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DIFFERENTIAL FORMS ON MANIFOLDS WITH A POLYNOMIAL STRUCTURE

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ABSTRACT. On a C^∞-manifold endowed with an integrable polynomial structure (with only simple roots of the characteristic polynomial), the decomposition of the tangent bundle which corresponds to the decomposition of the characteristic polynomial, induces a decomposition of the bundle of complex p-forms. This decomposition enables us to introduce derivation operators (of degree 1) of three types on the graded commutative algebra of differential forms. For these operators, we prove Poincaré Lemma (that every closed form is exact).

All objects under considerations are supposed to be smooth (of class C^∞). M will denote a manifold, TM and T^*M denote the tangent and cotangent bundle of M, respectively, $T^{\mathbb{C}}M$ and $T^{\mathbb{C}*}M$ the corresponding complexifications. Let us denote by $\Omega^p M$ the bundle of differential p-forms on M.

1. Let (M, f) be a polynomial structure, i. e. a smooth m-dimensional manifold M endowed with a $(1,1)$-tensor field f satisfying an equation $R(f) = 0$ where R is a polynomial. Suppose that R is a *minimal* polynomial of the endomorphism $f_x \colon T_xM \to T_xM$ of the tangent space at each point $x \in M$. In what follows we shall suppose that R has only *simple roots*. A polynomial structure f on M is called *integrable* if there are local coordinates in a neighborhood U (abbreviated as nbd) of any point x in which the matrix representation of f_x has a constant canonical form on U ([3], [5], [6], [9]).

The decomposition of $R(\xi)$ over $\mathbb{C}$ into pairwise distinct prime factors

$$R(\xi) = \prod_{i=1}^{r}(\xi - b_i) \prod_{j=1}^{s}(\xi - c_j) \prod_{j=1}^{s}(\xi - \bar{c}_j) \tag{1}$$

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with $b_i \in \mathbb{R}$, $c_j \in \mathbb{C}$ gives decompositions of both the complex tangent bundle and the complex cotangent bundle as follows. Let us denote

$$\begin{aligned} H_i &= \ker(f - b_i I)\,, & i &= 1,\dots,r\,, \\ H_{r+j} &= \ker\bigl(\xi^2 - (c_j + \bar{c}_j)\xi + c_j\bar{c}_j\bigr)\,, & j &= 1,\dots,s\,, \end{aligned} \tag{2}$$

where I denotes the identity isomorphism of $T^{\mathbb{C}}M$. $(H_1,\dots,H_{r+s})$ is an *almost-product structure associated with* f. Let P_t be a projector projecting onto the subspace H_t, $t = 1,\dots,r+s$. Further, let

$$\begin{aligned} \mathcal{H}_i &= H_i^{\mathbb{C}} = \ker(f^{\mathbb{C}} - b_i I) & &\text{for } i = 1,\dots,r\,, \\ \mathcal{H}_{r+j} &= \ker(f^{\mathbb{C}} - c_j I) & &\text{for } j = 1,\dots,s\,, \\ \mathcal{H}_{r+s+j} &= \ker(f^{\mathbb{C}} - \bar{c}_j I) = \overline{\mathcal{H}}_{r+j} & &\text{for } j = 1,\dots,s\,. \end{aligned} \tag{3}$$

It follows that $H^{\mathbb{C}}_{r+j} = \mathcal{H}_{r+j} \oplus \overline{\mathcal{H}}_{r+j}$, $j = 1,\dots,s$. Now the complex tangent bundle has decomposition $T^{\mathbb{C}}M = \bigoplus_{t=1}^{n} \mathcal{H}_t$, $n = r + 2s$, and $(\mathcal{H}_1,\dots,\mathcal{H}_{r+2s})$ is a *complex almost product structure associated with* f. The subbundle H_{r+j}, $j \in \{1,\dots,s\}$, is equipped with the endomorphism J_j satisfying $J_j^2 = -I_j^2$, $I_j = \mathrm{id}\,|_{H_{r+j}}$, $j = 1,\dots,s$,

$$J_j = \frac{2f - (c_j + \bar{c}_j)}{\sqrt{-(c_j - \bar{c}_j)^2}} P_{r+j}\,, \qquad j = 1,\dots,s\,. \tag{4}$$

That is, J_j is an almost complex structure on H_{r+j}, and the complex linear extension of J_j to $H^{\mathbb{C}}_{r+j}$ acts on $\mathcal{H}_{r+j}$ (respectively, on $\overline{\mathcal{H}}_{r+j}$) as multiplication by i (respectively by $-$i). The $(1,1)$-tensor field $\Phi = \sum_j J_j P_{r+j}$ satisfies $\Phi^3 + \Phi = 0$ (and is called an *almost contact structure associated with* f, or an f-structure ([7], [12])).

