Mario Lamberger; Jörg M. Thuswaldner

Distribution properties of digital expansions arising from linear recurrences

Mathematica Slovaca, Vol. 53 (2003), No. 1, 1--20

Persistent URL: http://dml.cz/dmlcz/133265

Math. Slovaca, 53 (2003), No. 1, 1–20

Mathematica Slovaca

DISTRIBUTION PROPERTIES OF DIGITAL EXPANSIONS ARISING FROM LINEAR RECURRENCES

MARIO LAMBERGER* — JÖRG M. THUSWALDNER**

(*Communicated by Stanislav Jakubec*)

ABSTRACT. Let $(G_j)_{j\geq 0}$ be a strictly increasing sequence of non-negative integers defined by a linear recurrence relation. Let $S_G(n)$ denote the G-ary sum-of-digits function. In this paper we establish several distribution results for $S_G(n)$ which include an analogue of a result of A. O. Gel'fond on the distribution of the q-ary sum-of-digits function in residue classes as well as an *Erdős-Kac-type theorem*. The main tool in the proofs of these results is an estimate for exponential sums of the form

$$\Big|\sum_{n<N} \exp\Big(2\pi\mathrm{i}\Big(\tfrac{r}{s}S_G(n)+yn\Big)\Big)\Big| \ll N^{\lambda},$$

where $r,s\in\mathbb{Z}$, $y\in\mathbb{R}$ and $\lambda<1$.

1. Introduction and statement of results

The aim of this paper is the investigation of certain properties of number systems defined by linear recurrences. Before we give an exact definition of these objects we want to review some earlier results on related topics.

In the present paper we are interested in representations of integers by linear recurrence number systems. Such representations were studied for instance by Pethő—Tichy [24] and Grabner—Tichy [13], [15]. In these papers the authors mainly emphasize the sum-of-digits function S_G of these representations. In particular, they establish an asymptotic formula for the summatory function of S_G, which turns out to be of a similar shape as the summatory function for the q-ary sum-of-digits function obtained by Delange [3]. Moreover,

2000 Mathematics Subject Classification: Primary 11A63, 11L07.
Keywords: linear recurrence number system, exponential sum estimate.

The first author was supported by the START-project Y96-MAT of the Austrian Science Fund. The second author was supported by the Austrian Science Fund project P-14200-MAT.

they are interested in the distribution modulo 1 of certain sequences defined via S_G (cf. also Grabner [14] and Larcher [20]). For similar results from the viewpoint of substitutions we refer to Dumont—Thomas [7], [8] and Dumont et al. [6]. Connections of linear recurrence number systems to automata theory and languages are outlined in Frougny [11].

Recently Drmota and his co-authors established several results on linear recurrence number systems. We mention here the papers Drmota—Gajdosik [4], [5]. Especially the first of these papers is strongly related to ours. In this paper the authors prove a local and a global limit theorem for S_G. Using this, one can easily derive that S_G is equidistributed in residue classes. In our paper we want to improve and extend this equidistribution result. Firstly, we obtain a better error term, on the other hand, we get distribution results even if the argument of $S_G(n)$ is restricted by congruence conditions. The reason, why this improvement is important, is that it makes it possible to apply sieve methods. This leads to a series of distribution results of S_G with respect to primes.

After this short review we want to recall the general setting. Let $G = (G_j)_{j\geq 0}$ be a strictly increasing sequence of positive integers and suppose that $G_0 = 1$. For an arbitrary non-negative integer n there exists a unique integer L such that $G_L \leq n < G_{L+1}$. Then the digits of n with respect to the base $(G_j)_{j\geq 0}$ can be computed using the greedy algorithm which leads to

$$n = \varepsilon_0(n)G_0 + \cdots + \varepsilon_L(n)G_L$$

where $0 \leq \varepsilon_\ell(n) < \frac{G_{L+1}}{G_L}$ holds. This expansion is unique and the digits satisfy the inequality

$$\varepsilon_0(n)G_0 + \cdots + \varepsilon_k(n)G_k < G_{k+1}$$

for $0 \leq k \leq L$.

In this case we define the sum-of-digits function of n with respect to $(G_j)_{j\geq 0}$ by

$$S_G(n) = \varepsilon_0(n) + \cdots + \varepsilon_L(n)\,.$$

Important examples of such G-ary digital expansions are the ordinary q-ary expansions with $G_j = q^j$ where $q \geq 2$ is an integer. Furthermore, we want to mention the Cantor number system (see e.g. [16], [19]) which is defined by $G_j = q_0 \dots q_j$ with positive integers $q_\ell \geq 2$.

In this work we are interested in digital expansions arising from sequences $(G_j)_{j\geq 0}$ satisfying a linear recurrence relation. To be more precise, we give the following definition (cf. [5], [25]):

DEFINITION 1.1. We will refer to a sequence $(G_j)_{j\geq 0}$ as a *finite linear recurrence base*, if the following conditions hold:

(1) $G_0 = 1$ and $G_k \geq a_1 G_{k-1} + \cdots + a_k G_0 + 1$ for $1 \leq k < d$.

