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A NOTE ON GENERAL RELATIONS

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ABSTRACT. Basic concepts concerning binary and ternary relations are extended to relations of arbitrary arities and then investigated.

The relations dealt with in the note are considered in the general sense as systems of maps. More precisely, by a relation we understand a subset $R \subseteqq G^H$, where G, H are sets (and G^H denotes the set of all maps of H into G). G and H are called the *carrier* and the *index set* of R, respectively. Relations with well-ordered index sets, the so called *relations of type* α, are studied in [6]. In this note, the fundamental concepts concerning binary and ternary relations are extended to general relations and discussed. Also some new concepts are defined and investigated. The same is done also in [7], but in the present note the concepts are extended to general relations in a more convenient way which ensures that all usual rules are preserved, which is not true in [7]. Moreover, the concepts introduced here extend the corresponding concepts defined for relations of type α in [6]. So the outcomes of the submitted contribution can be considered as generalizations of those of [6].

The aim of this paper is only to sketch a possibility of extending the study of binary and ternary relations to general relations. Therefore, in the few propositions presented, we describe only the fundamental behaviour of concepts extended or newly introduced. The proofs of propositions that can be carried out from definitions by quite simple considerations are usually omitted.

We denote by $\mathbb{N}$ the set of all positive integers (i.e. finite cardinal numbers greater than 0).

DEFINITION 1. Let H be a set with $\operatorname{card} H \geqq 2$. A *b-decomposition* of H is a pair $\left(\{K_i\}_{i=1}^3, \varphi\right)$, where $\{K_i\}_{i=1}^3$ is a sequence of three sets satisfying

(a) $\bigcup\limits_{i=1}^{3} K_i = H$,

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(b) $K_i \cap K_j = \emptyset$, whenever $i, j \in \{1,2,3\}$, $i \neq j$,
(c) $\operatorname{card} K_1 = \operatorname{card} K_2 > 0$,

and $\varphi\colon K_1 \to K_2$ is a bijection.

DEFINITION 2. Let H be a set with $\operatorname{card} H \geqq 3$. Let $n \in \mathbb{N}$ be a number for which there exists a cardinal number p with $p \cdot n = \operatorname{card} H$. By a *$t_n$-decomposition* of H we understand a pair $\left(\{K_i\}_{i=1}^{n}, \{\varphi_i\}_{i=1}^{n-1}\right)$, where $\{K_i\}_{i=1}^{n}$ is a sequence of n sets satisfying

(a) $\bigcup\limits_{i=1}^{n} K_i = H$,
(b) $K_i \cap K_j = \emptyset$ whenever $i, j \in \{1, \dots, n\}$, $i \neq j$,
(c) $\operatorname{card} K_i = p$ for each $i \in \{1, \dots, n\}$,

and $\{\varphi_i\}_{i=1}^{n-1}$ is a sequence of bijections $\varphi_i\colon K_i \to K_{i+1}$, $i = 1, \dots, n-1$.

For any map $f\colon H \to G$ and any subset $K \subseteqq H$, we denote by $f|_K$ the restriction of f to K. The abbreviation w.r.t. will be written instead of the phrase "with respect to".

DEFINITION 3. Let G, H be sets, and let $\mathcal{K} = \left(\{K_i\}_{i=1}^{3}, \varphi\right)$ be a b-decomposition of H. The relation $\left\{f \in G^H;\ f|_{K_1} = f|_{K_2} \circ \varphi\right\}$ is called *diagonal* w.r.t. $\mathcal{K}$ and denoted by $E_{\mathcal{K}}$.

DEFINITION 4. Let $R \subseteqq G^H$ be a relation, and let $\mathcal{K} = \left(\{K_i\}_{i=1}^{3}, \varphi\right)$ be a b-decomposition of H. Then we define a relation $R_{\mathcal{K}}^{-1} \subseteqq G^H$ by

$$f \in R_{\mathcal{K}}^{-1} \iff \exists g \in R:\ f|_{K_1} = g|_{K_2} \circ \varphi,\ f|_{K_2} = g|K_1 \circ \varphi^{-1},\ f|_{K_3} = g|_{K_3}.$$

$R_{\mathcal{K}}^{-1}$ is called the *inversion* of R w.r.t. $\mathcal{K}$.