The following can be verified:

THEOREM 1.1. *The polynomial structure (with characteristic polynomial having simple roots) is integrable if and only if the associated complex almost product structure is integrable (which means that $\mathcal{H}_t \oplus \mathcal{H}_w$ are integrable for all the pairs $1 \leqq t < w \leqq n$).*

Let us define, for $t = 1,\dots,r+2s$,

$$\mathcal{C}_t = \bigl\{\omega \in T^{*\mathbb{C}}M \mid \langle X,\omega\rangle = 0 \text{ for all } X \in \mathcal{H}_k\,,\ k \neq t\,,\ k = 1,\dots,r+2s\bigr\}\,. \tag{5}$$

For any multi-index $\alpha = (a_1,\dots,a_n)$, $n = r + 2s$ of weight $|\alpha| = \sum_i a_i = p$, let us introduce the subspace

$$\mathcal{C}^\alpha = \underbrace{\mathcal{C}_1 \wedge \dots \wedge \mathcal{C}_1}_{a_1\text{-times}} \wedge \dots \wedge \underbrace{\mathcal{C}_n \wedge \dots \wedge \mathcal{C}_n}_{a_n\text{-times}}\,.$$

Now the complexification of the cotangent bundle $T^{*\mathbb{C}}M = \Lambda^1(M)$ has a decomposition $T^{*\mathbb{C}}M = \bigoplus_{t=1}^{n} \mathcal{C}_t$, and the bundle of complex differential p-forms on M, $\Lambda^p(M) = \big(\Omega^p(M)\big)^{\mathbb{C}}$ is decomposable as follows: $\Lambda^p(M) = \bigoplus_{|\alpha|=p} \mathcal{C}^\alpha$.

2. In what follows, we suppose that the polynomial structure f on M is integrable. The following can be proved:

THEOREM 2.1. *The structure f is integrable if and only if the following condition is satisfied:*

(i) *if* $\omega \in \mathcal{C}^\alpha$, $\alpha = (a_1, \dots, a_n)$, $n = r + 2s$ *then* $\mathrm{d}\omega \in \bigoplus_{j=1}^{n} \mathcal{C}^\beta$ *where* $\beta = (a_1 + \delta_1^j, \dots, a_n + \delta_n^j)$.

EXAMPLE 2.1. A complex m-manifold M can be viewed in particular as a real $2m$-dimensional manifold equipped with an integrable almost complex structure, $J^2 = -I$ and vice versa, any integrable almost complex structure on M turns a real manifold M_{2m} into a complex manifold M_m. The tangent bundle of an almost complex (respectively, complex) manifold admits the decomposition

$$T^{\mathbb{C}}M = \ker(J - \mathrm{i}\,I) \oplus \ker(J + \mathrm{i}\,I)$$

and the algebra of complex differential forms can be bigraded as $\Lambda(M) = \bigoplus_{p,q\ 0}^{m} \mathcal{C}^{(p,q)}$. By integrability [4], $\mathrm{d}\mathcal{C}^{(p,q)} \subset \mathcal{C}^{(p+1,q)} \oplus \mathcal{C}^{(p,q+1)}$ for $p, q = 0, \dots, m$. Elements of $\mathcal{C}^{(p,q)}$ are called *forms of type* (p, q).

Let us set $\tilde{n} = \sum_{i=1}^{r} \dim H_i$, $\tilde{\tilde{n}} = \sum_{j=1}^{s} \dim H_{r+j}$. In the case of an integrable structure, there exist local coordinates $(x_1, \dots, x_m)$ in some nbd U of an arbitrary point x such that on U, H_i is the span of

$$\left\{ \frac{\partial}{\partial x_k}\,;\ \sum_{t<i} \dim H_t + 1 \leqq k \leqq \sum_{t\leqq i} \dim H_t \right\}, \qquad i = 1, \dots, r,$$

and H_{r+j} is the span of

$$\left\{ \frac{\partial}{\partial x_k}\,;\ \tilde{n} + \sum_{t<j} \dim H_{r+t} + 1 \leqq k \leqq \tilde{n} + \sum_{t\leqq j} \dim H_{r+t} \right\}, \qquad j = 1, \dots, s.$$

