Ján Jakubík; Judita Lihová

Systems of intervals of partially ordered sets

Mathematica Slovaca, Vol. 46 (1996), No. 4, 355--361

Persistent URL: http://dml.cz/dmlcz/133014

Math. Slovaca, **46** (1996), No. 4, 355–361

Mathematica Slovaca

Dedicated to the memory of Professor Milan Kolibiar

SYSTEMS OF INTERVALS OF PARTIALLY ORDERED SETS

JÁN JAKUBÍK* — JUDITA LIHOVÁ**

(*Communicated by Tibor Katriňák*)

ABSTRACT. Let $\operatorname{Int} P$ be the system of all nonempty intervals of a partially ordered set P, ordered by inclusion. In the present paper, we show that the characterization of partially ordered sets P with $\operatorname{Int} P$ selfdual given in [Czechoslovak Math. J. **44** (1994), 523–533] remains valid without assuming that each interval of P contains a finite maximal chain.

For a partially ordered set P we denote by $\operatorname{Int}_0 P$ the system of all intervals of P including the empty set. Next we put $\operatorname{Int} P = \operatorname{Int}_0 P \setminus \{\emptyset\}$. Both $\operatorname{Int}_0 P$ and $\operatorname{Int} P$ are partially ordered by inclusion.

In the case of a lattice L, the system $\operatorname{Int}_0 P$ was investigated in the papers [1]–[7], [9], [10]. In [1], it was proved that for a finite lattice L, $\operatorname{Int}_0 L$ is selfdual if and only if either $\operatorname{card} L \leqq 2$, or $\operatorname{card} L = 4$ and L has two atoms.

Also, in [1], the problem was proposed whether there exists an infinite lattice L such that $\operatorname{Int}_0 L$ is selfdual.

A negative answer to this problem was given in [8] by showing that if P is any partially ordered set with $\operatorname{card} P > 4$, then $\operatorname{Int}_0 P$ is not selfdual.

In [10], there is presented the characterization of partially ordered sets P satisfying the condition that every interval of P contains a finite maximal chain and having a selfdual system $\operatorname{Int} P$.

In the present note, it will be shown that the characterization given in [10] remains valid without assuming that each interval of P contains a finite maximal chain.

AMS Subject Classification (1991): Primary 06A99.
Key words: interval of a partially ordered set, selfdual partially ordered system.

Supported by Grant GA SAV 1230/94.

The system of all convex subsets of a partially ordered set was dealt with in [12]; the condition for this system to be selfdual was found.

1. Preliminaries

For a partially ordered set P we apply the same notation as in [10; Section 1].

Let U and V be equivalence relations on P. Consider the following conditions for U and V (cf. [10]):

(i) For every $a \in P$ there are elements $u_1, u_2 \in \operatorname{Min} P$ and $v_1, v_2 \in \operatorname{Max} P$ such that $u_1 \leqq v_1$, $u_2 \leqq v_2$ and $[a]U = \langle u_1, v_1 \rangle$, $[a]V = \langle u_2, v_2 \rangle$.

(ii) $U \cap V$ is the least equivalence on P (i.e., the equality).

(iii) For every $a, b \in P$ with $a \leqq b$ there exist $z_1, z_2 \in \langle a, b \rangle$ satisfying aUz_1Vb, aVz_2Ub.

These conditions imply that

(ii′) For any $a, b \in P$, $[a]U \cap [b]V$ is either empty or a one-element set.

(ii″) For each $a \in P$, $[a]U \cap [a]V = \{a\}$.

(iv) Given $a, b \in P$ with $a \leqq b$, the elements z_1, z_2 from (iii) are uniquely determined.

1.1. THEOREM. (cf. [10]) *Let P be a partially ordered set satisfying the condition*

(∗) *every interval of P contains a finite maximal chain.*

Then the partially ordered set $\operatorname{Int} P$ *is selfdual if and only if there exist equivalence relations U and V on P such that conditions* (i), (ii) *and* (iii) *are valid.*

P r o o f. Cf. [10; 2.7 and 3.8]. □

1.2. THEOREM. *Let P be a partially ordered set. Then the following conditions are equivalent:*

(a) $\operatorname{Int} P$ *is selfdual.*

(b) *There exist equivalence relations U and V on P satisfying conditions* (i), (ii) *and* (iii).

The implication (a) $\implies$ (b) is contained in [10; 3.8] (in the proof of 3.8 the condition (∗) was not applied). The inverse implication will be proved below.

2. Proof of implication $(b) \implies (a)$

In this section, we suppose that P is a partially ordered set, and that U, V are equivalence relations on P satisfying conditions (i), (ii) and (iii).

We apply the following construction from [10; Section 2].