(2) There are integers $a_i \geq 0$ $(1 \leq i \leq d-1)$ and $a_d > 0$ such that for each $n \geq 0$

$$G_{n+d} = a_1 G_{n+d-1} + \cdots + a_d G_n . \qquad (1)$$

(3) The coefficients $a_1, \ldots, a_d$ of the linear recurrence satisfy the *Parry-condition* (cf. Parry [23]), i.e.

$$(a_k, a_{k+1}, \ldots, a_d) \leq (a_1, a_2, \ldots, a_{d-k+1})$$

for $1 < k \leq d$, where $\leq$ denotes the lexicographic order.

Remark. A lot of papers treating digital expansions which stem from finite linear recurrences restrict themselves to the following class:

$$\begin{aligned} G_{n+d} &= a_1 G_{n+d-1} + \cdots + a_d G_n \qquad && \text{for} \quad n \geq 0 , \\ G_k &= a_1 G_{k-1} + \cdots + a_k G_0 + 1 \qquad && \text{for} \quad 0 \leq k < d , \end{aligned}$$

where $a_1 \geq a_2 \geq \cdots \geq a_d > 0$ are non-negative integers. From Brauer [1] we know that in this case the dominant root of the corresponding characteristic polynomial is a *Pisot number* $\alpha \in (a_1, a_1+1)$. Remember that a Pisot number is an algebraic integer whose conjugates lie all in the interior of the unit circle. Note that this class is a special case of the class of finite linear recurrent bases defined in Definition 1.1.

An example of a finite linear recurrence with non-decreasing coefficients is $G_{n+d} = G_{n+d-1} + G_n$, which corresponds to $a_1 = a_d = 1$ and $a_i = 0$ for $2 \leq i \leq d-1$.

Remark. We also want to note that Parry's condition (3) of Definition 1.1 is equivalent to demand that the following holds for all $n \geq 0$ and $1 \leq k < d$ (cf. Steiner [25]):

$$G_{n+d-k} > \sum_{i=k+1}^{d} a_i G_{n+d-i} .$$

This version of the condition was used in Drmota—Gajdosik [4], [5].

The present paper is devoted to the study of distribution properties of $S_G(n)$. The starting point of our work is a paper of A. O. Gel'fond [12] who proved the following result in the case $G_j = q^j$.

Let $r, a \in \mathbb{Z}$, $m, s \in \mathbb{N}$ and assume that $\gcd(s, q-1) = 1$; then one has

$$\left|\{n < N : \ n \equiv a \ (m) , \ S_q(n) \equiv r \ (s)\}\right| = \frac{N}{ms} + \mathcal{O}(N^{\mu}) ,$$

where $\mu < 1$ only depends on q and s.

(A special case of this result can be already found in [9].)

In several papers, extensions and generalizations of this result were obtained. (cf. for instance Mauduit—Sárközy [21], [22], Kim [18] or Thuswaldner [26]).

We now want to state our main results. They are all devoted to the structure of the set of all integers for which the sum-of-digits function S_G fulfills certain congruence properties. Thus we set

$$\mathcal{U}_{r,s}(N) := \{n < N :\ S_G(n) \equiv r\ (s)\}\,.$$

For convenience we set $e(z) := \mathrm{e}^{2\pi\mathrm{i}z}$ and $\|z\| := \min(\{z\}, 1-\{z\})$, where $\{z\}$ denotes the fractional part of z.

The main effort in our paper is needed to establish the following result on exponential sums, from which all other theorems can be deduced.

THEOREM 1.1. *Let $(G_j)_{j\ge 0}$ be given by a finite linear recurrence as in Definition* 1.1, *let $r \in \mathbb{Z}$, $s \in \mathbb{N}$ and suppose that* $\gcd(a_1 + \cdots + a_d - 1, s) = 1$, *$r \not\equiv 0\ (s)$ and $y \in \mathbb{R}$. Then*

$$\Big|\sum_{n<N} e\Big(\frac{r}{s}S_G(n) + yn\Big)\Big| \ll N^{\lambda}$$

where the implied constant depends only on $a_1, \dots, a_d$, s and

$$\lambda = \frac{1}{2}\left(1 + \frac{\log\Big(\alpha - \frac{\pi^2\alpha^{j_0}}{2j_1(s(\Delta+2)^2)}\Big)}{\log\alpha}\right) < 1\,. \tag{2}$$

Here, α denotes the dominating root of the characteristic polynomial of $(G_j)_{j\ge 0}$ and j_0 and j_1 can be calculated explicitly and depend only on $(G_j)_{j\ge 0}$ and $\Delta := \max\{a_d, 2\}$ (see Sections 3 *and* 4*).*

A first application of Theorem 1.1 is the analogue of the above mentioned result of Gel'fond.

