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ON A CHARACTERIZATION OF PROBABILITY MEASURES ON BOOLEAN ALGEBRAS AND SOME ORTHOMODULAR LATTICES

HELMUT LÄNGER* — MACIEJ MĄCZYŃSKI**

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ABSTRACT. Inequalities of Bell-type in Boolean algebras and in horizontal sums of such algebras are considered. Using combinatorial methods, results concerning the number of valid Bell-type inequalities are derived. Finally, the problem of extending mappings from certain subsets of finite Boolean algebras B to (finitely additive) probability measures on B is discussed.

1. Introduction

Probability measures on Boolean algebras can be characterized by axioms of probability theory expressed in terms of Boolean operations on the domain of their definition. Another way of approaching to this problem is to consider probability measures as real-valued mappings on Boolean algebras which satisfy a set of inequalities. The characterization of probability measures by inequalities is of special importance for some physical applications related to experiments which intend to determine whether a system is classical or non-classical. In this way, we obtain the so-called Bell or Clauser-Horne inequalities, well-known in theoretical physics. (See [1], [3] or [4] for a full discussion of these inequalities. In [2], the physical applications of generalized Bell inequalities are discussed.) Such inequalities can be defined as inequalities of the form $0 \leq L \leq 1$, where L is a linear combination, with real coefficients, of probabilities of individual events $p_i = p(a_i)$, $i = 1, 2, \ldots, n$, as well as of probabilities of some intersections of these events (e.g., $p_{ij} = p(a_i \wedge a_j)$), which are called *correlation probabilities.* In special cases, only inequalities with integer or even with ± 1 coefficients are considered. In order to have a physical application, such inequalities should be valid in every Boolean algebra, for every probability measure and every choice

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of events a_i. If there are statistical data coming from an experiment for which such an inequality is violated, then we draw the conclusion that the system is not classical, i.e., the events cannot form a Boolean algebra. A general method for proving linear inequalities involving probabilities has been developed in 1958 by Rényi [5], but no application of this method to Bell-type inequalities has been discussed. In this paper, we would like to apply the method of Rényi to the inequalities of the form $0 \leq L \leq 1$. We present a very simple proof of the characterization theorem, independent of the proof of Rényi. We also develop a method allowing us to estimate the number of such inequalities which are valid in a Boolean algebra. We extend our results to orthomodular lattices which are built up from Boolean algebras (the so-called horizontal sums of Boolean algebras). We shall show that there are Bell-type inequalities which are valid in all Boolean algebras, but which are not valid in some orthomodular lattices. In the last section of this paper, we prove theorems giving necessary and sufficient conditions for a real-valued mapping defined on the set of intersections over all subsets of some fixed generating set of a finite Boolean algebra in order to have an extension to a probability measure on the whole Boolean algebra. From this theorem, we obtain a theorem on extension to two-valued homomorphisms, related to Sikorski's theorem on extension to homomorphisms ([7]).

The organization of this paper is as follows: In Section 2, we prove some combinatorial lemmas to be used in the following sections. In Section 3, we prove a theorem on verification of Bell-type probability inequalities (on Boolean algebras), and we give some estimates of the number of such inequalities. In Section 4, we extend our results to some orthomodular lattices. In Section 5, we prove theorems on the extension of mappings to probability measures and to homomorphisms.

In the following, let n denote a positive integer and put $N := \{1, \dots, n\}$.

2. Some combinatorial lemmas

Here, we state some combinatorial lemmas. The results are not new. But for the sake of completeness of our paper, we state these lemmas together with some ideas of proof.

LEMMA 1. *Exactly half of the subsets of a finite non-empty set are of even cardinality.*

Proof. Induction on the cardinality of the base set. □

Lemma 2. *For a non-negative integer m and a real number a it holds*

$$\sum_{i=0}^{m}\binom{m}{i}(-1)^i=\delta_{m0}\,,\qquad \sum_{i=0}^{[\frac{m}{2}]}\binom{m}{2i}a^{2i}=\frac{(1+a)^m+(1-a)^m}{2}$$

and

$$\sum_{i=1}^{[\frac{m+1}{2}]}\binom{m}{2i-1}a^{2i-1}=\frac{(1+a)^m-(1-a)^m}{2}\,.$$

(Here $b^0:=1$ for every $b\in\mathbb{R}$.)

