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CONDITIONAL ENTROPY AND ROKHLIN METRIC

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ABSTRACT. In the present paper we introduce a pseudo-metric on an m-equivalence class $[\mathcal{N}]$ of fuzzy sub-σ-algebras having finitely many atoms. Identification of elements of $[\mathcal{N}]$ which are equivalent modulo 0 converts the pseudo-metric into a metric which we call the Rokhlin metric. In the classical crisp case all sub-σ-algebras with finitely many atoms belong to the same equivalence class and the Rokhlin metric in the generalized fuzzy setting reduces to the classical Rokhlin metric.

1. Introduction

Riečan and Dvurečenskij [14] suggested a new model for quantum mechanics, based on Piasecki's ideas [11], which was further developed in [13]. Subsequently, Markechová defined an entropy and a conditional entropy of complete fuzzy partitions ([9]) and that of stochastical complete repartition ([8]) of an F-probability measure space (cf. [3], [10]). Using triangular norms (cf. [1], [16]), the entropy of a fuzzy process is defined and studied in [2], (cf. [7]). In [5], [17], [18] we developed a cohesive approach to the fuzzification of the entropy theory using the concept of atoms in a fuzzy σ-algebra; an appropriate generalization of concepts leads to a satisfactory theory circumventing lacunae to such a theory in other approaches ([2], [8], [9]). In [17], a metric ϱ on a fuzzy measure algebra $\tilde{\mathcal{M}}$ ([19]) is introduced, and it is proved that $(\tilde{\mathcal{M}}, \varrho)$ is a complete metric space, which is convex if and only if $\tilde{\mathcal{M}}$ is nonatomic.

The object of this paper is to extend the concept of Rokhlin metric based on conditional entropy to the more general setting of fuzzy sub-σ-algebras. Sections 2 and 3 deal with the prerequisites for the results proved in Section 4. In Section 2 the definitions of an F-measure space $(X, \mathcal{M}, m)$, atoms of a fuzzy sub-σ-algebra $\mathcal{N}$ of $\mathcal{M}$, the concepts of the m-refinement of $\mathcal{N}$, the

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m-equivalence of two fuzzy sub-σ-algebras having finitely many atoms, and some basic results are given. The notions of the entropy $H(\mathcal{N})$ of $\mathcal{N}$ and the conditional entropy $H(\mathcal{N}_1 \mid \mathcal{N}_2)$, $\mathcal{N}_1, \mathcal{N}_2 \in \mathcal{F}(\mathcal{M})$, are described in Section 3 using the convex function $x \log x$, $x \in [0,1]$. In Section 4, some propositions are proved which lead to the definition of a pseudo-metric d on the m-equivalence class $[\mathcal{N}]$ containing $\mathcal{N}$. By identifying elements of $[\mathcal{N}]$ which are equivalent modulo 0, we obtain the Rokhlin metric on $[\mathcal{N}]$ following the terminology of the classical crisp case.

2. F-measure space, atoms and m-equivalence

2.1. A *fuzzy set* in a nonempty set X is an element of I^X, where $I \equiv [0,1]$. A fuzzy set which assigns the value t, $t \in I$, to each x in X is denoted by $\mathbf{t}$.

If λ_i belongs to I^X, the sequence $\{\lambda_i(x)\}_{i=1}^{\infty}$ is monotonic increasing and converges to $\lambda(x)$ for each x in X, then we say that $\{\lambda_i\}_{i=1}^{\infty}$ *increases to* λ in I^X and write $\lambda_i \uparrow \lambda$.

We shall denote by $\mathbb{N}$ the set of natural numbers and by $\mathbb{R}$ the set of real numbers.

2.2. ([6]) A *fuzzy σ-algebra* $\mathcal{M}$ on a nonempty set X is a subfamily of I^X satisfying:

A1. $\mathbf{1} \in \mathcal{M}$,

A2. $\lambda \in \mathcal{M} \implies \mathbf{1} - \lambda \in \mathcal{M}$,

A3. if $\{\lambda_i\}_{i=1}^{\infty}$ is a sequence in $\mathcal{M}$, then $\bigvee_{i=1}^{\infty} \lambda_i \equiv \sup \lambda_i \in \mathcal{M}$.

An arbitrary intersection of fuzzy σ-algebras on X is a fuzzy σ-algebra on X. If $\mathcal{N}_1$, $\mathcal{N}_2$ are fuzzy σ-algebras on X, then $\mathcal{N}_1 \vee \mathcal{N}_2$ stands for the smallest fuzzy σ-algebra on X containing $\mathcal{N}_1 \cup \mathcal{N}_2$.