Let $\langle a, b\rangle \in \operatorname{Int} P$. In view of (i), (ii) and (iii), there exist uniquely determined elements

$u \in \operatorname{Min} P \cap [a]V$;
$v \in \operatorname{Max} P \cap [b]U$;
$z_1 \in \langle a, b\rangle$ with aUz_1Vb;
$c \in \langle u, z_1\rangle$ with $uUcVz_1$;
$d \in \langle z_1, v\rangle$ with z_1UdVv.

We put $\varphi(\langle a, b\rangle) = \langle c, d\rangle$. (Cf. Fig. 1.)

The following two lemmas have been proved in [10] without applying the condition $(*)$.

2.1. Lemma. ([10; 2.2.]) *The mapping φ is one-to-one.*

2.2. Lemma. ([10; 2.3.]) *The mapping φ is onto* $\operatorname{Int} P$.

2.3. Lemma. *The equivalence classes corresponding to the relations U, V are convex subsets of P.*

Proof. Let us suppose, e.g., that $p \leqq s \leqq q$, pUq. By (iii), there exists $r \in \langle p, s\rangle$ such that pVr and rUs. Using (iii) again we obtain that there exists $t \in \langle r, q\rangle$ satisfying rVt, tUq. Then, in view of pUq, we have pUt and, clearly, pVt. Hence $p = t$. This implies $p = r$, pUs. □

Let a, b and z_1 be as above. There exists a uniquely determined element $z_2 \in \langle a, b\rangle$ with aVz_2Ub.

2.4. Lemma. *Let $x \in \langle a, b\rangle$, aVx. Then $x \leqq z_2$.*

Proof. There exists $x_0 \in \langle x, b\rangle$ with xVx_0Ub. Thus

$$x_0UbUz_2, \qquad x_0VxVaVz_2,$$

hence $x_0 = z_2$. Therefore $x \leqq z_2$. □

In the previous lemma, we can replace z_2 and V by z_1 and U.

2.5. Lemma. *Let $x \in \langle a, b\rangle$. There exist uniquely determined elements $x_1 \in \langle a, z_1\rangle$, $x_2 \in \langle a, z_2\rangle$, $x_1' \in \langle z_1, b\rangle$, $x_2' \in \langle z_2, b\rangle$ with*

$$aUx_1Vx, \quad aVx_2Ux, \quad xUx_1'Vb, \quad xVx_2'Ub. \tag{1}$$

Proof. The existence and uniqueness of $x_1, x_2 \in \langle a, x\rangle$, $x_1', x_2' \in \langle x, b\rangle$ satisfying (1) is a consequence of (i) – (iii). Then, in view of 2.4 and its dual, we have $x_1 \in \langle a, z_1\rangle$ and $x_2 \in \langle a, z_2\rangle$, $x_1' \in \langle z_1, b\rangle$, $x_2' \in \langle z_2, b\rangle$. □

2.6. Lemma. *Let $x \in \langle a, b\rangle$, and let $x_1 \in \langle a, z_1\rangle$, $x_2 \in \langle a, z_2\rangle$, $x_1' \in \langle z_1, b\rangle$, $x_2' \in \langle z_2, b\rangle$ be as in 2.5. Then $x_1 = \inf_{\langle a,b\rangle}\{x, z_1\}$, $x_2 = \inf_{\langle a,b\rangle}\{x, z_2\}$, $x_1' = \sup_{\langle a,b\rangle}\{x, z_1\}$, $x_2' = \sup_{\langle a,b\rangle}\{x, z_2\}$, $x = \sup_{\langle a,b\rangle}\{x_1, x_2\}$, $x = \inf_{\langle a,b\rangle}\{x_1', x_2'\}$.*

P r o o f. Let us prove $x_1 = \inf_{\langle a,b\rangle}\{x, z_1\}$, $x = \sup_{\langle a,b\rangle}\{x_1, x_2\}$. The other relations can be verified analogously. Let $\overline{x}$ be a lower bound of $\{x, z_1\}$ in $\langle a, b\rangle$. Since $a \leqq \overline{x} \leqq z_1$ and aUz_1, we have $aU\overline{x}$ by 2.3. In view of 2.4, the relations $\overline{x} \in \langle a, x\rangle$, aUx_1Vx, $aU\overline{x}$ imply $\overline{x} \leqq x_1$.