THEOREM 1.2. *Let $(G_j)_{j\ge 0}$ be given by a finite linear recurrence as in Definition* 1.1 *and suppose that* $\gcd(a_1+\cdots+a_d-1, s) = 1$. *Then the sum-of-digits function is well distributed in residue classes, i.e. if $r, a \in \mathbb{Z}$, $m, s \in \mathbb{N}$, then*

$$\big|\{n \in \mathcal{U}_{r,s}(N) :\ n \equiv a\ (m)\}\big| = \frac{N}{ms} + \mathcal{O}(N^{\lambda})$$

with λ as in (2).

Theorems on the distribution of certain sets in residue classes can often be used to establish results on the distribution of primes in these sets. We will establish two results in this direction. Namely, we will show an analogue to Gel'fond [12; Théorème II] as well as an Erdős-Kac-type theorem. First we give the analogue of Gel'fond's result.

COROLLARY 1.1. *Let* $(G_j)_{j\geq 0}$ *be given by a finite linear recurrence as in Definition* 1.1 *and suppose that* $\gcd(a_1+\cdots+a_d-1,s)=1$. *Let* $P_f(N)$ *be the set of those elements of* $\mathcal{U}_{r,s}(N)$ *which are not divisible by an* f*th power of a prime. Then*

$$|P_f(N)| = \frac{N}{s\zeta(f)} + \mathcal{O}(N^{\lambda_1})$$

with $\lambda_1 < 1$ *only depends on* $(G_j)_{j\geq 0}$ *and* s. *Here* ζ *denotes the* Riemann zeta function.

The Erdős-Kac-type theorem reads as follows.

COROLLARY 1.2. *Let* $(G_j)_{j\geq 0}$ *be given by a finite linear recurrence as in Definition* 1.1, $r\in\mathbb{Z}$, $s\in\mathbb{N}$ *and suppose that* $\gcd(a_1+\cdots+a_d-1,s)=1$. *Let* Q_N *be the frequency defined by*

$$Q_N(X) := \frac{1}{|\mathcal{U}_{r,s}(N)|}\left|\left\{n\in\mathcal{U}_{r,s}(N):\ \omega(n)-\log\log N \leq X\sqrt{\log\log N}\right\}\right|,$$

where $\omega(n)$ *denotes the number of distinct prime factors of* n. *Then we have*

$$\left|Q_N(X) - \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{X} \mathrm{e}^{-u^2/2}\,\mathrm{d}u\right| \ll \frac{\log\log\log N}{\sqrt{\log\log N}}$$

uniformly in $X\in\mathbb{R}$ *and* $N\in\mathbb{N}$, $N\geq 8$.

The next application is an analogue of a result on sumsets, whose version for q-ary number systems has been proved in Mauduit—Sárközy [21].

COROLLARY 1.3. *Let* $(G_j)_{j\geq 0}$ *be given by a finite linear recurrence as in Definition* 1.1, $r\in\mathbb{Z}$, $s\in\mathbb{N}$ *and suppose that* $\gcd(a_1+\cdots+a_d-1,s)=1$. *If* $\mathcal{A},\mathcal{B}\subset\{1,\dots,N\}$, *then*

$$\left|\left|\left\{(a,b):\ a\in\mathcal{A},\ b\in\mathcal{B},\ S_G(a+b)\equiv r\ (s)\right\}\right| - \frac{|\mathcal{A}||\mathcal{B}|}{s}\right| \ll N^{\lambda_3}\sqrt{|\mathcal{A}||\mathcal{B}|}$$

with $\lambda_3 < 1$ *uniformly in* N, s, $\mathcal{A}$ *and* $\mathcal{B}$.

Finally, from a result of Hooley [17] we get the following *Barban-Davenport-Halberstam-type theorem* as a consequence of Theorem 1.2 (cf. also Fouvry—Mauduit [10; Proposition 2]).

COROLLARY 1.4. *Let* $(G_j)_{j\geq 0}$ *be given by a finite linear recurrence as in Definition* 1.1 *and suppose that* $\gcd(a_1+\cdots+a_d-1,s)=1$. *If* $r,a\in\mathbb{Z}$, $m,s\in\mathbb{N}$ *and* $M\leq x$, *then*

$$\sum_{m<M}\sum_{a=0}^{m-1}\left(\left|\left\{n\in\mathcal{U}_{r,s}(N):\ n\equiv a\ (m)\right\}\right| - \frac{N}{ms}\right)^2 = \mathcal{O}(Mx) + \mathcal{O}_A\left(x^2(\log 2x)^{-A}\right)$$

holds for all $A>0$.

2. Preparatory lemmas

In this section we collect some auxiliary results, which will be used later on in the paper.