Proof. Binomial Theorem. □

Lemma 3.

$$\sum_{i=1}^{n}\binom{n}{i}\binom{n+1}{i-1}=\binom{2n+1}{n-1},$$
$$\sum_{i=0}^{n}\binom{n}{i}\binom{n+1}{i}=\binom{2n+1}{n},$$
$$\sum_{i=0}^{n}\binom{n}{i}\binom{n+1}{i+1}=\binom{2n+1}{n}.$$

Proof. The corresponding sum is the coefficient of x^n in

$$(x+1)^n\big(x(x+1)^{n+1}\big),\quad (x+1)^n(x+1)^{n+1}\quad\text{and}\quad (x+1)^n\frac{(x+1)^{n+1}-1}{x},$$

respectively. □

Lemma 4.

$$f\mapsto\Big(K\mapsto\sum_{I\subseteq K}f(I)\Big)\quad\textit{and}\quad g\mapsto\Big(K\mapsto\sum_{I\subseteq K}(-1)^{|K|-|I|}g(I)\Big)$$

are mutually inverse bijections between $\mathbb{R}^{2^N}$ and $\mathbb{R}^{2^N}$.

$$f\mapsto\Big(K\mapsto\sum_{K\subseteq I\subseteq N}f(I)\Big)\quad\textit{and}\quad g\mapsto\Big(K\mapsto\sum_{K\subseteq I\subseteq N}(-1)^{|I|-|K|}g(I)\Big)$$

are mutually inverse bijections between $\mathbb{R}^{2^N}$ and $\mathbb{R}^{2^N}$.

Proof. It suffices to show that the composition of two corresponding mappings always yields the identity mapping on $\mathbb{R}^{2^N}$. This can be done by interchanging sum signs and applying Lemma 2. □

3. Inequalities for probabilities on Boolean algebras

For the rest of the paper, by a probability measure on a Boolean algebra (or, more generally, on an ortholattice), we will always understand a finitely additive normed measure.

LEMMA 5. *Let $I \subseteq N$. Then there exist a finite Boolean algebra B, $a_1, \ldots, a_n \in B \setminus \{0,1\}$, and a probability measure p on B such that for all $K \subseteq N$ it holds*

$$p\Big(\bigwedge_{i\in K} a_i \wedge \bigwedge_{i\in N\setminus K} a_i'\Big) = \delta_{KI}\,.$$

Proof. Put $B := 2^{2^N}$ and $a_i := \{K \subseteq N \mid i \in K\}$ for all $i \in N$, and define $p\colon B \to \{0,1\}$ by

$$p(x) := \begin{cases} 1 & \text{if } I \in x\,, \\ 0 & \text{otherwise} \end{cases} \qquad (x \in B).$$

Then

$$\bigwedge_{i\in K} a_i \wedge \bigwedge_{i\in N\setminus K} a_i' = \{K\}$$

for all $K \subseteq N$. The rest of the proof is clear. □

THEOREM 6. *Let $f\colon 2^N \to \mathbb{R}$. Then the following are equivalent:*

(i)

$$\sum_{I\subseteq N} f(I)p\Big(\bigwedge_{i\in I} a_i\Big) \in [0,1] \tag{1}$$

for all Boolean algebras B, all $a_1, \ldots, a_n \in B$, and all probability measures p on B.

(ii)

$$\sum_{I\subseteq K} f(I) \in [0,1]$$

for all $K \subseteq N$.