To prove the second relation, take any $y \in \langle a, b\rangle$ with $y \geqq x_1$, $y \geqq x_2$. By 2.5, there exist $y_1 \in \langle a, z_1\rangle$, $y_2 \in \langle a, z_2\rangle$ satisfying aUy_1Vy, aVy_2Uy. But $x_1 \in \langle a, y\rangle$ and aUx_1, therefore $x_1 \leqq y_1$ by 2.4. Similarly, $x_2 \leqq y_2$. Since $x_1 \leqq y$, there exists p such that x_1VpUy. However, $p \in \langle a, y\rangle$ and pUy, so that $p \geqq y_2$ by the dual of 2.4. Analogously, there exists $q \in \langle y_1, y\rangle$ satisfying x_2Uq. Further, let us take $r \in \langle x_1, q\rangle$ such that x_1VrUq and $s \in \langle x_2, p\rangle$ with x_2UsVp. As $r \in \langle x_1, y\rangle$ and x_1Vr, in view of 2.4, we have $r \leqq p$. Analogously, $s \leqq q$. Now $rUqUs$, $sVpVr$, which yields $r = s$. Finally, the relations sUx_2Ux, rVx_1Vx, $r = s$ imply $r = x$. But $y \geqq r$, and the proof is complete. □

2.7. Lemma. *Under the notation as above, let $a' \in \langle a, b\rangle$. Then $\varphi(\langle a', b\rangle) \supseteq \varphi(\langle a, b\rangle)$.*

P r o o f. In view of 2.6, there exists

$$a_1' = \inf_{\langle a,b\rangle}\{a', z_1\}.$$

(Cf. Fig. 2.) Next, if we consider the elements c, a, u, z_1 and a_1', then 2.6 yields that there exists

$$p_1 = \inf_{\langle u,z_1\rangle}\{a_1', c\}.$$

We have $p_1Va_1'Va'$.

There exists $u' \in \operatorname{Min} P \cap [p_1]V$. Next there exists $c' \in \langle u', c\rangle$ such that $u'Uc'Vc$.

By 2.6, there exists $q_1 \in \langle z_1, b\rangle$ with

$$q_1 = \sup_{\langle a,b\rangle}\{a', z_1\}.$$

By considering the elements d, b, z_1, v and q_1 and by applying 2.6, we get that there exists $q_2 \in \langle d, v\rangle$ with

$$q_2 = \sup_{\langle z_1,v\rangle}\{q_1, d\}.$$

In view of the definition of the mapping φ, we have

$$\varphi(\langle a', b\rangle) = \langle c', q_2\rangle .$$

Hence, $\varphi(\langle a', b\rangle) \supseteq \varphi(\langle a, b\rangle)$. □

By analogous consideration, we obtain:

2.7'. Lemma. *Under the notation as above, let $b' \in \langle a, b\rangle$. Then $\varphi(\langle a, b'\rangle) \supseteq \varphi(\langle a, b\rangle)$.*

2.8. Lemma. *Let $\langle a', b'\rangle \subseteq \langle a, b\rangle$. Then $\varphi(\langle a', b'\rangle) \supseteq \varphi(\langle a, b\rangle)$.*

Proof. This is an immediate consequence of 2.7 and 2.7'. □

By considering the mapping φ, we can give an explicit description of φ^{-1} as follows.

Let $\langle c, d\rangle$ be an interval of P. Then there exist uniquely determined elements (cf. Fig. 1)

$u \in \operatorname{Min} P \cap [c]U$;
$v \in \operatorname{Max} P \cap [d]V$;
$z_1 \in \langle c, d\rangle$ with cVz_1Ud;
$a \in \langle u, z_1\rangle$ with $uVaUz_1$;
$b \in \langle z_1, v\rangle$ with z_1VbUv.

From the definition of φ, we obtain:

2.9. Lemma. $\varphi^{-1}(\langle c, d\rangle) = \langle a, b\rangle$.

In other words, the construction of φ^{-1} is the same as the construction of φ with the distinction that the roles of U and V are interchanged.

Hence, by the same method as we used for φ, we obtain

2.10. Lemma. *Let $\langle c, d\rangle$ and $\langle c', d'\rangle$ be intervals in P such that $\langle c', d'\rangle \subseteq \langle c, d\rangle$. Then $\varphi^{-1}(\langle c', d'\rangle) \supseteq \varphi^{-1}(\langle c, d\rangle)$.*

Proof of 1.2. As we already remarked above, the implication (a) $\Longrightarrow$ (b) was proved in [10].

Let (b) be valid, and let φ be as above. Then, in view of 2.1, 2.2, 2.8 and 2.10, we infer that φ is a dual isomorphism of $\operatorname{Int} P$. □

The following example shows that a partially ordered set P with $\operatorname{Int} P$ self-dual need not satisfy the condition $(*)$.

