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REMARKS ON THE INTEGRABILITY IN BANACH SPACES

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Z. Lipecki in [8] pointed out that the proof of Theorem 1 in [10] is invalid. In fact, the measure μ constructed there is countably additive only in the strong operator topology, see [11]. In the proof of Theorem 2 below, using the Dvoretzky—Rogers theorem, see Theorem IV.1.2 in [2], we construct the required measure countably additive in the uniform operator topology. Thus Theorems 1 and 2 below give a correct proof of Theorem 1 in [10]. Although our Theorem 1 is equivalent to Theorem 6 in [10], we give a very simple proof of it. Finally in Theorem 3, which is a complement to [9], we characterize the integrability of a measurable function using its weak (in [9] called scalar) integrability.

Let $\mathcal{P}$ be a δ-ring of subsets of a non empty set T, let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be Banach spaces (both real, or complex) and let $L(\boldsymbol{X}, \boldsymbol{Y})$ be the Banach space of all bounded linear operators from $\boldsymbol{X}$ to $\boldsymbol{Y}$. We say that a set function $\boldsymbol{m}$: $\mathcal{P}\to L(\boldsymbol{X}, \boldsymbol{Y})$ is an operator valued measure countably additive in the strong operator topology, if for every $\boldsymbol{x}\in\boldsymbol{X}$ the set function $E\to\boldsymbol{m}(E)\boldsymbol{x}$, $E\in\mathcal{P}$, is a countably additive vector measure. In [3] we started to develop a Lebesgue type integration theory for functions on T with values in $\boldsymbol{X}$ with respect to such a measure. The basic quantity of the theory is the semivariation $\hat{\boldsymbol{m}}$ of the measure $\boldsymbol{m}$, which is defined by the equality

$$\begin{aligned}\hat{\boldsymbol{m}}(E)=\sup&\left\{\left|\sum_{i=1}^{r}\boldsymbol{m}(E\cap E_i)\boldsymbol{x}_i\right|,\ \boldsymbol{x}_i\in\boldsymbol{X},\ |\boldsymbol{x}_i|\leqq 1,\ E_i\in\mathcal{P},\right.\\ &\left.E_i\cap E_j=\emptyset\ \text{for}\ i\neq j,\quad i,j=1,\dots,r,\ \ r=1,2,\dots\right\}\\ =&\sup_{|y^*|\leqq 1} v(y^*\boldsymbol{m}, E),\quad E\in\sigma(\mathcal{P}),\end{aligned}$$

where $\sigma(\mathcal{P})$ denotes the smallest σ-ring containing $\mathcal{P}$, and $v(y^*\boldsymbol{m}, .)$, $y^*\in\boldsymbol{Y}^*=$ the dual of $\boldsymbol{Y}$, is the variation of the measure $A\to y^*\boldsymbol{m}(A)\in\boldsymbol{X}^*$, $A\in\mathcal{P}$. We immediately see that $\hat{\boldsymbol{m}}(\emptyset)=0$, $\hat{\boldsymbol{m}}$ is monotone, subadditive and has the Fatou property: $E_n\in\sigma(\mathcal{P})$, $n=1, 2, \dots$ and $E_n\nearrow E$ implies $\hat{\boldsymbol{m}}(E_n)\nearrow\hat{\boldsymbol{m}}(E)$. We say that $\hat{\boldsymbol{m}}$ is continuous on $\mathcal{P}$ if $E_n\in\mathcal{P}$, $n=1, 2, \dots$ and $E_n\searrow\emptyset$ implies $\hat{\boldsymbol{m}}(E_n)\to 0$. It is easy to see that $\hat{\boldsymbol{m}}$ is continuous on $\mathcal{P}$ if and only if it is locally exhaustive on $\mathcal{P}$, i.e., $A\in\mathcal{P}$; $E_n\in\mathcal{P}$, $n=1, 2, \dots$ pairwise disjoint implies $\hat{\boldsymbol{m}}(A\cap E_n)\to 0$.

The basic assumption of the theory is the requirement of finiteness of the semivariation $\hat{m}$ on $\mathcal{P}$. In Theorem 5 in [3] we proved that if the semivariation $\hat{m}$ is continuous on $\mathcal{P}$, if $f\colon T \to \mathbf{X}$ is a bounded measurable function, and $A \in \mathcal{P}$, then the function $f . \chi_A$ is integrable. As Theorem 6 in [10] shows, the result is in a sense the best possible. We now give a very simple proof of Theorem 6 from [10].