(iii) *There exists a function $g\colon 2^N \to [0,1]$ such that for all $I \subseteq N$ it holds*

$$f(I) = \sum_{K\subseteq I} (-1)^{|I|-|K|} g(K)\,.$$

P r o o f . If B is a Boolean algebra, $a_1, \dots, a_n \in B$, and p is a probability measure on B, then

$$\begin{aligned}\sum_{I\subseteq N} f(I)p\Big(\bigwedge_{i\in I} a_i\Big) &= \sum_{I\subseteq N} f(I)p\Big(\bigvee_{I\subseteq K\subseteq N}\Big(\bigwedge_{i\in K} a_i \wedge \bigwedge_{i\in N\setminus K} a_i'\Big)\Big)\\ &= \sum_{I\subseteq N} f(I)\sum_{I\subseteq K\subseteq N} p\Big(\bigwedge_{i\in K} a_i \wedge \bigwedge_{i\in N\setminus K} a_i'\Big)\\ &= \sum_{K\subseteq N} p\Big(\bigwedge_{i\in K} a_i \wedge \bigwedge_{i\in N\setminus K} a_i'\Big)\sum_{I\subseteq K} f(I).\end{aligned}$$

The rest of the proof now follows from Lemmas 5 and 4. □

R e m a r k . From Lemma 4, it follows that, if (iii) holds, then g is uniquely determined, namely,

$$g(I) = \sum_{K\subseteq I} f(K)$$

for all $I \subseteq N$.

THEOREM 7. *There exist exactly 2^{2^n} different inequalities of the form (1) with integer coefficients $f(I)$, $I \subseteq N$, which are valid in all Boolean algebras, namely,*

$$\sum_{I\subseteq N}\Big(\sum_{K\subseteq I}(-1)^{|I|-|K|}g(K)\Big)p\Big(\bigwedge_{i\in I} a_i\Big) \in [0,1],$$

where g runs through $\{0,1\}^{2^N}$. The maximal possible value of these coefficients is 2^{n-1}, the minimum possible value is -2^{n-1}.

P r o o f . We use Theorem 6. Observe that the bijections mentioned in Lemma 4 induce bijections between $\mathbb{Z}^{2^N}$ and $\mathbb{Z}^{2^N}$ and that $[0,1]\cap\mathbb{Z} = \{0,1\}$. The last assertion of Theorem 7 follows by the fact that

$$\sum_{K\subseteq I}(-1)^{|I|-|K|}g(K) = (-1)^{|I|}\Big(\sum_{K\subseteq I,\ |K|\text{ even}} g(K) - \sum_{K\subseteq I,\ |K|\text{ odd}} g(K)\Big)$$

for all $I \subseteq N$ and by Lemma 1. □

We now want to obtain lower and upper bounds for the number of inequalities of the form (1) which are valid in all Boolean algebras and which have coefficients $f(I) \in \{-1,0,1\}$, $I \subseteq N$, with $f(\emptyset) = 0$.

THEOREM 8. *The following sets are equipotent:*

$$\Big\{f\colon 2^N \to \{-1,0,1\} \mid f(\emptyset)=0,\ \sum_{I\subseteq K} f(I)\in\{0,1\} \text{ for all } K\subseteq N\Big\}, \tag{2}$$

$$\Big\{g\colon 2^N \to \{0,1\} \mid g(\emptyset)=0,\ \sum_{I\subseteq K} (-1)^{|K|-|I|} g(I)\in\{-1,0,1\} \text{ for all } K\subseteq N\Big\}, \tag{3}$$

$$\Big\{A\subseteq 2^N \mid \emptyset\notin A,\ \big||\{I\in A\cap 2^K \mid |I| \text{ even}\}| - |\{I\in A\cap 2^K \mid |I| \text{ odd}\}|\big| \le 1 \text{ for all } K\subseteq N\Big\}. \tag{4}$$

P r o o f. That (2) and (3) are equipotent follows from Lemma 4. That (3) and (4) are equipotent follows by observing the well-known bijection between the indicator functions on a fixed set and the subsets of this set. □

THEOREM 9. $\dfrac{7^n+4^n-4\cdot 3^n+6\cdot 2^n-(-1)^n+4}{8}$ *is a lower bound for the cardinality of* (2).

