) (c) $\tau_i = \tau_o$ ([5], Cor. 2.6). From Th. 3.6 we obtain $\tau_o = \tau_M = \tau_i$. (o)-convergence is topological by Th. 3.6 and [5], Th. 4.14.

(ii) By (i) the lattice operations in L are continuous in τ_M. As $\tau_M = \tau_i$ is compact, and $\mathfrak{U}_{D(M)}$ is totally bounded, $(L, \mathfrak{U}_{D(M)})$ is a complete uniform space which is compact. Hence the lattice operations in L are uniformly continuous. Also the map $a \mapsto a'$ is uniformly continuous and $a_\alpha \downarrow a \Rightarrow a_\alpha \xrightarrow{\tau_M} a$. Hence $(L, \mathfrak{U}_{D(M)})$ is a uniform logic.

(iii) Since (L, τ_o) is a compact completely regular T_2 space, there is one and only one uniformity on L compatible with τ_o ([9], p. 290). Therefore τ_o is metrizable iff $\mathfrak{U}_{D(M)}$ is metrizable. By [13], Th. 2, p. 55 $\mathfrak{U}_{D(M)}$ is metrizable iff L is a separable logic. In this case $(L, \tau_M) \equiv (L, \tau_o)$ is a totally bounded metric space, hence it contains a countable dense subset ([3], 3.2.68, p. 103).

Remark 3.8. By [13], if $(L, \mathfrak{U})$ is a uniform logic, then $\mathfrak{U}$ is induced by a separating set of (o)-continuous outer R-valuations on L. If L is separable, then $\mathfrak{U}$ is induced by a faithful (o)-continuous outer R-valuation on L (see [13], Th. 4, p. 59 and Cor. 3, p. 61 for definitions and proofs).

4. Coarser (weaker) topology induced by measures

Let a, b be elements of a logic L. We say that *a is compatible with b* (written $a \leftrightarrow b$) if $a = (a \wedge b) \vee (a \wedge b')$. Owing to the orthomodularity, the compatibility relation is symmetric in L. The *centre of a logic* L is the set $C(L) = \{b \in L \mid a \leftrightarrow b$ for every $a \in L\}$.

Suppose that L is a logic and $m: L \to \langle 0, \infty)$ is a (nontrivial) measure on L. Denote by $\mathfrak{U}_{D^*(m)}$ the *uniformity induced by the family of pseudo-metrics*

$$D^*(m) = \{\varrho_{m, x \vee} \mid x \in C(L)\} \cup \{\varrho_{m, x \wedge} \mid x \in C(L)\},$$

where $\varrho_{m, x \vee}$, $\varrho_{m, x \wedge}$ are defined as in sec. 3. Let τ_m^* denote the *topology on L compatible with* $\mathfrak{U}_{D^*(m)}$. Clearly $a_\alpha \xrightarrow{\tau_m^*} a$ iff $m(a_\alpha \vee x) \to m(a \vee x)$, $m(a_\alpha \wedge x) \to m(a \wedge x)$ for every $x \in C(L)$ and hence the topology τ_m^* is coarser (weaker) than τ_m.

Lemma 4.1. *Let L be a logic and let $m: L \to [0, \infty)$ be a measure Then*

(i) *$\mathfrak{U}_{D^*(m)}$ is totally bounded.*

(ii) *τ_m^* is T_2 iff for every $a, b \in L$, $a \neq b$ there exists $x \in C(L)$ such that either $m(a \vee x) \neq m(b \vee x)$ or $m(a \wedge x) \neq m(b \wedge x)$. In this case τ_m^* is Tychonoff.*

(iii) *If τ_m^* is T_2, then m is faithful.*

(iv) *If $C(L)$ is countable, then $\mathfrak{U}_{D^*(m)}$ is pseudo-metrizable and the compatible pseudo-metric topology is separable.*

Proof. (i) $\mathfrak{U}_{D^*(m)}$ and τ_m^* are induced by the family $\Gamma = \{m_{x\vee} \mid x \in C(L)\} \cup \{m_{x\wedge} \mid x \in C(L)\}$ of bounded functions, where $m_{x\vee}(a) = m(x \vee a)$, $m_{x\wedge}(a) = m(x \wedge a)$ $(a \in L)$.