2.3. ([6]) An *F-probability measure* on a fuzzy σ-algebra $\mathcal{M}$ is a function $m \colon \mathcal{M} \to I$ such that

M1. $m(\mathbf{1}) = 1$,

M2. for $\lambda \in \mathcal{M}$, $m(\mathbf{1} - \lambda) = 1 - m(\lambda)$,

M3. for $\lambda, \mu \in \mathcal{M}$, $m(\lambda \vee \mu) + m(\lambda \wedge \mu) = m(\lambda) + m(\mu)$,

M4. if $\{\lambda_i\}_{i=1}^{\infty}$ is a sequence in $\mathcal{M}$ such that $\lambda_i \uparrow \lambda$, $\lambda \in \mathcal{M}$, then $m(\lambda) = \sup_i m(\lambda_i)$.

The triple $(X, \mathcal{M}, m)$ is called an *F-probability measure space*, elements of $\mathcal{M}$ are referred to as *F-measurable sets.*

2.4. ([17]) Let $(X, \mathcal{M}, m)$ be an F-probability measure space. For $\lambda, \mu \in \mathcal{M}$ define

$$\lambda = \mu \pmod{m} \iff m(\lambda) = m(\mu) = m(\lambda \vee \mu).$$

The relation "$=\pmod{m}$" is an equivalence relation on $\mathcal{M}$; $\tilde{\mathcal{M}}$ denotes the set of all equivalence classes induced by this relation and $\tilde{\mu}$ denotes the equivalence class determined by μ.

We define $\lambda, \mu \in \mathcal{M}$ *m-disjoint* if $\lambda \wedge \mu = 0 \pmod{m}$, i.e. $m(\lambda \wedge \mu) = 0$.

If λ_i, $i \in \mathbb{N}$, are pairwise m-disjoint F-measurable sets in $(X, \mathcal{M}, m)$, then

$$m\left(\bigvee_{i=1}^{\infty} \lambda_i\right) = \sum_{i=1}^{\infty} m(\lambda_i).$$

2.5. ([18]) Let $(X, \mathcal{M}, m)$ be an F-probability measure space and let $\mathcal{N}$ be a fuzzy sub-σ-algebra of $\tilde{\mathcal{M}}$. An element $\tilde{\mu} \in \tilde{\mathcal{N}}$ is called an *atom* of $\mathcal{N}$ if $m(\mu) > 0$ and, for any $\tilde{\lambda} \in \tilde{\mathcal{N}}$,

$$m(\lambda \wedge \mu) = m(\lambda) \neq m(\mu) \implies m(\lambda) = 0.$$

We shall denote the set of all atoms of $\mathcal{N}$ by $\overline{\mathcal{N}}$, and by $\mathcal{F}(\mathcal{M})$ the family of fuzzy sub-σ-algebras of $\mathcal{M}$ having finitely many atoms.

The following are proved in [18]:

(i) Distinct atoms are pairwise m-disjoint.

(ii) If $\overline{\mathcal{N}}_1 = \{\lambda_i : 1 \le i \le k\}$, $\overline{\mathcal{N}}_2 = \{\mu_j : 1 \le j \le q\}$, then $\overline{\mathcal{N}_1 \vee \mathcal{N}_2} = \{\lambda_i \wedge \mu_j : \lambda_i \in \overline{\mathcal{N}}_1, \ \mu_j \in \overline{\mathcal{N}}_2 \text{ and } m(\lambda_i \wedge \mu_j) > 0\}$.

2.6. ([18]) Let $(X, \mathcal{M}, m)$ be an F-probability measure space and let $\mathcal{N}_1$, $\mathcal{N}_2$ be fuzzy sub-σ-algebras of $\mathcal{M}$. Then $\mathcal{N}_2$ is called an *m-refinement* of $\mathcal{N}_1$, written as $\mathcal{N}_1 \le_m \mathcal{N}_2$, if, for each $\mu \in \overline{\mathcal{N}}_2$, there exists $\lambda \in \overline{\mathcal{N}}_1$ such that $m(\lambda \wedge \mu) = m(\mu)$.