P r o o f. The case $n=1$ is trivial. Now, let $n>1$, and denote the set (4) by M. We use Lemma 1. M has exactly 1, 2^n-1 and $(2^{n-1}-1)2^{n-1}$ members of cardinality 0, 1 and 2, respectively. The three-element members of M either contain (as elements) two distinct non-empty subsets A, B of N of even cardinality and one subset of $A\cup B$ of odd cardinality, or they contain (as elements) two distinct subsets C, D of N of odd cardinality and one non-empty subset of $C\cup D$ of even cardinality. Thus the number of three-element members of M can be calculated as follows:

$$\frac{1}{2}\Bigg(\sum_{i=1}^{[\frac{n}{2}]}\binom{n}{2i}\sum_{j=0}^{i}\binom{2i}{2j}\sum_{k=0}^{[\frac{n-2i}{2}]}\binom{n-2i}{2k}2^{2i+2k-1}$$

$$+\sum_{i=1}^{[\frac{n}{2}]}\binom{n}{2i}\sum_{j=1}^{i}\binom{2i}{2j-1}\sum_{k=1}^{[\frac{n-2i+1}{2}]}\binom{n-2i}{2k-1}2^{2i+2k-2}-2\cdot\sum_{i=1}^{[\frac{n}{2}]}\binom{n}{2i}2^{2i-1}$$

$$+\sum_{i=1}^{[\frac{n+1}{2}]}\binom{n}{2i-1}\sum_{j=0}^{[\frac{2i-1}{2}]}\binom{2i-1}{2j}\sum_{k=1}^{[\frac{n-2i+2}{2}]}\binom{n-2i+1}{2k-1}\big(2^{2i+2k-3}-1\big)$$

$$+\sum_{i=1}^{[\frac{n+1}{2}]}\binom{n}{2i-1}\sum_{j=1}^{i}\binom{2i-1}{2j-1}\sum_{k=0}^{[\frac{n-2i+1}{2}]}\binom{n-2i+1}{2k}\big(2^{2i+2k-2}-1\big)$$

$$- \sum_{i=1}^{[\frac{n+1}{2}]} \binom{n}{2i-1} (2^{2i-2} - 1) \Bigg) .$$

Theorem 9 now follows by applying Lemma 2. □

THEOREM 10. $\frac{3 \cdot 2^{n-1} + 1}{2^n + 2} \binom{2^n}{2^{n-1}}$ *is an upper bound for the cardinality of* (2).

P r o o f . The case $n = 1$ is trivial. Now, let $n > 1$. Using Lemma 1 we count the elements of the set

$$\Big\{ A \subseteq 2^N \mid \emptyset \notin A ,\ \big| |\{ I \in A \mid |I| \text{ even} \}| - |\{ I \in A \mid |I| \text{ odd} \}| \big| \le 1 \Big\} ,$$

which includes (4):

$$\sum_{i=1}^{2^{n-1}-1} \binom{2^{n-1}-1}{i} \binom{2^{n-1}}{i-1} + \sum_{i=0}^{2^{n-1}-1} \binom{2^{n-1}-1}{i} \binom{2^{n-1}}{i} + \sum_{i=0}^{2^{n-1}-1} \binom{2^{n-1}-1}{i} \binom{2^{n-1}}{i+1} .$$

Application of Lemma 3 completes the proof of the theorem. □

THEOREM 11. $\frac{3 \cdot 2^{n-1} + 1}{2^n + 2} \binom{2^n}{2^{n-1}} \sim \frac{3}{\sqrt{2\pi}} 2^{2^n - \frac{n}{2}}$ *for* $n \to \infty$.

P r o o f . Stirling's formula. □

R e m a r k . For $n = 1, \ldots, 5$ the lower and upper bounds mentioned in Theorems 9 and 10 read as follows:

n	lower bound	upper bound
1	2	2
2	7	7
3	44	91
4	304	17875
5	2132	866262915

4. Inequalities for probabilities on some orthomodular lattices

In this section, we shall show that not all inequalities valid in Boolean algebras are valid in all orthomodular lattices. We shall formulate sufficient conditions in order that (1) be valid in some special class of orthomodular lattices called *horizontal sums of Boolean algebras.* Namely, we say that an ortholattice L is the horizontal sum of the Boolean algebras B_k, $k \in K$, if the algebras B_k are subalgebras of L such that L is the set-theoretical union of all B_k, the sets $B_k \setminus \{0,1\}$, $k \in K$, are pairwise disjoint and whenever two elements a, b of L do not lie in the same B_k, then $a \vee b = 1$ and $a \wedge b = 0$. It is clear that L is then orthomodular (i.e., the distributivity laws hold in some special cases) and for $k > 1$, in general, not distributive. We give an example of an inequality which holds in all Boolean algebras but not in all orthomodular lattices. This shows that there exist inequalities which are characteristic for the class of Boolean algebras, but not for the class of orthomodular lattices.