(ii) It is evident from the definition of τ_m^*.

(iii) Suppose that τ_m^* is T_2 and $a \in L$, $a \neq 0$. If there exists $x \in C(L)$ such that $m(a \wedge x) \neq m(0 \wedge x) = 0$, then $m(a) \neq 0$. If there exists $y \in C(L)$ such that $m(a \vee y) \neq m(0 \vee y) = m(y)$, then from $m(a \vee y) + m(a \wedge y) = m(a) + m(y)$ we have $m(a) - m(a \wedge y) > 0$ and hence $m(a) > 0$.

(iv) Let $C(L)$ be countable. Then the family of pseudometrics $D^*(m)$ is countable, which implies the pseudo-metrizability of $\mathfrak{U}_{D^*(m)}$. But a totally bounded pseudo-metric space is separable (see [3], (3.2.68), (3.2.69), p. 153).

Lemma 4.2. *Let L be a logic and m be a measure on L.*

(i) $a_\alpha \xrightarrow{\tau_m^*} a \Rightarrow a'_\alpha \xrightarrow{\tau_m^*} a'$.

(ii) $a_\alpha \xrightarrow{\tau_m^*} a \Rightarrow \forall x \in C(L)$: $a_\alpha \vee x \xrightarrow{\tau_m^*} a \vee x$, $a_\alpha \wedge x \xrightarrow{\tau_m^*} a \wedge x$.

(iii) $a_\alpha \xrightarrow{\tau_m^*} a$, $b_\alpha \xrightarrow{\tau_m^*} b$, $a_\alpha \perp b_\alpha$, $a \perp b \Rightarrow a_\alpha \vee b_\alpha \xrightarrow{\tau_m^*} a \vee b$.

(iv) $m(a_\alpha \Delta a) \to 0$ iff $a_\alpha \xrightarrow{\tau_m^*} a$, $a_\alpha \vee a \xrightarrow{\tau_m^*} a$, $a_\alpha \wedge a \xrightarrow{\tau_m^*} a$.

Proof. (i) Follows from $m(a') = m(1) - m(a)$.

(ii) Let $x, y \in C(L)$ and $a_\alpha \xrightarrow{\tau_m^*} a$, then $m((a_\alpha \vee x) \vee y)) = m(a_\alpha \vee x \vee y) \to$ $\to m(a \vee x \vee y)$, $m((a_\alpha \vee x) \wedge y)) = m(a_\alpha \wedge y) \vee (x \wedge y)) = m(a_\alpha \wedge y) +$ $+ m(x \wedge y) - m(a_\alpha \wedge (x \wedge y)) \to m(a \wedge y) + m(x \wedge y) - m(a \wedge x \wedge y) =$ $= m((a \vee x) \wedge y)$, hence $a_\alpha \vee x \xrightarrow{\tau_m^*} a \vee x$. By (i) we get also $a_\alpha \wedge x \xrightarrow{\tau_m^*} a \wedge x$.

(iii) Let $a_\alpha \xrightarrow{\tau_m^*} a$, $b_\alpha \xrightarrow{\tau_m^*} b$, $a_\alpha \perp b_\alpha$, $a \perp b$. For any $x \in C(L)$ we have $m(a_\alpha \vee b_\alpha) \wedge x) = m(a_\alpha \wedge x) + m(b_\alpha \wedge x) \to m(a \wedge x) + m(b \wedge x) =$ $= m((a \vee b) \wedge x)$, $m((a_\alpha \vee b_\alpha) \vee x) = m(a_\alpha \vee b_\alpha) + m(x) - m((a_\alpha \vee b_\alpha) \wedge$ $\wedge x) \to m(a) + m(b) - m((a \vee b) \wedge x) = m(a \vee b \vee x)$.