The fuzzy sub-σ-algebras $\mathcal{N}_1$ and $\mathcal{N}_2, \mathcal{N}_1, \mathcal{N}_2 \in \mathcal{F}(\mathcal{M})$, are called *$m$-equivalent*, written as $\mathcal{N}_1 \approx_m \mathcal{N}_2$, if

$$m\Big(\lambda \wedge \Big(\bigvee\{\mu : \mu \in \overline{\mathcal{N}}_2\}\Big)\Big) = m(\lambda) \qquad \text{for each} \quad \lambda \in \overline{\mathcal{N}}_1,$$

and

$$m\Big(\mu \wedge \Big(\bigvee\{\lambda : \lambda \in \overline{\mathcal{N}}_1\}\Big)\Big) = m(\mu) \qquad \text{for each} \quad \mu \in \overline{\mathcal{N}}_2.$$

The following are proved:

(i) The relation of m-equivalence of fuzzy sub-σ-algebras is an equivalence relation in $\mathcal{F}(\mathcal{M})$.

We denote by $[\mathcal{N}]$ the equivalence class containing $\mathcal{N}$ in $\mathcal{F}(M)$.

(ii) For $\mathcal{N}_1, \mathcal{N}_2 \in \mathcal{F}(\mathcal{M})$

(a) $\mathcal{N}_i \leq_m \mathcal{N}_1 \vee \mathcal{N}_2$, $i = 1, 2$;

(b) $\mathcal{N}_1 \approx_m \mathcal{N}_2 \implies \mathcal{N}_1 \approx_m \mathcal{N}_1 \vee \mathcal{N}_2$.

3. Entropy and conditional entropy

3.1. ([18]) Let $(X, \mathcal{M}, m)$ be an F-probability measure space and $\mathcal{N} \in \mathcal{F}(\mathcal{M})$. The *entropy* $H(\mathcal{N})$ of $\mathcal{N}$ is defined by

$$H(\mathcal{N}) = -\sum_{\mu \in \overline{\mathcal{N}}} g\big(m\ (\mu)\big)\,,$$

where the function $g\colon [0,1] \to \mathbb{R}$ is given by

$$g(x) = \begin{cases} x \log x\,, & x > 0\,, \\ 0\,, & \text{otherwise.} \end{cases}$$

Here, the empty sum is defined to be zero.

3.2. ([5]) Let $\mathcal{N}_1, \mathcal{N}_2 \in \mathcal{F}(\mathcal{M})$ and $\overline{\mathcal{N}}_1 = \{\lambda_i :\ 1 \leq i \leq p\}$, and $\overline{\mathcal{N}}_2 = \{\mu_j :\ 1 \leq j \leq q\}$. Define the *conditional entropy* $H(\mathcal{N}_1 \mid \mathcal{N}_2)$ by

$$H(\mathcal{N}_1 \mid \mathcal{N}_2) = -\sum_j \sum_i m(\mu_j) g\big(m(\lambda_i \mid \mu_j)\big)\,,$$

where

$$m(\lambda_i \mid \mu_j) = \frac{m(\lambda_i \wedge \mu_j)}{m(\mu_j)}\,.$$

3.3. *It is observed that:*

(i) *For* $\mathcal{N}_1, \mathcal{N}_2 \in \mathcal{F}(\mathcal{M})$, $H(\mathcal{N}_1) \geq 0$ *and* $H(\mathcal{N}_1 \mid \mathcal{N}_2) \geq 0$ ([5], [18]).

(ii) *For* $\mathcal{N}_1 = \{\mathbf{0}, \mathbf{1}\}$ *and* $\mathcal{N} \in \mathcal{F}(\mathcal{M})$, $H(\mathcal{N} \mid \mathcal{N}_1) = H(\mathcal{N})$.

(iii) *The function* g *is convex, and so, for any convex combination* $\sum_j \alpha_j x_j$ (*i.e.* $\alpha_j \geq 0$ *for all* j *and* $\sum_j \alpha_j = 1$) *of elements* $x_j \in [0,1]$,

$$g\Big(\sum_j \alpha_j x_j\Big) \leq \sum_j \alpha_j g(x_j) \qquad (\textit{cf.}\ [4])\,.$$

4. The Rokhlin metric

Throughout this section $(X, \mathcal{M}, m)$ denotes an F-probability measure space and $\mathcal{N}$ a fuzzy sub-σ-algebra of $\mathcal{M}$.