LEMMA 12. *Let $f\colon 2^N \to \mathbb{R}$, let L be the horizontal sum of the Boolean algebras B_k, $k \in K$ ($K \neq \emptyset$), assume $0,1 \notin K$, let $a_1,\dots,a_n \in L$, let p be a probability measure on L, and put $I_0 := \{i \in N \mid a_i = 0\}$, $I_1 := \{i \in N \mid a_i = 1\}$, $I_k := \{i \in N \mid a_i \in B_k \setminus \{0,1\}\}$ for all $k \in K$, and $M := \{k \in K \mid I_k \neq \emptyset\}$. Then*

$$\begin{aligned}
&\sum_{I \subseteq N} f(I)\, p\Big(\bigwedge_{i \in I} a_i\Big) \\
&= \sum_{S \subseteq I_1} f(S) + \sum_{k \in M} \sum_{\emptyset \neq T \subseteq I_k} p\Big(\bigwedge_{i \in T} a_i \wedge \bigwedge_{i \in I_k \setminus T} a_i'\Big) \sum_{\emptyset \neq I \subseteq T} \sum_{S \subseteq I_1} f(I \cup S) \\
&= \sum_{k \in M} \sum_{T \subseteq I_k} p\Big(\bigwedge_{i \in T} a_i \wedge \bigwedge_{i \in I_k \setminus T} a_i'\Big) \sum_{I \subseteq T} \sum_{S \subseteq I_1} f(I \cup S) - \big(|M| - 1\big) \sum_{S \subseteq I_1} f(S).
\end{aligned}$$

Proof.

$$\begin{aligned}
\sum_{I \subseteq N} f(I)\, p\Big(\bigwedge_{i \in I} a_i\Big) &= \sum_{I \subseteq N \setminus I_0} f(I) p\Big(\bigwedge_{i \in I} a_i\Big) \\
&= \sum_{I \subseteq N \setminus (I_0 \cup I_1)} \sum_{S \subseteq I_1} f(I \cup S) p\Big(\bigwedge_{i \in I} a_i\Big) \\
&= \sum_{S \subseteq I_1} f(S) + \sum_{\emptyset \neq I \subseteq N \setminus (I_0 \cup I_1)} \sum_{S \subseteq I_1} f(I \cup S) p\Big(\bigwedge_{i \in I} a_i\Big)
\end{aligned}$$

$$= \sum_{S\subseteq I_1} f(S) + \sum_{k\in M} \sum_{\emptyset\neq I\subseteq I_k} \sum_{S\subseteq I_1} f(I\cup S) p\Big(\bigwedge_{i\in I} a_i\Big)$$

$$= \sum_{S\subseteq I_1} f(S) + \sum_{k\in M} \sum_{\emptyset\neq I\subseteq I_k} \sum_{S\subseteq I_1} f(I\cup S) \sum_{I\subseteq T\subseteq I_k} p\Big(\bigwedge_{i\in T} a_i \wedge \bigwedge_{i\in I_k\setminus T} a_i'\Big)$$

$$= \sum_{S\subseteq I_1} f(S) + \sum_{k\in M} \sum_{\emptyset\neq T\subseteq I_k} p\Big(\bigwedge_{i\in T} a_i \wedge \bigwedge_{i\in I_k\setminus T} a_i'\Big) \sum_{\emptyset\neq I\subseteq T} \sum_{S\subseteq I_1} f(I\cup S)$$

$$= \sum_{S\subseteq I_1} f(S) + \sum_{k\in M} \sum_{T\subseteq I_k} p\Big(\bigwedge_{i\in T} a_i \wedge \bigwedge_{i\in I_k\setminus T} a_i'\Big)\Big(\sum_{I\subseteq T} \sum_{S\subseteq I_1} f(I\cup S) - \sum_{S\subseteq I_1} f(S)\Big)$$

$$= \sum_{k\in M} \sum_{T\subseteq I_k} p\Big(\bigwedge_{i\in T} a_i \wedge \bigwedge_{i\in I_k\setminus T} a_i'\Big) \sum_{I\subseteq T} \sum_{S\subseteq I_1} f(I\cup S) - \big(|M|-1\big) \sum_{S\subseteq I_1} f(S).$$