(iv) Suppose that $m(a_\alpha \Delta a) \to 0$. Then $(m(a_\alpha \vee a) - m(a)) + (m(a) -$ $- m(a_\alpha \wedge a)) \to 0$, and hence $m(a_\alpha \vee a) \to m(a)$, $m(a_\alpha \wedge a) \to m(a)$. Let $x \in$ $\in C(L)$, then from $m(a \vee x) \leqslant m(a_\alpha \vee a \vee x) = m(a_\alpha \vee a) + m(x) -$ $- m((a_\alpha \vee a) \wedge x) \leqslant m(a_\alpha \vee a) + m(x) - m(a \wedge x) \to m(a) + m(x) -$ $- m(a \wedge x) = m(a \vee x)$ we obtain that $m(a_\alpha \vee a \vee x) \to m(a \vee x)$. Further, from $m(a \wedge x) \leqslant m((a_\alpha \vee a) \wedge x) = m(a_\alpha \vee a) + m(x) - m(a_\alpha \vee a \vee x) \to$ $\to m(a) + m(x) - m(a \vee x) = m(a \wedge x)$ we obtain that $m((a_\alpha \vee a) \wedge x) \to$ $\to m(a \wedge x)$. This proves that $a_\alpha \vee a \xrightarrow{\tau_m^*} a$. By duality, using (i), we prove that $a_\alpha \wedge a \xrightarrow{\tau_m^*} a$. Now since $a_\alpha \wedge a \leqslant a_\alpha \leqslant a_\alpha \vee a$ for every α, it follows that $a_\alpha \xrightarrow{\tau_m^*} a$.

Conversely, let $a_\alpha \xrightarrow{\tau_m^*} a$, $a_\alpha \vee a \xrightarrow{\tau_m^*} a$, $a_\alpha \wedge a \xrightarrow{\tau_m^*} a$. Then $m(a_\alpha \Delta a) =$ $= m(a_\alpha \vee a) - m(a_\alpha \wedge a) \to m(a) - m(a) = 0$.

Lemma 4.3. *Let L be a logic and m be a measure on L. Then*

(i) *If m is subadditive, then $\tau_m^* \subset \tau_{\varrho_m} = \tau_m$.*

(ii) *If L is a Boolean algebra, then $\tau_m^* = \tau_{\varrho_m} = \tau_m$.*

(iii) *If m is (o)-continuous, then $\tau_m^* \subset \tau_o$.*

Proof. (i) It is evident from the facts that $\tau_m^* \subset \tau_m$ and $\tau_m = \tau_{\varrho_m}$ for a subadditive measure.

(ii) If L is a Boolean algebra, then $C(L) = L$, and hence $\tau_m^* = \tau_m$. As m is subadditive, we also have $\tau_{\varrho_m} = \tau_m$.

(iii) $a_\alpha \xrightarrow{(o)} a$ implies $a_\alpha \Delta a \xrightarrow{(o)} 0$, and since m is (o)-continuous, we get $m(a_\alpha \Delta a) \to 0$. Thus $\tau_m^* \subset \tau_o$ by (iv) of Lemma 4.2.

Let M be a set of measures on a logic L. We denote by $\mathfrak{U}_{D^*(M)}$ *the uniformity induced by the family of pseudo-metrics*

$$D^*(M) = \{\varrho_{m,x\vee} \mid m \in M,\ x \in C(L)\} \cup \{\varrho_{m,x\wedge} \mid m \in M,\ x \in C(L)\}.$$

Let τ_M^* denote the *topology on L compatible with* $\mathfrak{U}_{D^*(M)}$. Clearly, $a_\alpha \xrightarrow{\tau_M^*} a$ iff $\forall m \in M\, \forall x \in C(L)$: $m(a_\alpha \vee x) \to m(a \vee x)$, $m(a_\alpha \wedge x) \to m(a \wedge x)$.

A set M of measures on a logic L is said to be ordering (for L) if $m(a) \leqslant m(b)$ *for all* $m \in M$ *implies* $a \leqslant b$ ($a, b \in L$). It is clear that an ordering set is separating. Indeed, $x \nleqslant 0$ implies that there is $m \in M$ such that $m(x) > m(0) = 0$.

Theorem 4.4. *Let L be a logic and let M be a set of measures on L. Then*

(i) $\mathfrak{U}_{D^*(M)}$ *is totally bounded.*

(ii) $\tau_M^* \subset \tau_M$.