DEFINITION 5. Let $R, S \subseteqq G^H$ be relations, and let $\mathcal{K} = \left(\{K_i\}_{i=1}^{3}, \varphi\right)$ be a b-decomposition of H. Then we define a relation $(RS)_{\mathcal{K}} \subseteqq G^H$ by

$$\begin{aligned} f \in (RS)_{\mathcal{K}} \iff \exists g \in R,\ \exists h \in S:\ & g|_{K_2} = h|_{K_1} \circ \varphi^{-1},\ g|_{K_3} = h|_{K_3}, \\ & f|_{K_1} = g|_{K_1},\ f|_{K_2 \cup K_3} = h|_{K_2 \cup K_3}. \end{aligned}$$

$(RS)_{\mathcal{K}}$ is called the *composition* of R and S w.r.t. $\mathcal{K}$.

Next, we set $R_{\mathcal{K}}^{1} = R$ and $R_{\mathcal{K}}^{m+1} = (R_{\mathcal{K}}^{m} R)_{\mathcal{K}}$ for each $m \in \mathbb{N}$. $R_{\mathcal{K}}^{m}$ is called the *mth power* of R w.r.t. $\mathcal{K}$.

DEFINITION 6. Let $R \subseteqq G^H$ be a relation, and let $\mathcal{K} = \left(\{K_i\}_{i=1}^{n}, \{\varphi_i\}_{i=1}^{n-1}\right)$ be a t_n-decomposition of H. Then we define a relation ${}^{1}R_{\mathcal{K}} \subseteqq G^H$ by

$$\begin{aligned} f \in {}^{1}R_{\mathcal{K}} \iff \exists g \in R:\ & f|_{K_i} = g|_{K_{i+1}} \circ \varphi_i \quad \text{for } i = 1, \dots, n-1, \\ & f|_{K_n} = g|_{K_1} \circ \varphi_1^{-1} \circ \varphi_2^{-1} \circ \dots \circ \varphi_{n-1}^{-1}. \end{aligned}$$

Next, we set ${}^{m+1}R_{\mathcal{K}} = {}^{1}({}^{m}R_{\mathcal{K}})_{\mathcal{K}}$ for each $m \in \mathbb{N}$. ${}^{m}R_{\mathcal{K}}$ is called the *mth cyclic transposition* of R w.r.t. $\mathcal{K}$.

For binary relations, i.e. relations whose index set is $\{1, 2\}$, the usual concepts of diagonality, inversion and composition coincide with the above concepts w.r.t. the b-decomposition $\left(\{K_i\}_{i=1}^{3}, \varphi\right)$, where $K_1 = \{1\}$, $K_2 = \{2\}$, $K_3 = \emptyset$ and $\varphi\colon \{1\} \to \{2\}$. Analogously to binary relations we get (cf. [6], [7]):

PROPOSITION 1. *Let $R, S, T, U \subseteqq G^H$ be relations, and let $\mathcal{K}$ be a b-decomposition of H. Then*