□

THEOREM 13. *Let $f\colon 2^N \to \mathbb{R}$ and $m \geq 1$, and assume*

$$\sum_{I\subseteq K} f(I) \in \big[\tfrac{m-1}{2m-1}, \tfrac{m}{2m-1}\big]$$

for all $K \subseteq N$. Then (1) holds in all horizontal sums L of at most m Boolean algebras.

P r o o f. Using Lemma 12 one obtains (with the terminology used in Lemma 12)

$$\sum_{I\subseteq N} f(I)\, p\Big(\bigwedge_{i\in I} a_i\Big) \leq |M|\frac{m}{2m-1} - \big(|M|-1\big)\frac{m-1}{2m-1} = \frac{|M|+m-1}{2m-1} \leq 1$$

and

$$\sum_{I\subseteq N} f(I)\, p\Big(\bigwedge_{i\in I} a_i\Big) \geq |M|\frac{m-1}{2m-1} - \big(|M|-1\big)\frac{m}{2m-1} = \frac{-|M|+m}{2m-1} \geq 0\,.$$

□

R e m a r k . Observe that the case $m = 1$ yields a part of Theorem 6. From the proof of Theorem 13, it follows that Theorem 13 remains valid if one replaces m with n, and then omits "at most n".

THEOREM 14. *Let* $f\colon 2^N \to \mathbb{R}$*, assume*

$$\sum_{I\subseteq K} f(I) \in [0,1]$$

for all $K \subseteq N$*, and suppose there exist a positive integer* m *and pairwise disjoint subsets* $K_0,\dots,K_m$ *of* N *such that* $K_1,\dots,K_m \neq \emptyset$ *and*

$$\sum_{I\subseteq K_0} f(I) + \sum_{i=1}^{m} \sum_{\emptyset\neq K\subseteq K_i} \sum_{I\subseteq K_0} f(K\cup I) \notin [0,1]\,.$$

Then (1) *holds in all Boolean algebras* B*, but not in all horizontal sums of* m *finite Boolean algebras.*

P r o o f . The first part follows from Theorem 6. Now, by Lemma 5, for every $i \in \{1,\dots,m\}$ there exist a finite Boolean algebra B_i, $b_j \in B_i\setminus\{0,1\}$ for all $j \in K_i$, and a probability measure q_i on B_i such that for all $T \subseteq K_i$

$$q_i\Big(\bigwedge_{j\in T} b_j \wedge \bigwedge_{j\in K_i\setminus T} b'_j\Big) = \delta_{TK_i}\,.$$

Let L denote the horizontal sum of $B_1,\dots,B_m$; put $b_j := 1$ for all $j \in K_0$ and $b_j := 0$ for all $j \in N\setminus(K_0\cup\dots\cup K_m)$, and let $q\colon L \to [0,1]$ denote the common extension of $q_1,\dots,q_m$. (Without loss of generality, we may assume that the algebras B_i have the same 0 and 1 and that the sets $B_i\setminus\{0,1\}$, $i = 1,\dots,m$, are mutually disjoint.) According to Lemma 12, we have

$$\sum_{I\subseteq N} f(I)\, q\Big(\bigwedge_{i\in I} b_i\Big) = \sum_{I\subseteq K_0} f(I) + \sum_{i=1}^{m} \sum_{\emptyset\neq K\subseteq K_i} \sum_{I\subseteq K_0} f(K\cup I) \notin [0,1]\,.$$

□

E x a m p l e . $p(a_1) + p(a_2) - p(a_1 \wedge a_2) \in [0,1]$ holds in every Boolean algebra, but not in $2^2 + 2^2$. Let a_1, a_2, a'_1, a'_2 denote the four non-trivial elements of $2^2 + 2^2$, and put $p(a_1) = p(a_2) = p(1) := 1$ and $p(0) = p(a'_1) = p(a'_2) := 0$. We have $f(\{1\}) + f(\{2\}) = 2 \notin [0,1]$. This corresponds to the case $m = 2$, $K_0 = \emptyset$, $K_1 = \{1\}$ and $K_2 = \{2\}$ of Theorem 14.