(iii) τ_M^* *is Hausdorff (and hence also Tychonoff) iff for every* $a, b \in L$, $a \neq b$ *there exist* $x \in C(L)$, $m \in M$ *such that either* $m(a \wedge x) \neq m(b \wedge x)$ *or* $m(a \vee x) \neq m(b \vee x)$.

(iv) *If M is ordering, then* $\tau_i \subset \tau_M^*$ *and* τ_M^* *is* T_2.

(v) *If all the measures in M are (o)-continuous, then* $\tau_M^* \subset \tau_o$.

Proof. (i)—(iii) is obvious.

(iv) If $c_\alpha \in \langle a, b \rangle$ and $c_\alpha \xrightarrow{\tau_M^*} c$, then for every $m \in M$, $m(c_\alpha) \in \langle m(a), m(b) \rangle$ and hence also $m(c) \in \langle m(a), m(b) \rangle$. Since M is ordering, we obtain $c \in \langle a, b \rangle$. Thus every $\langle a, b \rangle \subset L$ is closed in τ_M^* and hence $\tau_i \subset \tau_M^*$.

(v) Follows from (iii) of Lemma 4.3.

Theorem 4.5. *Let L be a complete logic in which* τ_i *is* T_2. *Let M be an ordering set of (o)-continuous measures on L. Then*

(i) $\tau_o = \tau_i = \tau_M^*$ *is a compact Tychonoff topology.*

(ii) $(L, \mathfrak{U}_{D^*(M)})$ *is a complete uniform space.*

(iii) $a_\alpha \xrightarrow{\tau_o} a \Rightarrow a'_\alpha \xrightarrow{\tau_o} a'$.

(iv) $a_\alpha \xrightarrow{\tau_o} a$, $b_\alpha \xrightarrow{\tau_o} b$, $a_\alpha \perp b_\alpha$, $a \perp b \Rightarrow a_\alpha \vee b_\alpha \xrightarrow{\tau_o} a \vee b$.

(v) $a_\alpha \xrightarrow{\tau_o} a \Rightarrow \forall x \in C(L)$: $a_\alpha \vee x \xrightarrow{\tau_o} a \vee x$, $a_\alpha \wedge x \xrightarrow{\tau_o} a \wedge x$.

Proof. (i) The facts that L is complete and τ_i is T_2 imply that τ_i is compact and $\tau_i = \tau_o$ (see [5], Cor. 2.6). (iv) and (v) of Theorem 4.4 imply that $\tau_i = \tau_M^* = \tau_o$.

(ii) τ_M^* compact implies that $(L, \mathfrak{U}_{D^*(M)})$ is a complete uniform space.

(iii)—(v) follow from Lemma 4.2 and from (i) of this theorem.

Theorem 4.6. *Let L be a complete (o)-continuous logic in which* τ_i *is* T_2. *Let M be an ordering set of (o)-continuous measures for L. Then*

(i) $\tau_o = \tau_i = \tau_M^* = \tau_M$.

(ii) $(L, \mathfrak{U}_{D^*(M)})$ *is a uniform logic.*

(iii) $a_\alpha \xrightarrow{\tau_o} a$ *iff* $\forall m \in M$: $m(a_\alpha) \to m(a)$.

(iv) $a_\alpha \xrightarrow{\tau_o} a$ *iff* $a_\alpha \xrightarrow{(o)} a$.

Proof. It follows from Theorems 4.5 and 3.7 and the compactness of τ_o.

Examples 4.7. (1) Let X be any uncountable set and let $L = 2^X$. We define for any $A, B \in L$: $A \leqslant B$ if $A \subset B$ and $A' = X \setminus A$. Then L is a complete Boolean algebra, which is (o)-continuous and τ_i is T_2. Putting for every $x \in X$: $\omega_x(A) = 1$ if $x \in A$ and $\omega_x(A) = 0$ if $x \notin A$, we obtain that $M = \{\omega_x \mid x \in X\}$ is an ordering set of (o)-continuous measures for L. Note that τ_o is not discrete and L is not separable.

(2) It is not difficult to construct a nonboolean logic with finitely many elements which has an ordering set of measures. Such a logic also satisfies the conditions of Th. 4.6.

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