With respect to the holonomic frame $\left(\frac{\partial}{\partial x_k}\right)$, the coordinate expression of the associated almost contact structure is

$$\Phi\left(\frac{\partial}{\partial x_i}\right) = 0, \qquad \Phi\left(\frac{\partial}{\partial x_j}\right) = \frac{\partial}{\partial x_{\tilde{\tilde{n}}+j}}, \qquad \Phi\left(\frac{\partial}{\partial x_{\tilde{\tilde{n}}+j}}\right) = \frac{\partial}{\partial x_j} \tag{6}$$

where $\tilde{n}+1 \leqq j \leqq \tilde{n}+\tilde{\tilde{n}}$. Let us introduce

$$\frac{\partial}{\partial z_q} = \frac{1}{2}\frac{\partial}{\partial x_{\tilde{n}+q}} - \mathrm{i}\,\frac{\partial}{\partial x_{\tilde{n}+\tilde{\tilde{n}}+q}}\,. \tag{7}$$

Then $\frac{\partial}{\partial x_k}$, $\frac{\partial}{\partial z_q}$ and the complex conjugates $\frac{\partial}{\partial \bar{z}_q}$ form a frame field for the complex tangent bundle (suitable parts of which are bases for $\mathcal{H}_i$, $\mathcal{H}_{j+r}$ and $\mathcal{H}_{j+r+s}$, respectively). Its dual co-basis $(\mathrm{d}x_k, \mathrm{d}z_q, \mathrm{d}\bar{z}_q)$ consists of

$$\mathrm{d}z_q = \mathrm{d}x_{\tilde{n}+q} + \mathrm{i}\,\mathrm{d}x_{\tilde{n}+\tilde{\tilde{n}}+q}\,, \qquad \mathrm{d}\bar{z}_q = \mathrm{d}x_{\tilde{n}+q} - \mathrm{i}\,\mathrm{d}x_{\tilde{n}+\tilde{\tilde{n}}+q}\,.$$

By the integrability conditions, for any differential form $\omega \in \mathcal{C}^{\alpha}$, $\alpha = (a_1, \dots, a_n)$, $n = r+2s$ the differential $\mathrm{d}\omega \in \mathcal{C}^{(a_1+1,\dots,a_n)} \oplus \dots \oplus \mathcal{C}^{(a_1,\dots,a_n+1)}$. We can introduce the derivation operators

$$\begin{aligned}
&\partial_i\colon \mathcal{C}^{(a_1,\dots,a_i,\dots,a_n)} \to \mathcal{C}^{(a_1,\dots,a_i+1,\dots,a_n)}\,, && 1 \leqq i \leqq r\,,\\
&\Delta_j\colon \mathcal{C}^{(a_1,\dots,a_{r+j},\dots,a_n)} \to \mathcal{C}^{(a_1,\dots,a_{r+j}+1,\dots,a_n)}\,, && 1 \leqq j \leqq s\,,\\
&\bar{\Delta}_j\colon \mathcal{C}^{(a_1,\dots,a_{r+s+j},\dots,a_n)} \to \mathcal{C}^{(a_1,\dots,a_{r+s+j}+1,\dots,a_n)}\,, && 1 \leqq j \leqq s\,,
\end{aligned} \tag{8}$$

uniquely determined by the formula

$$\mathrm{d}\omega = \partial_1\omega + \dots + \partial_r\omega + \Delta_1\omega + \dots + \Delta_s\omega + \bar{\Delta}_1\omega + \dots + \bar{\Delta}_s\omega\,.$$

By evaluation in local coordinates $\bar{\Delta}_j\bar{\omega} = \overline{\Delta_j\omega}$.

EXAMPLE 2.2. On a complex manifold we obtain the decomposition $\mathrm{d} = \partial_1 + \partial_2$.

Let D stand for the operator ∂_i, or Δ_j, or $\bar{\Delta}_j$, respectively. We say that a differential p-form on a polynomial manifold (M, f) (respectively, on an open subset $\mathcal{V} \subset (M, f)$) is

(i) D-*closed* if $\mathrm{D}\omega = 0$,

(ii) D-*exact* if $\omega = \mathrm{D}\alpha$ for some $(p-1)$-form α on M (respectively, on $\mathcal{V}$).