5. Extension of mappings to probability measures and to homomorphisms

For the rest of the paper, let B denote a finite Boolean algebra with $|B| > 1$ and A a generating set of B.

THEOREM 15. *Let* $p\colon \{\bigwedge C \mid C \subseteq A\} \to \mathbb{R}$. *Then* p *can be extended to a probability measure on* B *if and only if* $p(1) = 1$,

$$\sum_{C \subseteq D \subseteq A} (-1)^{|D|-|C|} p\Big(\bigwedge D\Big) \geq 0$$

for all $C \subseteq A$, *and*

$$\sum_{C \subseteq D \subseteq A} (-1)^{|D|} p\Big(\bigwedge D\Big) = 0$$

for all $C \subseteq A$ *with*

$$\bigwedge C \wedge \Big(\bigvee (A \setminus C)\Big)' = 0 .$$

If such an extension exists, then it is unique.

Proof. First assume that such an extension $\bar{p}$ exists. Then $p(1) = \bar{p}(1) = 1$. Now, for all $C \subseteq A$ we have

$$p\Big(\bigwedge C\Big) = \bar{p}\Big(\bigwedge C\Big) = \sum_{C \subseteq D \subseteq A} \bar{p}\Big(\bigwedge D \wedge \Big(\bigvee (A \setminus D)\Big)'\Big),$$

and hence, because of Lemma 4, also

$$\bar{p}\Big(\bigwedge C \wedge \Big(\bigvee (A \setminus C)\Big)'\Big) = \sum_{C \subseteq D \subseteq A} (-1)^{|D|-|C|} p\Big(\bigwedge D\Big).$$

Conversely, assume that the three conditions are satisfied. Define

$$\bar{p}\Big(\bigvee_{C \in M} \Big(\bigwedge C \wedge \Big(\bigvee (A \setminus C)\Big)'\Big)\Big) := \sum_{C \in M} \sum_{C \subseteq D \subseteq A} (-1)^{|D|-|C|} p\Big(\bigwedge D\Big)$$

for all $M \subseteq 2^A$. From the fact that $\bigwedge C \wedge \big(\bigvee (A \setminus C)\big)'$, $C \subseteq A$, are mutually disjoint and from the last condition, it follows that $\bar{p}$ is well-defined and finitely additive. Clearly, $\bar{p}(0) = 0$. Using Lemma 4 and the first condition one obtains

$$\begin{aligned}\bar{p}(1) &= \bar{p}\Big(\bigvee_{C \subseteq A} \Big(\bigwedge C \wedge \Big(\bigvee (A \setminus C)\Big)'\Big)\Big)\\ &= \sum_{C \subseteq A} \sum_{C \subseteq D \subseteq A} (-1)^{|D|-|C|} p\Big(\bigwedge D\Big) = p\Big(\bigwedge \emptyset\Big) = p(1) = 1 .\end{aligned}$$

Because of the second condition, $\bar{p}$ is non-negative. Using again Lemma 4 we obtain for arbitrary $C \subseteq A$

$$\bar{p}\Big(\bigwedge C\Big) = \bar{p}\Big(\bigvee_{C\subseteq D\subseteq A}\Big(\bigwedge D \wedge \Big(\bigvee(A\setminus D)\Big)'\Big)\Big)$$
$$= \sum_{C\subseteq D\subseteq A}\ \sum_{D\subseteq E\subseteq A} (-1)^{|E|-|D|}\, p\Big(\bigwedge E\Big) = p\Big(\bigwedge C\Big).$$