$$
\begin{array}{rl}
(1) & E_{\mathcal{K}} = (E_{\mathcal{K}})_{\mathcal{K}}^{-1}, \\
(2) & R = (R_{\mathcal{K}}^{-1})_{\mathcal{K}}^{-1}, \\
(3) & (R \cup S)_{\mathcal{K}}^{-1} = R_{\mathcal{K}}^{-1} \cup S_{\mathcal{K}}^{-1}, \\
(4) & (R \cap S)_{\mathcal{K}}^{-1} = R_{\mathcal{K}}^{-1} \cap S_{\mathcal{K}}^{-1}, \\
(5) & (R - S)_{\mathcal{K}}^{-1} = R_{\mathcal{K}}^{-1} - S_{\mathcal{K}}^{-1}, \\
(6) & R \subseteqq S \implies R_{\mathcal{K}}^{-1} \subseteqq S_{\mathcal{K}}^{-1}, \\
(7) & R = (RE_{\mathcal{K}})_{\mathcal{K}} = (E_{\mathcal{K}}R)_{\mathcal{K}}, \\
(8) & \big((RS)_{\mathcal{K}}\big)_{\mathcal{K}}^{-1} = (S_{\mathcal{K}}^{-1}R_{\mathcal{K}}^{-1})_{\mathcal{K}}, \\
(9) & \big((RS)_{\mathcal{K}}T\big)_{\mathcal{K}} = \big(R(ST)_{\mathcal{K}}\big)_{\mathcal{K}}, \\
(10) & R \subseteqq S,\ T \subseteqq U \implies (RT)_{\mathcal{K}} \subseteqq (SU)_{\mathcal{K}}, \\
(11) & \big((R \cup S)T\big)_{\mathcal{K}} = (RT)_{\mathcal{K}} \cup (ST)_{\mathcal{K}},\ \big(T(R \cup S)\big)_{\mathcal{K}} = (TR)_{\mathcal{K}} \cup (TS)_{\mathcal{K}}, \\
(12) & \big((R \cap S)T\big)_{\mathcal{K}} \subseteqq (RT)_{\mathcal{K}} \cap (ST)_{\mathcal{K}},\ \big(T(R \cap S)\big)_{\mathcal{K}} \subseteqq (TR)_{\mathcal{K}} \cap (TS)_{\mathcal{K}}, \\
(13) & \big((R - S)T\big)_{\mathcal{K}} \supseteqq (RT)_{\mathcal{K}} - (ST)_{\mathcal{K}},\ \big(T(R - S)\big)_{\mathcal{K}} \supseteqq (TR)_{\mathcal{K}} - (TS)_{\mathcal{K}}, \\
(14) & (R_{\mathcal{K}}^{m}R_{\mathcal{K}}^{n})_{\mathcal{K}} = R_{\mathcal{K}}^{m+n},\ (R_{\mathcal{K}}^{m})_{\mathcal{K}}^{n} = R_{\mathcal{K}}^{mn} \text{ for any } m, n \in \mathbb{N}.
\end{array}
$$

The concept of cyclic transposition has originally been introduced for relations of type α in [6]. The cyclic transposition defined above for general relations extends all considerable types of the cyclic transposition of relations of type α from [6]. The following lemma and proposition show that the laws satisfied for cyclic transpositions of relations of type α (see [6]) are satisfied also for cyclic transpositions of general relations.

LEMMA. *Let $R \subseteqq G^H$ be a relation, and let $\mathcal{K} = \left(\{K_i\}_{i=1}^{n}, \{\varphi_i\}_{i=1}^{n-1}\right)$ be a t_n-decomposition of H. Then for any $m \in \mathbb{N}$, $m < n$, we have*

$$
\begin{aligned}
f \in {}^{m}R_{\mathcal{K}} \iff \exists g \in R :\ & f|_{K_i} = g|_{K_{i+m}} \circ \varphi_{i+m-1} \circ \varphi_{i+m-2} \circ \cdots \circ \varphi_i \\
& \qquad \textit{for} \quad i = 1, \ldots, n - m, \\
& f|_{K_i} = g|_{K_{i-n+m}} \circ \varphi_{i-n+m} \circ \varphi_{i-n+m+1} \circ \cdots \circ \varphi_{i-1} \\
& \qquad \textit{for} \quad i = n - m + 1, \ldots, n.
\end{aligned}
$$

P r o o f . For $m = 1$ the assertion follows immediately from Definition 6. Suppose that the assertion is valid for a number $m \in \mathbb{N}$, $m < n-1$. Then

$$\begin{aligned} h \in {}^{m+1}R_{\mathcal{K}} \iff \exists f \in {}^{m}R_{\mathcal{K}} :\ & h\big|_{K_i} = f\big|_{K_{i+1}} \circ \varphi_i \quad \text{for} \quad i = 1, \dots, n-1, \\ & h\big|_{K_n} = f\big|_{K_1} \circ \varphi_1^{-1} \circ \varphi_2^{-1} \circ \dots \circ \varphi_{n-1}^{-1} . \end{aligned}$$