PROPOSITION 4.1. ([5]) *If $\mathcal{N}_1$, $\mathcal{N}_2$, $\mathcal{N}_3$ be elements of $[\mathcal{N}]$, then*

$$H(\mathcal{N}_1 \vee \mathcal{N}_2 \mid \mathcal{N}_3) = H(\mathcal{N}_1 \mid \mathcal{N}_3) + H(\mathcal{N}_2 \mid \mathcal{N}_1 \vee \mathcal{N}_3).$$

COROLLARY 4.2. *Let $\mathcal{N}_1, \mathcal{N}_2 \in [\mathcal{N}]$. Then*

$$H(\mathcal{N}_1 \vee \mathcal{N}_2) = H(\mathcal{N}_1) + H(\mathcal{N}_2 \mid \mathcal{N}_1).$$

Consequently $H(\mathcal{N}_1 \vee \mathcal{N}_2) \geq H(\mathcal{N}_1)$.

Proof. Follows from Proposition 4.1 and 3.3(ii). □

PROPOSITION 4.3. *If $\mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_3 \in [\mathcal{N}]$, then*

$$H(\mathcal{N}_1 \mid \mathcal{N}_2 \vee \mathcal{N}_3) \leq H(\mathcal{N}_1 \mid \mathcal{N}_2).$$

Proof. Let $\overline{\mathcal{N}}_1 = \{\lambda_i : 1 \leq i \leq p\}$, $\overline{\mathcal{N}}_2 = \{\mu_j : 1 \leq j \leq q\}$, and $\overline{\mathcal{N}}_3 = \{\nu_k : 1 \leq k \leq r\}$. Since $\mathcal{N}_1 \vee \mathcal{N}_2 \approx_m \mathcal{N}_3$, we have

$$m(\lambda_i \wedge \mu_j) = m\Big((\lambda_i \wedge \mu_j) \wedge \bigvee_k \nu_k\Big) = m\Big(\lambda_i \wedge \Big(\bigvee_k (\mu_j \wedge \nu_k)\Big)\Big)$$

$$= m\Big(\lambda_i \wedge \Big(\bigvee_k \eta_{jk}\Big)\Big) = \sum_k m(\lambda_i \wedge \eta_{jk}),$$

where $\eta_{jk} = \mu_j \wedge \nu_k$, $1 \leq j \leq q$, $1 \leq k \leq r$. Hence

$$\begin{aligned}
H(\mathcal{N}_1 \mid \mathcal{N}_2) &= -\sum_j \sum_i m(\mu_j)\, g\left(\frac{m(\lambda_i \wedge \mu_j)}{m(\mu_j)}\right) \\
&= -\sum_j \sum_i m(\mu_j)\, g\left(\sum_k \frac{m(\lambda_i \wedge \eta_{jk})}{m(\mu_j)}\right) \\
&= -\sum_j \sum_i m(\mu_j)\, g\left(\sum_k \frac{m(\eta_{jk})}{m(\mu_j)} \cdot \frac{m(\lambda_i \wedge \eta_{jk})}{m(\eta_{jk})}\right) \\
&\geq -\sum_j \sum_i m(\mu_j) \sum_k \frac{m(\eta_{jk})}{m(\mu_j)}\, g\left(\frac{m(\lambda_i \wedge \eta_{jk})}{m(\eta_{jk})}\right) \\
&= -\sum_j \sum_i \sum_k m(\eta_{jk})\, g\left(\frac{m(\lambda_i \wedge \eta_{jk})}{m(\eta_{jk})}\right) \\
&= H(\mathcal{N}_1 \mid \mathcal{N}_2 \vee \mathcal{N}_3).
\end{aligned}$$

□

PROPOSITION 4.4. *For* $\mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_3 \in [\mathcal{N}]$,

$$H(\mathcal{N}_1 \mid \mathcal{N}_2) + H(\mathcal{N}_2 \mid \mathcal{N}_3) \geq H(\mathcal{N}_1 \mid \mathcal{N}_3).$$

Proof. Using Corollary 4.2 and Proposition 4.3, we obtain

$$\begin{aligned} H(\mathcal{N}_1 \mid \mathcal{N}_2) + H(\mathcal{N}_2 \mid \mathcal{N}_3) &= H(\mathcal{N}_1 \vee \mathcal{N}_2) + H(\mathcal{N}_2 \vee \mathcal{N}_3) - H(\mathcal{N}_2) - H(\mathcal{N}_3) \\ &= H(\mathcal{N}_1 \vee \mathcal{N}_2) + H(\mathcal{N}_3 \mid \mathcal{N}_2) - H(\mathcal{N}_3) \\ &\geq H(\mathcal{N}_1 \vee \mathcal{N}_2) + H(\mathcal{N}_3 \mid \mathcal{N}_1 \vee \mathcal{N}_2) - H(\mathcal{N}_3) \\ &= H(\mathcal{N}_1 \vee \mathcal{N}_2 \vee \mathcal{N}_3) - H(\mathcal{N}_3) \\ &\geq H(\mathcal{N}_1 \vee \mathcal{N}_3) - H(\mathcal{N}_3) = H(\mathcal{N}_1 \mid \mathcal{N}_3). \end{aligned}$$