As in the case of the exterior differential d, any D-exact form is D-closed but the converse is not true globally. We will prove that the converse holds locally.

3. We will use the following generalization of the Poincaré Lemma ([8]) for p-forms depending smoothly on real parameters.

LEMMA 3.1. *Let M be a C^∞-manifold, let $U \subset M$ be an open subset, and let $\omega(t_1, \dots, t_k)$ be a p-form on M depending (smoothly) on real parameters $t_1, \dots, t_k$. Let $\mathrm{d}\omega(t_1, \dots, t_k) = 0$. Then for any $x \in U$ there exists a nbd $\mathcal{V}_x$ and a $(p-1)$-form $\alpha(t_1, \dots, t_k)$ on $\mathcal{V}_x$ such that $\mathrm{d}\alpha = \omega$ on $\mathcal{V}_x$.*

Proof. The proof is similar to the classical case [8; pp. 98–100]. □

Similarly, the Grothendieck Lemma (sometimes incorrectly called the Dolbeaut' Lemma) ([1; p. 71, 9.4]), which generalizes the Poincaré Lemma in a complex setting, can be re-formulated as follows:

LEMMA 3.2. *Let M be a complex manifold, let $U \subset M$ be an open subset, and let $\omega(t_1, \dots, t_k) \in \mathcal{C}^{(p,q)}(U)$ be a C^∞ complex form on U of type (p, q), $p \geqq 1$, which depends on real parameters. Let $\partial_1 \omega(t_1, \dots, t_k) = 0$. Then for any $x \in U$ there exists a nbd $\mathcal{V}_x$ and a form α of type $(p-1, q)$, $\alpha(t_1, \dots, t_k) \in \mathcal{C}^{(p-1,q)}(\mathcal{V}_x)$ such that*

$$\partial_1 \alpha(t_1, \dots, t_k) = \omega(t_1, \dots, t_k).$$

For a (real) manifold equipped with a polynomial structure we shall prove a similar result.

THEOREM 3.1. *Let (M, f) be a polynomial m-manifold for which the characteristic polynomial $R(\xi)$ admits only simple roots. Let $U \subset M$ be open, and let $\omega \in \mathcal{C}^\alpha$ be a p-form where $\alpha = (a_1, \dots, a_n)$, $a_i \geqq 1$, $n = r + 2s$ (respectively, $a_{r+j} \geqq 1$, or $a_{r+s+j} \geqq 1$). Let $\partial_i \omega = 0$ (respectively, $\Delta_j \omega = 0$, or $\bar{\Delta}_i \omega = 0$). Then for any point $x \in U$ there exists a nbd $\mathcal{V}_x \subset U$ and a differential $(p-1)$-form $\alpha \in \mathcal{C}^\beta(U)$, $\beta = (a_1, \dots, a_i - 1, \dots, a_n)$ (respectively, $\beta = (a_1, \dots, a_{r+j} - 1, \dots, a_n)$ or $\beta = (a_1, \dots, a_{r+s+j} - 1, \dots, a_n)$) such that $\partial_i \alpha = \omega$ (respectively, $\Delta_j \alpha = \omega$, or $\bar{\Delta}_j \alpha = \omega$) on $\mathcal{V}_x$.*

Proof. Let $\partial_i \omega = 0$, $1 \leqq i \leqq r$. Locally, we can write

$$\begin{aligned} \omega &= \sum f_{i_1, \dots, j_1, \dots, k_w}(x_1, \dots, x_m)\, \mathrm{d}x_{i_1} \wedge \dots \wedge \mathrm{d}z_{j_1} \wedge \dots \wedge \mathrm{d}\bar{z}_{k_w} \\ &= \sum \mathrm{d}x_{i_1} \wedge \dots \wedge \mathrm{d}x_{i_l} \wedge \Big(\sum f_{i_1, \dots, j_1, \dots, k_w}(x)\, \mathrm{d}x_{i_l+1} \wedge \dots \wedge \mathrm{d}x_{i_l+a_i} \Big) \\ &\qquad \wedge \mathrm{d}x_{i_l+a_{i+1}} \wedge \dots \wedge \mathrm{d}z_{j_1} \wedge \dots \wedge \mathrm{d}\bar{z}_{k_w} \end{aligned} \tag{9}$$