Consequently,

$$\begin{aligned} h \in {}^{m+1}R_{\mathcal{K}} \iff \exists g \in R :\ & \\ & h\big|_{K_i} = g\big|_{K_{i+m+1}} \circ \varphi_{i+m} \circ \varphi_{i+m-1} \circ \dots \circ \varphi_{i+1} \circ \varphi_i \\ & \qquad \text{for} \quad i = 1, \dots, n-m-1, \\ & h\big|_{K_{n-m}} = g\big|_{K_1} \circ \varphi_1^{-1} \circ \varphi_2^{-1} \circ \dots \circ \varphi_{n-m}^{-1} \circ \varphi_{n-m} , \\ & h\big|_{K_i} = g\big|_{K_{i-n+m+1}} \circ \varphi_{i-n+m+1}^{-1} \circ \varphi_{i-n+m+2}^{-1} \circ \dots \circ \varphi_i^{-1} \circ \varphi_i \\ & \qquad \text{for} \quad i = n-m+1, \dots, n-1, \\ & h\big|_{K_n} = g\big|_{K_{m+1}} \circ \varphi_m \circ \varphi_{m-1} \circ \dots \circ \varphi_1 \circ \varphi_1^{-1} \circ \varphi_2^{-1} \circ \dots \circ \varphi_{n-1}^{-1} . \end{aligned}$$

This yields

$$\begin{aligned} h \in {}^{m+1}R_{\mathcal{K}} \iff \exists g \in R :\ & \\ & h\big|_{K_i} = g\big|_{K_{i+m+1}} \circ \varphi_{i+m} \circ \varphi_{i+m-1} \circ \dots \circ \varphi_i \\ & \qquad \text{for} \quad i = 1, \dots, n-m-1, \\ & h\big|_{K_i} = g\big|_{K_{i-n+m+1}} \circ \varphi_{i-n+m+1}^{-1} \circ \varphi_{i-n+m+2}^{-1} \circ \dots \circ \varphi_{i-1}^{-1} \\ & \qquad \text{for} \quad i = n-m, \dots, n . \end{aligned}$$

Hence the assertion is valid for $m+1$, and the proof is complete. □

PROPOSITION 2. *Let $R, S \subseteqq G^H$ be relations, and let $\mathcal{K}$ be a t_n-decomposition of H. Let $m, k \in \mathbb{N}$. Then*

(1) $R = {}^{n}R_{\mathcal{K}}$,
(2) ${}^{m}(R \cup S)_{\mathcal{K}} = {}^{m}R_{\mathcal{K}} \cup {}^{m}S_{\mathcal{K}}$,
(3) ${}^{m}(R \cap S)_{\mathcal{K}} = {}^{m}R_{\mathcal{K}} \cap {}^{m}S_{\mathcal{K}}$,
(4) ${}^{m}(R - S)_{\mathcal{K}} = {}^{m}R_{\mathcal{K}} - {}^{m}S_{\mathcal{K}}$,
(5) $R \subseteqq S \implies {}^{m}R_{\mathcal{K}} \subseteqq {}^{m}S_{\mathcal{K}}$,
(6) ${}^{k}({}^{m}R_{\mathcal{K}})_{\mathcal{K}} = {}^{m+k}R_{\mathcal{K}}$.

P r o o f .

(1) Let $h \in {}^{n}R_{\mathcal{K}}$. Then there exists $f \in {}^{n-1}R_{\mathcal{K}}$ such that $h\big|_{K_i} = f\big|_{K_{i+1}} \circ \varphi_i$ for $i = 1, \dots, n-1$ and $h\big|_{K_n} = f\big|_{K_1} \circ \varphi_1^{-1} \circ \varphi_2^{-1} \circ \dots \circ \varphi_{n-1}^{-1}$. By Lemma, from $f \in {}^{n-1}R_{\mathcal{K}}$ it follows that there exists $g \in R$ such that $f\big|_{K_1} = g\big|_{K_n} \circ$

$\varphi_{n-1} \circ \varphi_{n-2} \circ \cdots \circ \varphi_1$ and $f\big|_{K_i} = g\big|_{K_{i-1}} \circ \varphi_{i-1}^{-1}$ for $i = 2, \ldots, n$. We have $h\big|_{K_i} = g\big|_{K_i} \circ \varphi_i^{-1} \circ \varphi_i = g\big|_{K_i}$ for $i = 1, \ldots, n-1$, and $h\big|_{K_n} = g\big|_{K_n} \circ \varphi_{n-1} \circ \varphi_{n-2} \circ \cdots \circ \varphi_1 \circ \varphi_1^{-1} \circ \varphi_2^{-1} \circ \cdots \circ \varphi_{n-1}^{-1} = g\big|_{K_n}$. Therefore, $h = g \in R$. We have proved the inclusion ${}^nR_{\mathcal{K}} \subseteqq R$. Reversing the arguments we can prove the converse.