Theorem 1. *Suppose that the semivariation $\hat{m}$ is not continuous on $\mathcal{P}$ (equivalently, not locally exhaustive on $\mathcal{P}$). Then there is a set $A \in \mathcal{P}$ and a bounded $\mathcal{P}$-elementary function $f\colon T \to \mathbf{X}$ such that the function $f . \chi_A$ is not integrable.*

Proof. By assumption there is an $\varepsilon > 0$ a set $A \in \mathcal{P}$, and a sequence of pairwise disjoint sets $E_n \in \mathcal{P}$, $n = 1, 2, \ldots$ such that $\hat{m}(A \cap E_n) > \varepsilon$ for each $n = 1, 2, \ldots$. According to the definition of the semivariation $\hat{m}$ for each $n = 1, 2, \ldots$ there is a $\mathcal{P}$-simple function $f_n\colon T \to \mathbf{X}$, $\sup_{t \in A \cap E_n} |f_n(t)| \leqq 1$ such that $\left| \int_{E_n \cap A} f_n \, \mathrm{d}\mathbf{m} \right| > \varepsilon$. Now $f = \sum_{n=1}^{\infty} d_n . \chi_{E_n}$ is $\mathcal{P}$-elementary, and $f . \chi_A$ cannot be integrable, since the indefinite integral $E \to \int_E f \, \mathrm{d}\mathbf{m}$, $E \in \sigma(\mathcal{P})$, of an integrable function f is a countably additive vector measure, see Theorem 3 in [3].

If the Banach space $\mathbf{Y}$ contains no subspace isomorphic to c_0, see pp. 160 and 161 in [1], then the finiteness of the semivariation $\hat{m}$ on $\mathcal{P}$ is equivalent to its continuity on $\mathcal{P}$, see the *-Theorem in [3] and the Corollary of Theorem 5 in [4]. We now show that the assumption $c_0 \not\subset \mathbf{Y}$ is essential for the finiteness of $\hat{m}$ to imply its continuity.

Theorem 2. *Let $\mathbf{X}$ be an infinite dimensional Banach space and let $\mathcal{P} = 2^N$ be the power set of the set N of positive integers. Then there exists a measure $\mathbf{m}\colon \mathcal{P} \to L(\mathbf{X}, c_0)$ countably additive in the uniform operator topology with finite but not continuous semivariation $\hat{m}$ on $\mathcal{P}$.*

Proof. Since $\mathbf{X}^*$, the dual of $\mathbf{X}$, is also infinite dimensional, according to the Dvoretzky—Rogers theorem (see Theorem IV. 1.2 in [2]) there is a sequence $x_n^* \in \mathbf{X}^*$, $n = 1, 2, \ldots$ such that the series $\sum_{n=1}^{\infty} x_n^*$ is unconditionally convergent in $\mathbf{X}^*$ and $\sum_{n=1}^{\infty} |x_n^*| = +\infty$. Without loss of generality we may suppose that $|x_n^*| \leqq 1$ for each $n = 1, 2, \ldots$. Put $n_0 = 0$ and let n_1 be the first positive integer such that $\sum_{i=1}^{n_1} |x_i^*| > 1$. Clearly $\sum_{i=1}^{n_1} |x_i^*| \leqq 2$. Similarly, let n_2 be the first positive integer such that $\sum_{i=n_1+1}^{n_2} |x_i^*| > 1$. Then obviously $n_2 > n_1 + 1$ and again $\sum_{i=n_1+1}^{n_2} |x_i^*| \leqq 2$. Continuing in this way we obtain a subsequence n_k, $k=1, 2, \ldots$ such that $1<\sum_{i=n_{k-1}+1}^{n_k}|x_i^*|\leqq 2$ for each $k=1, 2, \ldots$, where $n_0=0$. Put $I_k=\{n_{k-1}+1, \ldots, n_k\}$ and let $e_k=(0, \ldots, 0, 1, 0, \ldots)\in c_0$, $k=1, 2, \ldots$. Clearly $T=\bigcup_{k=1}^{\infty} I_k$ and $I_k\cap I_j=\emptyset$ for $k\neq j$, $j, k=1, 2, \ldots$. For $i\in I_k$ put $y_i=e_k$ and $U_i x=x^*x \,.\, y_i\in c_0$, $x\in X$. Obviously $U_i\in L(X, c_0)$ for each $i=1, 2, \ldots$ and $\sum_{i\in E} U_i x=\sum_{i\in E} x_i^* x \,.\, y_i\in c_0$ for any $E\in\mathcal{P}$ and $x\in X$. Evidently $\sum_{i\in E} U_i$: $X\to c_0$ is linear and $\left|\sum_{i\in E} U_i\right|\leqq\left|\sum_{i\in E} x^*_i\right|$. Hence if we put $m(E)=\sum_{i\in E} U_i$ for $E\in\mathcal{P}$, then m: $\mathcal{P}\to L(X, c_0)$ is countably additive in the uniform operator topology.