where the first sum runs over $1 \leqq i_1 < \dots \leqq \tilde{n}$, $1 \leqq j_1 < \dots \leqq \tilde{\tilde{n}}$, $1 \leqq k_1 < \dots \leqq \tilde{\tilde{n}}$, the third sum runs over $(i_l + 1, \dots, i_l + a_i)$, $l = a_1 + \dots + a_{i-1}$, and the second sum over the remaining indices. Let us introduce $\Omega(x) \in (\mathcal{C}_i)^{a_i}(M) = \mathcal{C}^{(0,\dots,a_i,\dots,0)}$, $\Omega = \sum f_{i_1,\dots,j_1,\dots,k_w}(x)\, \mathrm{d}x_{i_l+1} \wedge \dots \wedge \mathrm{d}x_{i_l+a_i}$. Then $\omega = \mathrm{d}x_{i_1} \wedge \dots \wedge \Omega \wedge \dots \wedge \mathrm{d}\bar{z}_{k_w}$ where Ω is regarded as an a_i-form on $\mathcal{H}_i$ in variables corresponding to the coordinates on $\mathcal{H}_i$ while the other coordinates play the role of parameters. Let us consider an integral manifold M_i of the distribution $\mathcal{H}_i$ (which exists by integrability of f). Denote by d_i the exterior derivative with respect to M_i. Then

$$\partial_i \omega = \pm\, \mathrm{d}x_{i_1} \wedge \dots \wedge \mathrm{d}x_{i_l} \wedge \mathrm{d}_i \Omega \wedge \mathrm{d}x_{i_{l+a_i+1}} \wedge \dots \wedge \mathrm{d}\bar{z}_{k_w}. \tag{10}$$

Since $\partial_i \Omega = 0$ we obtain $\mathrm{d}_i \omega = 0$. By Lemma 3.1, there exists locally an $(a_i - 1)$-form $\tilde{\Omega} \in (\mathcal{C}_i)^{a_i - 1}(M_i)$ in the same variables, depending on the same parameters such that $\mathrm{d}_i \tilde{\Omega} = \Omega$. Now let us consider $\tilde{\Omega}$ as a form in all the variables $x_1, \dots, x_m$, $\tilde{\Omega} \in (\mathcal{C}_i)^{a_i - 1}(M)$. Then

$$\alpha = \sum \mathrm{d}x_{i_1} \wedge \dots \wedge \tilde{\Omega} \wedge \dots \wedge \mathrm{d}\bar{z}_{k_w} \tag{11}$$

is the required form satisfying $\partial_i \alpha = \omega$.

Similarly, in the case $\Delta_j \omega = 0$, after a suitable replacement of coefficients in the coordinate formula for ω, we obtain a form Ω on $\mathcal{H}_{r+j}$ in variables corresponding to $\mathcal{H}_{r+j}$ which depends on real parameters. By integrability, through each point there passes a unique maximal integral submanifold M_{r+j} of the distribution $\mathcal{H}_{r+j}$. The restriction $J_j P_{r+j}$ of Φ onto M_{r+j} is an integrable almost complex structure on the almost complex manifold M_{r+j}. Therefore M_{r+j} is a complex manifold ([1], [4]), and we can apply Lemma 3.2 considering Ω as a form on M or on M_i alternatively. Denote by $\mathcal{P}^{(r+j)}$ the projector onto the dual space $\mathcal{H}^*_{r+j}$. Let us introduce an operator $\tilde{\partial}_{r+j}$ by $\tilde{\partial}_{r+j}\tau = \mathcal{P}^{(r+j)}(\mathrm{d}_{r+j}\tau)$ for $\tau \in \mathcal{H}^*_{r+j}$. Obviously, $\Delta_j \omega = 0$ if and only if $\tilde{\partial}_{r+j}\Omega = 0$. Locally, there exists $\tilde{\Omega} \in (\mathcal{C}_{r+j})^{a_{r+j}-1}$ such that $\tilde{\partial}_{r+j}(\tilde{\Omega}) = \Omega$. In some nbd of a given point, the form α introduced by

$$\alpha = \sum \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}z_{j_w} \wedge \tilde{\Omega} \wedge \cdots \wedge \mathrm{d}\bar{z}_{k_w} \tag{12}$$

satisfies $\Delta_j \alpha = \omega$.

Let $\bar{\Delta}_j \omega = 0$. Then $\bar{\Delta}_j \bar{\omega} = 0$ and locally, there is a form β satisfying $\Delta_j \beta = \bar{\omega}$. Consequently, $\alpha = \bar{\beta}$ satisfies $\bar{\Delta}_j \alpha = \omega$. □

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