The assertions (2)–(6) follow immediately from Definition 6. □

In [7], we also defined concepts of diagonality, inversion, composition and cyclic transposition for general relations with regard to certain decompositions of the index set. But these decompositions do not contain any bijections between components, and therefore only the images of entire components are used for defining the concepts. This deficiency has as a consequence that laws like (2), (4) in Proposition 1 or (1), (3) in Proposition 2 are not satisfied in general. On the other hand, the s-decompositions and t-decompositions introduced for well-ordered index sets in [6] implicitly contain bijections between two of the components, and so they are special cases of the b-decompositions defined above. Similarly, all considerable types of the c-decompositions from [6] are special cases of the above t_n-decompositions.

DEFINITION 7. Let $R \subseteqq G^H$ be a relation, and let $\mathcal{K}$ be a b-decomposition of H. Then R is called

reflexive (*irreflexive*) w.r.t. $\mathcal{K}$ if $E_{\mathcal{K}} \subseteqq R$ ($R \cap E_{\mathcal{K}} = \emptyset$),
symmetric (*asymmetric, antisymmetric*) w.r.t. $\mathcal{K}$ if $R_{\mathcal{K}}^{-1} \subseteqq R$ ($R \cap R_{\mathcal{K}}^{-1} = \emptyset$, $R \cap R_{\mathcal{K}}^{-1} \subseteqq E_{\mathcal{K}}$),
transitive (*atransitive*) w.r.t. $\mathcal{K}$ if $R_{\mathcal{K}}^2 \subseteqq R$ ($R \cap R_{\mathcal{K}}^m = \emptyset$ for every $m \in \mathbb{N}$, $m \geqq 2$).

PROPOSITION 3. *Let $R, S \subseteqq G^H$ be relations, and let $\mathcal{K}$ be a b-decomposition of H. Then*

(a) *if R, S are reflexive w.r.t. $\mathcal{K}$, then so are $R \cup S$, $R \cap S$, $R_{\mathcal{K}}^{-1}$, $(RS)_{\mathcal{K}}$;*
(b) *if R, S are irreflexive (symmetric) w.r.t. $\mathcal{K}$, then so are $R \cup S$, $R \cap S$, $R_{\mathcal{K}}^{-1}$;*
(c) *if R, S are transitive w.r.t. $\mathcal{K}$, then so are $R \cap S$ and $R_{\mathcal{K}}^{-1}$;*
(d) *if R, S are asymmetric (antisymmetric, atransitive), then so are $R \cap S$ and $R_{\mathcal{K}}^{-1}$;*
(e) *if R, S are symmetric w.r.t. $\mathcal{K}$, then the condition $(RS)_{\mathcal{K}} = (SR)_{\mathcal{K}}$ is necessary and sufficient for $(RS)_{\mathcal{K}}$ to be symmetric w.r.t. $\mathcal{K}$, too.*
(f) *$\bigcup\limits_{k=1}^{\infty} R_{\mathcal{K}}^k$ is the least (with respect to the set inclusion) of all relations $T \subseteqq G^H$ that are transitive w.r.t. $\mathcal{K}$ and fulfil $R \subseteqq T$.*

P r o o f . It follows directly from Definition 7 and Proposition 1. Let us prove, say, (f). To this end, let $\mathcal{K} = \left(\{K_i\}_{i=1}^3, \varphi\right)$. Denote $Q = \bigcup\limits_{k=1}^{\infty} R_{\mathcal{K}}^k$, and let $f \in Q_{\mathcal{K}}^2$. Then there exist $g, h \in Q$ such that $g|_{K_2} = h|_{K_1} \circ \varphi^{-1}$, $g|_{K_3} = h|_{K_3}$, $f|_{K_1} = g|_{K_1}$ and $f|_{K_2 \cup K_3} = h|_{K_2 \cup K_3}$. As $g, h \in Q$, there are $m, n \in \mathbb{N}$ so that $g \in R_{\mathcal{K}}^m$, $h \in R_{\mathcal{K}}^n$. Consequently, $f \in \left(R_{\mathcal{K}}^m R_{\mathcal{K}}^n\right)_{\mathcal{K}} = R_{\mathcal{K}}^{m+n} \subseteqq Q$. Hence $Q_{\mathcal{K}}^2 \subseteqq Q$. Let $T \subseteqq G^H$ be a relation, transitive w.r.t. $\mathcal{K}$ and such that $R \subseteqq T$. Then $R_{\mathcal{K}}^n \subseteqq T_{\mathcal{K}}^n = T$ for each $n \in \mathbb{N}$, thus $Q \subseteqq T$. Since $Q \supseteqq R$, (f) is proved. □