Lemma 2.1. *Let* $G_0, G_1, \dots, G_{d-1}$ *be positive integers and* $G_{n+d} = \sum_{i=1}^{d} a_i G_{n+d-i}$ *for all* $n \geq 0$. *Under the assumption that* $\gcd\{i \geq 0 : \ a_i \neq 0\} = 1$, *the characteristic polynomial* $\chi_G(z)$ *of the recurrence has a unique real root* $\alpha > 1$ *of maximal modulus which satisfies*

$$G_n = C\alpha^n + \mathcal{O}\big(\alpha^{(1-\delta)n}\big)$$

for some $\delta > 0$ *and a real constant* $C > 0$.

Lemma 2.2. *Suppose that* $G = (G_i)_{i \geq 0}$ *satisfies the conditions of Definition* 1.1.

If $0 \leq h < a_i$ *(for* $1 \leq i \leq d$*),* $0 \leq m < G_{n+d-i}$ *and* $n \geq 0$, *then*

$$k = a_1 G_{n+d-1} + \dots + a_{i-1} G_{n+d-i+1} + h G_{n+d-i} + m$$

has the digits

$$\begin{aligned} \varepsilon_l(k) &= \varepsilon_l(m) \qquad (0 \leq l < n+d-i)\,, \\ \varepsilon_{n+d-i}(k) &= h\,, \\ \varepsilon_l(k) &= a_{n+d-l} \qquad (n+d-i < l < n+d)\,. \end{aligned}$$

Suppose that k *has the digital expansion* $k = \varepsilon_L(k) G_L + \dots + \varepsilon_0(k) G_0$.

If $n + d \leq L$, $h < \varepsilon_{n+d}(k)$, *and* $m < G_{n+d}$, *then*

$$k' = \varepsilon_L(k) G_L + \varepsilon_{L-1} G_{L-1} + \dots + \varepsilon_{n+d+1}(k) G_{n+d+k} + h G_{n+d} + m$$

has the digits

$$\begin{aligned} \varepsilon_l(k') &= \varepsilon_l(m) \qquad (0 \leq l < n+d)\,, \\ \varepsilon_{n+d}(k') &= h\,, \\ \varepsilon_l(k') &= \varepsilon_l(k) \qquad (n+d < l \leq L)\,. \end{aligned}$$

The proofs of these results can be found in Drmota—Gajdosik [4].

We will also make use of the following lemma, which is an application of a general method due to Coquet et al. [2].

Lemma 2.3. *Let* $g \colon \mathbb{N}_0 \to \mathbb{C}$ *be a function satisfying* $g(0) = 1$, $|g(k)| \leq 1$ *and*

$$g(k) = \prod_{n=0}^{L(k)} g\big(\varepsilon_n(k) G_n\big)\,, \qquad \text{where} \quad k = \sum_{n=0}^{L(k)} \varepsilon_n(k) G_n\,.$$

Assume further that

$$\left|\frac{1}{G_n}\sum_{k<G_n} g(k)\right| \le \frac{1}{f(G_n)},$$

where $f\colon [1,\infty) \to [0,\infty)$ *is a continuous non-decreasing function satisfying* $f(u) \le u$.

Then

$$\left|\frac{1}{N}\sum_{k<N} g(k)\right| \le \frac{C}{f(\sqrt{N})},$$

where C *is a constant only depending on* $(G_n)_{n\ge 0}$.

For a proof of this lemma see also Grabner—Tichy [13].

In the next section we will need the following easy estimate.

Lemma 2.4. *For* $\varphi_1, \varphi_2, \varphi_3 \in \mathbb{R}$ *define* $\varphi := \max(\|\varphi_1\|, \|\varphi_2\|, \|\varphi_3\|)$. *Then the inequality*

$$|e(\varphi_1) + e(\varphi_2) + e(\varphi_3) + 1| \le 4 - \frac{1}{2}(\pi\varphi)^2$$

holds.

Proof. Because of the estimate $\cos(t) \le 1 - t^2/6$ valid for $t \in [-\pi, \pi]$, we get

$$|e(\varphi_1) + e(\varphi_2) + e(\varphi_3) + 1| \le |3 + e(\varphi)| = \sqrt{10 + 6\cos(2\pi\varphi)} \le \sqrt{16 - (2\pi\varphi)^2}$$
$$\le 4 - \frac{(\pi\varphi)^2}{2}.$$

□

Let G_n satisfy the conditions of Definition 1.1. Then we consider the exponential sums

$$S_n := \sum_{k<G_n} e\Big(\frac{r}{s}S_G(k) + yk\Big) \qquad (n \in \mathbb{N}).$$

We denote by $\mathcal{I}$ the set

$$\mathcal{I} := \{1 \le i \le d:\ a_i \ne 0\} = \{\rho_1, \rho_2, \dots, \rho_{|\mathcal{I}|}\}$$

of the indices of non-vanishing coefficients of the recurrence (1). Note that $\rho_1 = 1$ and $\rho_{|\mathcal{I}|} = d$.