Let $\mathbb{R}$ be the set of all reals with the natural linear order, $X = \mathbb{R}$, and let Y be the interval $\langle 0, 1\rangle$ of X. Put $P = X \times Y$. For (x_1, y_1) and (x_2, y_2) in P we put $(x_1, y_1) \leqq (x_2, y_2)$ if $y_2 - y_1 \geqq |x_2 - x_1|$. Next we define binary relations U and V on P by

$$(x_1, y_1) U (x_2, y_2) \iff y_1 - x_1 = y_2 - x_2 ,$$
$$(x_1, y_1) V (x_2, y_2) \iff y_1 + x_1 = y_2 + x_2 .$$

Then U and V are equivalence relations on P satisfying conditions (i), (ii) and (iii). Hence, P is selfdual. P does not satisfy $(*)$; in fact, there does not exist any prime interval in P.

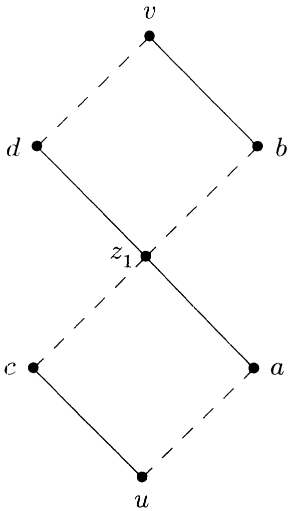

FIGURE 1.

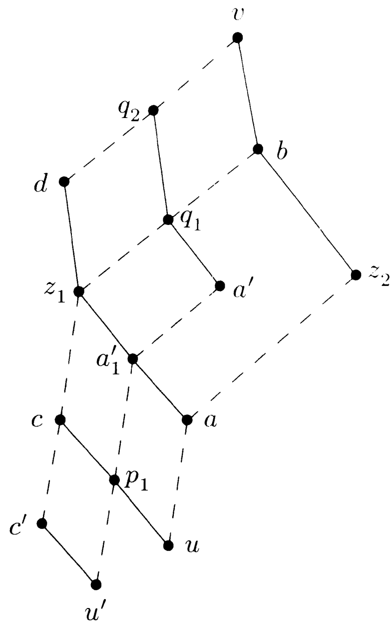

FIGURE 2.

REFERENCES

[1] IGOSHIN, V. I.: *Selfduality of lattices of intervals of finite lattices.* In: International conference on algebra dedicated to the memory of A. I. Maltsev, Summaries of lectures on model theory and algebraic systems, Inst. matem. Sibir. otdel. AN SSSR, Novosibirsk, 1989, p. 48. (Russian)

[2] IGOSHIN, V. I.: *Lattices of intervals and lattices of convex sublattices of lattices.* Uporyad. Mnozhestva Reshetki (Saratov) **6** (1980), 69–76. (Russian)

[3] IGOSHIN, V. I.: *Identities in interval lattices of lattices.* In: Contributions to Lattice Theory. Colloq. Math. Soc. János Bolyai 33, Szeged, 1980(1983), pp. 491–501.

[4] IGOSHIN, V. I.: *On lattices with restrictions on their interval lattices.* In: Lectures in Universal Algebra. Colloq. Math. Soc. János Bolyai 43, Szeged, 1983(1986), pp. 209–2[illegible]

[5] IGOSHIN, V. I.: *Algebraic characterization of interval lattices*, Uspekhi Mat. Nauk **40** (1985), 205–206. (Russian)

[6] IGOSHIN, V. I.: *Interval properties of quasivarieties of lattices.* XVII. Vsesoyuz. algebr. konf. Summaries of lectures, Kishinev. (Russian)

[7] IGOSHIN, V. I.: *Semimodularity in interval lattices*, Math. Slovaca **38** (1988), 305–308. (Russian)

[8] JAKUBÍK, J.: *Selfduality of the system of intervals of a partially ordered set*, Czechoslovak Math. J. **41** (1991), 135–140.

[9] KOLIBIAR, M.: *Intervals, convex sublattices and subdirect representations of lattices.* In: Universal Algebra and Applications. Banach Center Publ. 9, Polish Acad. Sci., Warsaw, 1982, pp. 335–339.

[10] LIHOVÁ, J.: *Posets having a selfdual interval poset*, Czechoslovak Math. J. **44** (1994), 523–533.

[11] SLAVÍK, V.: *On lattices with isomorphic interval lattices*, Czechoslovak Math. J. **35** (1985), 550-554.

[12] ZELINA, M.: *Selfduality of the system of convex subsets of a partially ordered set*, Comment. Math. Univ. Carolin. **34** (1993), 593–595.

Received December 9, 1994
Revised February 27, 1995

* *Mathematical Institute*
Slovak Academy of Sciences
Grešákova 6
SK–040 01 Košice
SLOVAKIA

** *Department of Geometry and Algebra*
Faculty of Science
Šafárik University
Jesenná 5
SK–041 54 Košice
SLOVAKIA