DEFINITION 8. Let $R \subseteqq G^H$ be a relation, and let $\mathcal{K}$ be a t_n-decomposition of H. Then R is called *cyclic* (*acyclic*) w.r.t. $\mathcal{K}$ if ${}^1R_{\mathcal{K}} \subseteqq R$ ($R \cap {}^1R_{\mathcal{K}} = \emptyset$).

PROPOSITION 4. *Let $R, S \subseteqq G^H$ be relations, and let $\mathcal{K}$ be a t_n-decomposition of H. Then*

(a) *if R, S are cyclic w.r.t. $\mathcal{K}$, then so are $R \cup S$, $R \cap S$, ${}^1R_{\mathcal{K}}$;*
(b) *if R, S are acyclic w.r.t. $\mathcal{K}$, then so are $R \cap S$ and ${}^1R_{\mathcal{K}}$;*
(c) *$\bigcup\limits_{k=1}^{n} {}^kR_{\mathcal{K}}$ is the least (with respect to the set inclusion) of all relations $T \subseteqq G^H$ that are cyclic w.r.t. $\mathcal{K}$ and fulfil $R \subseteqq T$.*

P r o o f . It follows directly from Definition 8 and Proposition 2. Let us prove, say, (c). To this end, denote $Q = \bigcup\limits_{k=1}^{n} {}^kR_{\mathcal{K}}$. Then $R \subseteqq Q$, and we have ${}^1Q_{\mathcal{K}} = \bigcup\limits_{k=1}^{n} {}^{k+1}R_{\mathcal{K}} = \bigcup\limits_{k=2}^{n+1} {}^kR_{\mathcal{K}} = \bigcup\limits_{k=1}^{n} {}^kR_{\mathcal{K}} = Q$. Hence Q is cyclic w.r.t. $\mathcal{K}$. Let $T \subseteqq G^H$ be a relation cyclic w.r.t. $\mathcal{K}$ such that $R \subseteqq T$. Then $Q = \bigcup\limits_{k=1}^{n} {}^kR_{\mathcal{K}} \subseteqq \bigcup\limits_{k=1}^{n} {}^kT_{\mathcal{K}} \subseteqq T$. This proves (c). □

Obviously, the concepts introduced in Definition 7 extend those well known for binary relations. But some of them also extend the concepts known for ternary relations (i.e. relations whose index set is $\{1,2,3\}$), see e.g. [2], [4], [5]. The concepts introduced in Definition 8 extend the usual concepts considered for ternary relations. More precisely, the usual cyclicity (acyclicity) of ternary relations coincides with the cyclicity (acyclicity) w.r.t. the t_3-decomposition $\left(\{K_i\}_{i=1}^3, \{\varphi_i\}_{i=1}^2\right)$ of the index set $\{1,2,3\}$, where $K_i = \{i\}$ for $i = 1,2,3$ and $\varphi_i \colon \{i\} \to \{i+1\}$ for $i = 1,2$. All concepts introduced in Definitions 7 and 8 extend the corresponding concepts (concerning relations of type α) from [6]. Next, let us note that each t_3-decomposition $\left(\{K_i\}_{i=1}^3, \{\varphi_i\}_{i=1}^2\right)$ of a given set can also be understood as its b-decomposition $\left(\{K_i\}_{i=1}^3, \varphi_1\right)$. Consequently, if $R \subseteqq G^H$ is a relation and $\mathcal{K}$ a t_3-decomposition of H, then for R one can consider the concepts w.r.t. $\mathcal{K}$, introduced both in Definition 7 and 8.