We first set up a recurrence relation for the exponential sums S_n. Keeping track of Lemma 2.2, we derive by direct calculations that

$$S_{n+d} = \sum_{j\in\mathcal{I}} A_{n+d,j} S_{n+d-j} \tag{3}$$

with

$$A_{n,j} = \sum_{m=0}^{a_j-1} e\left(\frac{r}{s}\left(\sum_{q=1}^{j-1} a_q + m\right) + y\left(\sum_{q=1}^{j-1} a_q G_{n-q} + mG_{n-j}\right)\right).$$

3. Construction of recurrences

In this section we will dwell upon certain recurrence relations which will be needed in the proof of Theorem 1.1. The recurrence relation (3) has the disadvantage that its coefficients $A_{n+d,j}$ depend on n. Thus we want to construct a finite recurrence with constant coefficients which generates a sequence T_n. This will be done in a way such that T_n is an upper bound for $|S_n|$. Furthermore, the characteristic polynomial χ_T of this recurrence will be of a shape that allows a nontrivial estimation of the asymptotic behaviour of T_n and, hence, of $|S_n|$. This will finally lead to a proof of Theorem 1.1.

The clue of the proof consists in comparing the characteristic polynomial χ_T with the characteristic polynomial of a recurrence relation for the sequence $(G_n)_{n\geq 0}$. Unfortunately, the characteristic polynomial of the recurrence (1) is not suitable for this comparison. For this matter we have to set up a new recurrence relation for the G_n by iterating (1) in a certain way.

Now we want to set up a recurrence relation for the sequence T_n. For this matter we will need the abbreviation

$$E_{n,j}(m) := e\left(\frac{r}{s}\left(\sum_{q=1}^{j-1} a_q + m\right) + y\left(\sum_{q=1}^{j-1} a_q G_{n-q} + mG_{n-j}\right)\right).$$

In what follows we have to distinguish two cases. Since the calculations corresponding to each of these cases are rather tedious, we devote a subsection to each of them. In both of these subsections we will need some notations. In particular, we write $\mathbf{j}_\ell := (j_1, \ldots, j_\ell)$ and $\mathbf{m}_\ell := (m_1, \ldots, m_\ell)$ as well as

$$(\mathbf{j}_\ell; \mathbf{m}_\ell) := (j_1, \ldots, j_\ell; m_1, \ldots, m_\ell).$$

Furthermore, define the set

$$\mathcal{M}_\ell := \left\{(\mathbf{j}_\ell; \mathbf{m}_\ell) : \ j_k \in \mathcal{I},\ 0 \leq m_k \leq a_{j_k} - 1,\ 1 \leq k \leq \ell\right\}.$$

3.1. The case $a_1 > 1$.

Our first task is to set up a new recurrence relation for the sequence S_n. For this matter let

$$\mathcal{N} := \left\{(\mathbf{j}_2; \mathbf{m}_2) : \ j_1 = j_2 = 1,\ 0 \leq m_1 \leq a_1 - 1,\ m_2 = 0\right\}.$$

We start from the recurrence relation (3). Iterating this relation two times yields

$$\begin{aligned} S_{n+d} &= \sum_{(\mathbf{j}_2;\mathbf{m}_2)\in\mathcal{M}_2} E_{n+d,j_1}(m_1)\, E_{n+d-j_1,j_2}(m_2)\, S_{n+d-j_1-j_2} \\ &= \sum_{(\mathbf{j}_2;\mathbf{m}_2)\in\mathcal{M}_2\setminus\mathcal{N}} E_{n+d,j_1}(m_1)\, E_{n+d-j_1,j_2}(m_2)\, S_{n+d-j_1-j_2} \\ &\quad + \sum_{(\mathbf{j}_2;\mathbf{m}_2)\in\mathcal{N}} E_{n+d,j_1}(m_1)\, E_{n+d-j_1,j_2}(m_2)\, S_{n+d-j_1-j_2}. \end{aligned}$$

In observing that $E_{n,1}(0) = 1$ for all $n \geq 0$, we derive

$$\begin{aligned} S_{n+d} = &\sum_{(\mathbf{j}_2;\mathbf{m}_2)\in\mathcal{M}_2\setminus\mathcal{N}} E_{n+d,j_1}(m_1)\, E_{n+d-j_1,j_2}(m_2)\, S_{n+d-j_1-j_2} \\ &+ \sum_{m=0}^{a_1-1} E_{n+d,1}(m)\, S_{n+d-2}\,. \end{aligned}$$