□

PROPOSITION 4.5. *For* $\mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_3 \in [\mathcal{N}]$,

$$H(\mathcal{N}_1 \vee \mathcal{N}_2 \mid \mathcal{N}_3) \leq H(\mathcal{N}_1 \mid \mathcal{N}_3) + H(\mathcal{N}_2 \mid \mathcal{N}_3).$$

Proof. Follows from Proposition 4.1 and Proposition 4.3. □

THEOREM 4.6. *For* $\mathcal{N}_1, \mathcal{N}_2 \in [\mathcal{N}]$, *define*

$$d(\mathcal{N}_1, \mathcal{N}_2) = H(\mathcal{N}_1 \mid \mathcal{N}_2) + H(\mathcal{N}_2 \mid \mathcal{N}_1).$$

Then d *is a pseudo-metric on* $[\mathcal{N}]$.

Proof. By definition, $d(\mathcal{N}_1 \mid \mathcal{N}_2) \geq 0$ and $d(\mathcal{N}_1 \mid \mathcal{N}_2) = d(\mathcal{N}_2 \mid \mathcal{N}_1)$. Evidently $d(\mathcal{N}_1, \mathcal{N}_1) = H(\mathcal{N}_1 \mid \mathcal{N}_1) = 0$. Finally, for $\mathcal{N}_1$, $\mathcal{N}_2, \mathcal{N}_3 \in [\mathcal{N}]$, by Proposition 4.4, we obtain

$$\begin{aligned} d(\mathcal{N}_1, \mathcal{N}_3) &= H(\mathcal{N}_1 \mid \mathcal{N}_3) + H(\mathcal{N}_3 \mid \mathcal{N}_1) \\ &\leq H(\mathcal{N}_1 \mid \mathcal{N}_2) + H(\mathcal{N}_2 \mid \mathcal{N}_3) + H(\mathcal{N}_3 \mid \mathcal{N}_2) + H(\mathcal{N}_2 \mid \mathcal{N}_1) \\ &= d(\mathcal{N}_1, \mathcal{N}_2) + d(\mathcal{N}_2, \mathcal{N}_3). \end{aligned}$$

□

PROPOSITION 4.7. *If* $\mathcal{N}_1, \mathcal{N}_2 \in [\mathcal{N}]$, *then*

$$H(\mathcal{N}_1 \mid \mathcal{N}_2) = 0 \iff \mathcal{N}_1 \leq_m \mathcal{N}_2.$$

Proof. Let $\overline{\mathcal{N}}_1 = \{\lambda_i : 1 \leq i \leq p\}$ and $\overline{\mathcal{N}}_2 = \{\mu_j : 1 \leq j \leq q\}$. Let $\mathcal{N}_1 \leq_m \mathcal{N}_2$. Then, for any $\mu_j \in \overline{\mathcal{N}}_2$, there exists $\lambda_i \in \overline{\mathcal{N}}_1$ such that $m(\lambda_i \wedge \mu_j) = m(\mu_j)$. Consequently $g(\lambda_i \mid \mu_j) = 0$ and so $H(\mathcal{N}_1 \mid \mathcal{N}_2) = 0$.

Conversely, let $H(\mathcal{N}_1 \mid \mathcal{N}_2) = 0$. Since $m(\mu_j) > 0$ for all $\mu_j \in \overline{\mathcal{N}}_2$, we obtain that $g(m(\lambda_i \mid \mu_j)) = 0$ for all i, j, $1 \le i \le p$, and $1 \le j \le q$. Hence either $m(\lambda_i \mid \mu_j) = 1$ or $m(\lambda_i \mid \mu_j) = 0$. If $m(\lambda_i \mid \mu_j) = 1$ then $m(\lambda_i \wedge \mu_j) = m(\mu_j)$. Let $m(\lambda_i \mid \mu_j) = 0$. Since $\mathcal{N}_1 \approx_m \mathcal{N}_2$, for $\mu_j \in \overline{\mathcal{N}}_2$, we get