Therefore $\bar{p}$ is an extension of p. From the first part of the proof, it follows that in case $\bar{p}$ exists, it is unique. □

THEOREM 16. *Let $p\colon \{\bigwedge C \mid C \subseteq A\} \to \{0,1\} \subseteq \mathbb{R}$. Then p can be extended to a probability measure on B if and only if $p(1) = 1$,*

$$\sum_{C\subseteq D\subseteq A,\ p(\bigwedge D)=1} (-1)^{|D|-|C|} \geq 0$$

for all $C \subseteq A$, and

$$\sum_{C\subseteq D\subseteq A,\ p(\bigwedge D)=1} (-1)^{|D|} = 0$$

for all $C \subseteq A$ with

$$\bigwedge C \wedge \Big(\bigvee(A\setminus C)\Big)' = 0 .$$

If such an extension exists, then it is unique and two-valued.

Proof. Theorem 16 follows from Theorem 15 and its proof. □

THEOREM 17. *Let $f\colon \{\bigwedge C \mid C \subseteq A\} \to \{0,1\} \subseteq B$. Then f can be extended to a homomorphism from B to $\{0,1\} \subseteq B$ if and only if $f(1) = 1$,*

$$\sum_{C\subseteq D\subseteq A,\ f(\bigwedge D)=1} (-1)^{|D|-|C|} \geq 0$$

for all $C \subseteq A$, and

$$\sum_{C\subseteq D\subseteq A,\ f(\bigwedge D)=1} (-1)^{|D|} = 0$$

for all $C \subseteq A$ with

$$\bigwedge C \wedge \Big(\bigvee(A\setminus C)\Big)' = 0 .$$

If such an extension exists, then it is unique.

Proof. Theorem 17 follows from Theorem 16 and from the fact that the homomorphisms from a Boolean algebra B_1 with $|B_1| > 1$ to $\{0,1\} \subseteq B_1$ are exactly the two-valued probability measures on B_1. □

THEOREM 18. (Sikorski's Theorem [6]). *Every homomorphism from a subalgebra of a Boolean algebra B_1 to a complete Boolean algebra B_1' can be extended to a homomorphism from B_1 to B_1'.*

THEOREM 19. *Let B_1 be a Boolean algebra with $|B_1| > 1$, D a finite subset of B_1, and $f\colon \big\{\bigwedge C \mid C \subseteq D\big\} \to \{0,1\} \subseteq B_1$. Then f can be extended to a homomorphism from B_1 to $\{0,1\} \subseteq B_1$ if and only if the three conditions of Theorem 17 are satisfied.*

Proof. Theorem 19 follows from Theorems 17 and 18. □

Remark. The last condition of Theorem 15 for a three-element subset $A = \{a,b,c\}$ of B and for $C := \{a\}$ reads as follows: $p(a) - p(a \wedge b) - p(a \wedge c) + p(a \wedge b \wedge c) = 0$ if $a \wedge b' \wedge c' = 0$. It is easy to see that in every Boolean algebra, it holds $p(a) - p(a \wedge b) - p(a \wedge c) + p(a \wedge b \wedge c) = p(a \wedge b' \wedge c')$. That this condition together with $p \geq 0$ and $p(1) = 1$ characterizes probability measures on Boolean algebras is the content of our concluding:

THEOREM 20. *Let B_1 be a Boolean algebra, and $p\colon B_1 \to [0,\infty)$. Then p is a probability measure on B_1 if and only if $p(1) = 1$ and $p(a) + p(a \wedge b \wedge c) = p(a \wedge b) + p(a \wedge c) + p(a \wedge b' \wedge c')$ for all $a,b,c \in B_1$.*

Proof. Assume that the last two conditions are satisfied. If $a,b \in B_1$ and $a \wedge b = 0$, then $p(a \vee b) = p(a \vee b) + p\big((a \vee b) \wedge a \wedge b\big) = p\big((a \vee b) \wedge a\big) + p\big((a \vee b) \wedge b\big) + p\big((a \vee b) \wedge a' \wedge b'\big) = p(a) + p(b)$. Finite additivity of p now follows by an induction argument. □

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