Iterating S_{n+d-2} in the last line of the previous equation $(d-1)$ times yields

$$\begin{aligned} S_{n+d} = &\sum_{(\mathbf{j}_2;\mathbf{m}_2)\in\mathcal{M}_2\setminus\mathcal{N}} E_{n+d,j_1}(m_1)\, E_{n+d-j_1,j_2}(m_2)\, S_{n+d-j_1-j_2} \\ &+ \sum_{\substack{(\mathbf{j}_{d-1};\mathbf{m}_{d-1})\in\mathcal{M}_{d-1}\\ m\in\{0,\ldots,a_1-1\}}} \left(E_{n+d,1}(m) \prod_{\ell=1}^{d-1} E_{n+d-2-j_1-\cdots-j_{\ell-1},j_\ell}(m_\ell)\right) \times \\ &\times S_{n+d-2-j_1-\cdots-j_{d-1}}\,. \end{aligned} \tag{4}$$

In a next step we want to extract four summands of the sums in (4). For this matter we adopt the following notation. Let $\sum^{(1)}$ be the sum over all $(\mathbf{j}_2;\mathbf{m}_2) \in \mathcal{M}_2 \setminus \mathcal{N}$ such that

$$(\mathbf{j}_2;\mathbf{m}_2) \neq (1,d;0,0)\,.$$

Furthermore, let $\sum^{(2)}$ be the sum over all $(\mathbf{j}_{d-1};\mathbf{m}_{d-1}) \in \mathcal{M}_{d-1}$, $m \in \{0,\ldots$ $\ldots,a_1-1\}$ such that

$$\begin{aligned} (\mathbf{j}_{d-1};\mathbf{m}_{d-1};m) &\neq (\underbrace{1,\ldots,1}_{d-1 \text{ times}};\underbrace{0,\ldots,0}_{d-1 \text{ times}};1)\,, \\ (\mathbf{j}_{d-1};\mathbf{m}_{d-1};m) &\neq (\underbrace{1,\ldots,1}_{d-1 \text{ times}};\underbrace{0,\ldots,0}_{d-2 \text{ times}},1;0)\,, \\ (\mathbf{j}_{d-1};\mathbf{m}_{d-1};m) &\neq (\underbrace{1,\ldots,1}_{d-1 \text{ times}};\underbrace{0,\ldots,0}_{d-1 \text{ times}};0)\,. \end{aligned}$$

With this notation we can rewrite (4) extracting the four summands which are not contained in the sums $\sum'$ and $\sum''$. Keeping in mind that $E_{n,1}(0) = 1$ for all $n \geq 0$, this yields

$$\begin{aligned} S_{n+d} = &\left(E_{n+d-1,d}(0) + E_{n+d,1}(1) + E_{n,1}(1) + 1\right) S_{n-1} \\ &+ \sum\nolimits^{(1)} E_{n+d,j_1}(m_1)\, E_{n+d-j_1,j_2}(m_2)\, S_{n+d-j_1-j_2} \\ &+ \sum\nolimits^{(2)} \left(E_{n+d,1}(m) \prod_{\ell=1}^{d-1} E_{n+d-2-j_1-\cdots-j_{\ell-1},j_\ell}(m_\ell)\right) S_{n+d-2-j_1-\cdots-j_{d-1}}\,. \end{aligned} \tag{5}$$

By the definition of G_n and S_n, substituting 1 for each $E_{n,j}$ in (5) yields the recurrence

$$G_{n+d} = 4G_{n-1} + \sum^{(1)} G_{n+d-j_1-j_2} + \sum^{(2)} G_{n+d-2-j_1-\cdots-j_{d-1}} \tag{6}$$

for the sequence $(G_n)_{n\geq 0}$. Let χ_G be the characteristic polynomial of this recurrence.

Now we will set up the recurrence for the desired sequence T_n. If we take absolute values in (5), we get

$$\begin{aligned}\tilde{T}_{n+d} := \big|E_{n+d-1,d}(0) + E_{n+d,1}(1) + E_{n,1}(1) + 1\big|\,\tilde{T}_{n-1}\\ + \sum^{(1)} \tilde{T}_{n+d-j_1-j_2} + \sum^{(2)} \tilde{T}_{n+d-2-j_1-\cdots-j_{d-1}}\,.\end{aligned} \tag{7}$$

Clearly, $\tilde{T}_n$ fulfills $|S_n| \leq \tilde{T}_n \leq G_n$. The only disadvantage of this sequence is that the coefficient of $\tilde{T}_{n-1}$ in (7) depends on n and y. In order to get rid of these dependencies, we want to estimate the quantity

$$\big|E_{n+d-1,d}(0) + E_{n+d,1}(1) + E_{n,1}(1) + 1\big| \tag{8}$$

uniformly in n and y.