Now we shall extend the known procedure by which a binary relation is assigned to any ternary relation and any element of the carrier – see e.g. [2], [5].

Definition 9. Let $R \subseteqq G^H$ be a relation, and let $\mathcal{K} = \big(\{K_i\}_{i=1}^3, \varphi\big)$ be a b-decomposition of H. Let $X = \{x_k\,;\ k \in K_3\}$ be a family of elements of G. Then we define a relation $R_{X,\mathcal{K}} \subseteqq G^{H-K_3}$ by

$$f \in R_{X,\mathcal{K}} \iff \exists g \in R:\ f(k) = g(k) \quad \text{for each } k \in K_1 \cup K_2 \quad \text{and}$$
$$g(k) = x_k \quad \text{for each } k \in K_3\,.$$

$R_{X,\mathcal{K}}$ is called the *X-projection* of R w.r.t. $\mathcal{K}$.

Notation. If $\mathcal{K} = \big(\{K_i\}_{i=1}^3, \varphi\big)$ is a b-decomposition of a set H, then we denote by $\tilde{\mathcal{K}}$ the b-decomposition of $H - K_3$ given by $\tilde{\mathcal{K}} = \big(\{\tilde{K}_i\}_{i=1}^3, \varphi\big)$, where $\tilde{K}_i = K_i$ for $i = 1, 2$, and $K_3 = \emptyset$.

Proposition 5. *Let $R, S \subseteqq G^H$ be relations, and let $\mathcal{K} = \big(\{K_i\}_{i=1}^3, \varphi\big)$ be a b-decomposition of H. Let $X = \{x_k\,;\ k \in K_3\}$ be a family of elements of G. Then*

(1) $(E_{\mathcal{K}})_{X,\mathcal{K}} = E_{\tilde{\mathcal{K}}}$,
(2) $(R_{\mathcal{K}}^{-1})_{X,\tilde{\mathcal{K}}} = (R_{X,\mathcal{K}})_{\tilde{\mathcal{K}}}^{-1}$,
(3) $(R \cup S)_{X,\mathcal{K}} = R_{X,\mathcal{K}} \cup S_{X,\mathcal{K}}$,
(4) $(R \cap S)_{X,\mathcal{K}} = R_{X,\mathcal{K}} \cap S_{X,\mathcal{K}}$,
(5) $\big((RS)_{\mathcal{K}}\big)_{X,\mathcal{K}} = (R_{X,\mathcal{K}} S_{X,\mathcal{K}})_{\tilde{\mathcal{K}}}$,
(6) $R \subseteqq S \implies R_{X,\mathcal{K}} \subseteqq S_{X,\mathcal{K}}$.

Proposition 6. *Let $R, S \subseteqq G^H$ be relations, and let $\mathcal{K} = \big(\{K_i\}_{i=1}^3, \varphi\big)$ be a b-decomposition of H. If $R_{X,\mathcal{K}} \subseteqq S_{X,\mathcal{K}}$ for each family $X = \{x_k\,;\ k \in K_3\}$ of elements of G, then $R \subseteqq S$.*

Proposition 7. *Let $R \subseteqq G^H$ be a relation, and let $\mathcal{K} = \big(\{K_i\}_{i=1}^3, \varphi\big)$ be a b-decomposition of H. If R has any one of the properties w.r.t. $\mathcal{K}$ introduced in Definition 7, then $R_{X,\mathcal{K}}$ has the same property w.r.t. $\tilde{\mathcal{K}}$ for each family $X = \{x_k\,;\ k \in K_3\}$ of elements of G, and vice versa.*

Also many other properties known for binary or ternary relations can be extended to general relations in the way shown above. Particularly, the concepts introduced for general relations in [7] could be redefined w.r.t. the decompositions considered and studied in this note. But the results obtained would be analogous to those of [7]. Further, relations having several properties extended simultaneously can be dealt with, too. Especially, for general relations we can define and study quasiorders, orders, tolerances or equivalences w.r.t. a b-decomposition of the index set considered. Or we can introduce and investigate cyclic orders for general relations w.r.t. a t_n-decomposition of the index set. The three direct operations of addition, multiplication and exponentiation for general relational systems are dealt with in [8].

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