Now according to the definition of the norm in X^* there are $x_i\in X$, $|x_i|\leqq 1$, $i=1, 2, \ldots$ such that $\sum_{i=n_{k-1}+1}^{n_k} x_i^* x_i>1$ for each $k=1, 2, \ldots$, From the definition of the semivariation $\hat{m}$ we have

$$\hat{m}(T)=\sup\left\{\left|\sum_{i=1}^{r} m(E_i)x_i\right|,\ x_i\in X,\ |x_i|\leqq 1,\ E_i\in\mathcal{P},\ E_i\cap E_j=\emptyset\right.$$

$$\left.\text{for } i\neq j,\ \bigcup_{i=1}^{r} E_i=T,\ i, j=1, \ldots, r,\ r=1, 2, \ldots\right\}$$

$$=\sup\left\{\left|\sum_{i=1}^{r}\sum_{t\in E_i} U_t x_i\right|, \ldots\right\}$$

$$=\sup\left\{\left|\sum_{i=1}^{r}\sum_{t\in E_i} x_t^* x_i \,.\, y_t\right|, \ldots\right\}$$

$$=\max_k\left\{\left|\sum_{t\in I_k} x^*_t x_{i_t}\right|, \ldots\right\},\ \text{where } i_t=i \text{ for } t\in E_i$$

$$\leqq\max_k\sum_{t\in I_k}|x_t^*|\leqq 2.$$

On the other hand $\hat{m}(I_k)\geqq\left|\sum_{i=n_{k-1}+1}^{n_k} x_i^* x_i\right|>1$ for each $k=1, 2, \ldots$. Since I_k, $k=1, 2, \ldots$ are pairwise disjoint elements of $\mathcal{P}$ with union equal to $T\in\mathcal{P}$, $\hat{m}$ is not continuous on $\mathcal{P}$. The theorem is proved.

The next theorem, a complement to [9], characterizes the integrability of a measurable function using its weak (in [9] called scalar) integrability. It may be proved similarly as Theorem 17 in [3].

Theorem 3. *A measurable function $f\colon T\to \boldsymbol{X}$ is integrable with respect to $\boldsymbol{m}\colon \mathscr{P}\to L(\boldsymbol{X}, \boldsymbol{Y})$ if and only if f is integrable with respect to $y^*\boldsymbol{m}\colon \mathscr{P}\to \boldsymbol{X}^*$ for each $y^*\in \boldsymbol{Y}^*$ and the scalar measures $\{\int f\,\mathrm{d}(y^*\boldsymbol{m}),\ y^*\in \boldsymbol{Y}^*,\ |y^*|\leqq 1\}$ are uniformly countably additive on $\sigma(\mathscr{P})$ (equivalently, uniformly exhaustive on $\mathscr{P}$).*

Note a certain similarity between integrable functions and the elements of $\mathscr{L}_1(\boldsymbol{m})$, see Definition 4 in [4]. Namely, a measurable function $f\colon T\to \boldsymbol{X}$ belongs to $\mathscr{L}_1(\boldsymbol{m})$ if and only if the function $|f|$ is integrable with respect to the measure $v(y^*\boldsymbol{m}, .)\colon \mathscr{P}\to[0, +\infty)$ for each $y^*\in \boldsymbol{Y}^*$ and the integrals $\{\int |f|\,\mathrm{d}v(y^*\boldsymbol{m}, .),\ y^*\in \boldsymbol{Y}^*,\ |y^*|\leqq 1\}$ are uniformly σ-additive on $\sigma(\mathscr{P})$ (equivalently, uniformly exhaustive on $\mathscr{P}$).

Let us note also that if $\mathscr{P}$ is generated by a ring $\mathscr{R}$, i.e., if $\mathscr{P}=\delta(\mathscr{R})$, or if $\mathscr{P}=\delta(\mathscr{C}_0)$ is the δ-ring of relatively compact Baire subsets of a locally compact Hausdorff topological space, then according to Theorem 11 and Lemma 7 in [6] the above mentioned uniform exhaustivity on $\mathscr{P}$ may be replaced by a uniform exhaustivity on $\mathscr{R}$, or on $\mathscr{C}_0$, respectively. ($\mathscr{C}_0$ denotes the lattice of all compact G_δ subsets).

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ЗАМЕТКИ ОБ ИНТЕГРИРУЕМОСТИ В ПРОСТРАНСТВАХ БАНАХА

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Резюме

Основным результатом работы является доказательство следующей

Теоремы 2. Пусть X бесконечномерное пространство банаха и пусть $\mathcal{P}$ семейство всех подмножеств множества натуральных чисел. Тогда существует мера $\boldsymbol{m}$: $\mathcal{P} \to L(\boldsymbol{X}, c_0)$ счетно аддитивная в равномерной операторной топологии, имеющая конечную полувариацию $\hat{\boldsymbol{m}}$ на $\mathcal{P}$, которая не является непрерывной сверху на пустом множестве.

Из этого результата вытекает корректное доказательство Теоремы 1 из [10].