Using the recurrence relation (1) we get

$$\begin{aligned}E_{n+d-1,d}(0) &= e\left(\frac{r}{s}\left(\sum_{q=1}^{d-1} a_q\right) + y\left(\sum_{q=1}^{d-1} a_q G_{n+d-1-q}\right)\right)\\ &= e\left(\frac{r}{s}\left(\sum_{q=1}^{d-1} a_q\right) + y\left(G_{n+d-1} - a_d G_{n-1}\right)\right).\end{aligned}$$

Obviously, (8) is of the form

$$|e(\varphi_1) + e(\varphi_2) + e(\varphi_3) + 1|$$

with

$$\begin{aligned}\varphi_1 &= \frac{r}{s}\sum_{q=1}^{d-1} a_q + y(G_{n+d-1} - a_d G_{n-1})\,,\\ \varphi_2 &= \frac{r}{s} + yG_{n+d-1}\,,\\ \varphi_3 &= \frac{r}{s} + yG_{n-1}\,.\end{aligned}$$

Observing that

$$\varphi_1 - \varphi_2 + a_d\varphi_3 = \frac{r}{s}(a_1 + \cdots + a_d - 1)\,,$$

the conditions $\gcd(a_1 + \cdots + a_d - 1, s) = 1$ and $r \not\equiv 0\,(s)$ ensure that

$$\|\varphi_1 - \varphi_2 + a_d\varphi_3\| \geq \frac{1}{s}\,.$$

Thus there exists an index $j \in \{1,2,3\}$ such that $\|\varphi_j\| \geq 1/\bigl(s(a_d+2)\bigr)$ and, hence, $\varphi = \max_j\{\|\varphi_j\|\} \geq 1/\bigl(s(a_d+2)\bigr)$.

Now Lemma 2.4 yields

$$\bigl|E_{n+d-1,d}(0) + E_{n+d,1}(1) + E_{n,1}(1) + 1\bigr| \leq 4 - \frac{1}{2}\left(\frac{\pi}{s(a_d+2)}\right)^2 .$$

After these considerations we are in the position to define the recursion T_n by

$$\begin{aligned} T_{n+d} = {} & \left(4 - \frac{1}{2}\left(\frac{\pi}{s(a_d+2)}\right)^2\right) T_{n-1} \\ & + \sum\nolimits^{(1)} T_{n+d-j_1-j_2} + \sum\nolimits^{(2)} T_{n+d-2-j_1-\cdots-j_{d-1}}. \end{aligned} \tag{9}$$

Furthermore, let χ_T denote the characteristic polynomial for this recurrence.

3.2. The case $a_1 = 1$.

In this case the choice $m_1 > 0$ used in the set $\mathcal{N}$ above is impossible. Thus we have to iterate in a different way than we did in the previous subsection. Set first

$$\mathcal{N} := \bigl\{(\mathbf{j}_2;\mathbf{m}_2) : \ j_1 \in \mathcal{I}\,,\ \ j_2 = 1\,,\ \ m_1 = m_2 = 0\bigr\}\,.$$

Like above, we begin by a 2-fold iteration, which leads to

$$\begin{aligned} S_{n+d} = {} & \sum_{(\mathbf{j}_2;\mathbf{m}_2)\in\mathcal{M}_2\setminus\mathcal{N}} E_{n+d,j_1}(0)\, E_{n+d-j_1,j_2}(0)\, S_{n+d-j_1-j_2} \\ & + \sum_{j\in\mathcal{I}} E_{n+d,j}(0)\, S_{n+d-1-j} \end{aligned}$$

by the definition of the set $\mathcal{N}$. We denote by M the difference $M := d - \rho_2$, where ρ_2 was the index of the second non-vanishing element in $\{a_1, \ldots, a_d\}$.

The last summand $S_{n+d-1-j}$ above will be iterated M times:

$$\begin{aligned} S_{n+d} = {} & \sum_{(\mathbf{j}_2;\mathbf{m}_2)\in\mathcal{M}_2\setminus\mathcal{N}} E_{n+d,j_1}(0)\, E_{n+d-j_1,j_2}(0)\, S_{n+d-j_1-j_2} \\ & + \sum_{\substack{(\mathbf{j}_M;\mathbf{m}_M)\in\mathcal{M}_M \\ j\in\mathcal{I}}} \left(E_{n+d,j}(0) \prod_{\ell=1}^{M} E_{n+d-1-j-j_1-\cdots-j_{\ell-1},j_\ell}(0)\right) \times \\ & \times S_{n+d-1-j-j_1-\cdots-j_M}\,. \end{aligned} \tag{10}$$