$$m\left(\mu_j \wedge \left(\bigvee_i \lambda_i\right)\right) = m(\mu_j),$$

or

$$\sum_i m(\mu_j \wedge \lambda_i) = m(\mu_j). \tag{4.6.1}$$

If possible, let us assume that there is a $\lambda_k \in \overline{\mathcal{N}}_1$ such that $0 < m(\lambda_k \wedge \mu_j) < m(\mu_j)$. Then $m(\mu_j) \cdot g(m(\lambda_k \mid \mu_j)) \ne 0$, which contradicts the hypothesis that $H(\mathcal{N}_1 \mid \mathcal{N}_2) = 0$. Hence, from (4.6.1), we deduce that there exists an i_0, $1 \le i_0 \le p$, such that $m(\lambda_{i_0} \wedge \mu_j) = m(\mu_j)$.

Thus $\mathcal{N}_1 \le_m \mathcal{N}_2$. □

PROPOSITION 4.8. *For fuzzy sub-σ-algebras $\mathcal{N}_1$, $\mathcal{N}_2$, $\mathcal{N}_3$ of $\mathcal{M}$, $\mathcal{N}_1 \le_m \mathcal{N}_2$ and $\mathcal{N}_2 \le_m \mathcal{N}_3$ imply that $\mathcal{N}_1 \le_m \mathcal{N}_3$.*

Proof. Let $\nu \in \overline{\mathcal{N}}_3$. Then, since $\mathcal{N}_2 \le_m \mathcal{N}_3$, there exists $\mu \in \overline{\mathcal{N}}_2$ such that $m(\mu \wedge \nu) = m(\nu)$. Also, since $\mathcal{N}_1 \le_m \mathcal{N}_2$, there exists $\lambda \in \overline{\mathcal{N}}_1$ such that $m(\lambda \wedge \mu) = m(\mu)$, and so $m(\lambda \vee \mu) = m(\lambda)$. Now, we have

$$\begin{aligned}
m(\nu) &= m(\mu \wedge \nu) = m(\mu \wedge \nu) + m(\lambda) - m(\lambda) \\
&= m((\mu \wedge \nu) \vee \lambda) + m(\mu \wedge \nu \wedge \lambda) - m(\lambda) \\
&= m((\mu \vee \lambda) \wedge (\nu \vee \lambda)) + m(\mu \wedge \nu \wedge \lambda) - m(\lambda) \\
&= m(\mu \vee \lambda) + m(\nu \vee \lambda) - m(\mu \vee \lambda \vee \nu) + m(\mu \wedge \nu \wedge \lambda) - m(\lambda) \\
&= m(\nu \vee \lambda) - m(\mu \vee \lambda \vee \nu) + m(\mu \wedge \nu \wedge \lambda) \\
&\le m(\mu \wedge \nu \wedge \lambda) \le m(\lambda \wedge \nu).
\end{aligned}$$

Thus $m(\nu) = m(\lambda \wedge \nu)$, i.e., $\mathcal{N}_1 \le_m \mathcal{N}_3$. □

Remark 4.9. For $\mathcal{N}_1$, $\mathcal{N}_2 \in [\mathcal{N}]$, define a relation $\sim$ as follows:

$$\mathcal{N}_1 \sim \mathcal{N}_2 \iff \mathcal{N}_1 \le_m \mathcal{N}_2 \text{ and } \mathcal{N}_2 \le_m \mathcal{N}_1.$$

In view of Proposition 4.8, $\sim$ is an equivalence relation on $[\mathcal{N}]$. We call this relation *equivalence modulo* 0. If we identify the equivalence class $\tilde{\mathcal{N}}$ induced by the relation $\sim$ with $\mathcal{N}$, then the pseudo-metric d defined in Theorem 4.6, becomes a metric on $[\mathcal{N}]$. Following the terminology of the classical crisp case we call this metric the *Rokhlin metric* (cf. [4], [12], [15]).

Thus we have the following:

THEOREM 4.10. *For $\mathcal{N}_1, \mathcal{N}_2 \in [\mathcal{N}]/\sim$, $d(\mathcal{N}_1, \mathcal{N}_2) = H(\mathcal{N}_1 \mid \mathcal{N}_2) + H(\mathcal{N}_2 \mid \mathcal{N}_1)$, is a metric on $[\mathcal{N}]/\sim$.*

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