Now we extract four summands of the sums in (10). Again we adopt the following notation. Let $\sum^{(1)}$ be the sum over all $(\mathbf{j}_2;\mathbf{m}_2) \in \mathcal{M}_2 \setminus \mathcal{N}$ such that

$$(\mathbf{j}_2;\mathbf{m}_2) \neq (1,d;0,0).$$

Furthermore, let $\sum^{(2)}$ be the sum over all $(\mathbf{j}_M;\mathbf{m}_M) \in \mathcal{M}_M$, $j \in \mathcal{I}$ such that

$$\begin{aligned}
(\mathbf{j}_M;\mathbf{m}_M;j) &\neq (\underbrace{1,\dots,1}_{M\text{ times}};\underbrace{0,\dots,0}_{M\text{ times}};\rho_2),\\
(\mathbf{j}_M;\mathbf{m}_M;j) &\neq (\underbrace{1,\dots,1}_{M-1\text{ times}},\rho_2;\underbrace{0,\dots,0}_{M\text{ times}};\rho_2),\\
(\mathbf{j}_M;\mathbf{m}_M;j) &\neq (\underbrace{1,\dots,1}_{M\text{ times}};\underbrace{0,\dots,0}_{M\text{ times}};1).
\end{aligned}$$

Remark. We note here that the case $d = \rho_2$ is excluded by the above way of iteration. Nevertheless, easy calculations show that by extracting $(1,d;0,0)$, $(d,1;0,0)$, $(d,d;0,0)$ and $(1,1;0,0)$ from the set $\mathcal{M}_2 \setminus \mathcal{N}$ one is able to group the same four coefficients at S_{n-d} like in equation (12) below for the case $d > \rho_2$.

With this notation we can rewrite (10) as

$$\begin{aligned}
S_{n+d} = {} & E_{n+d-1,d}(0)\, S_{n-1} + E_{n+d,\rho_2}(0)\, S_{n-1}\\
& + E_{n+d,\rho_2}(0)\cdot E_{n,\rho_2}(0)\, S_{n-\rho_2} + S_{n+\rho_2-2}\\
& + \sum\nolimits^{(1)} E_{n+d,j_1}(0)\, E_{n+d-j_1,j_2}(0)\, S_{n+d-j_1-j_2}\\
& + \sum\nolimits^{(2)} \Big(E_{n+d,j}(0) \prod_{\ell=1}^{M} E_{n+d-1-j-j_1-\dots-j_{\ell-1},j_\ell}(0)\Big) S_{n+d-1-j-j_1-\dots-j_M}.
\end{aligned} \tag{11}$$

Here we notice that $a_1 = 1$ implies $A_{n+d,1} = E_{n+d,1}(0) = 1$ for the recurrence (3). Thus, in general, we deal with

$$S_{n+d} = S_{n+d-1} + \sum_{j\in\mathcal{I}\setminus\{1\}} E_{n+d,j}(0)\, S_{n+d-j}$$

which allows us to group the coefficients extracted in (11) at the common element $S_{n-\rho_2}$. Thus,

$$
\begin{aligned}
S_{n+d} = {} & \big(E_{n+d-1,d}(0) + E_{n+d,\rho_2}(0) + E_{n+d,\rho_2}(0)E_{n,\rho_2}(0) + 1\big) S_{n-\rho_2} \\
& + \sum\nolimits^{(1)} E_{n+d,j_1}(0)\, E_{n+d-j_1,j_2}(0)\, S_{n+d-j_1-j_2} \\
& + \sum\nolimits^{(2)} \left(E_{n+d,j}(0) \prod_{\ell=1}^{M} E_{n+d-1-j-j_1-\cdots-j_{\ell-1},j_\ell}(0) \right) S_{n+d-1-j-j_1-\cdots-j_M} \\
& + \big(E_{n+d-1,d}(0) + E_{n+d,\rho_2}(0)\big) \times \\
& \times \sum\nolimits^{(3)} \prod_{\ell=1}^{\rho_2-1} E_{n-1-j_1-\cdots-j_{\ell-1},j_\ell}(0)\, S_{n-1-j_1-\cdots-j_{\rho_2-1}} \\
& + \sum\nolimits^{(4)} \prod_{\ell=1}^{2\rho_2-2} E_{n+\rho_2-2-j_1-\cdots-j_{\ell-1},j_\ell}(0)\, S_{n+\rho_2-2-j_1-\cdots-j_{2\rho_2-2}}\,,
\end{aligned}
\tag{12}
$$

where we introduced the new notations

$$
\sum\nolimits^{(3)} := \sum_{\substack{(\mathbf{j}_{\rho_2-1};\mathbf{m}_{\rho_2-1})\in\mathcal{M}_{\rho_2-1}\\ (j_1,\dots,j_{\rho_2-1})\neq(1,\dots,1)}} \qquad \text{and} \qquad \sum\nolimits^{(4)} := \sum_{\substack{(\mathbf{j}_{2\rho_2-2};\mathbf{m}_{2\rho_2-2})\in\mathcal{M}_{2\rho_2-2}\\ (j_1,\dots,j_{2\rho_2-2})\neq(1,\dots,1)}} .
$$

Substituting 1 for each $E_{n,j}$ in (12) yields the recurrence

$$
\begin{aligned}
G_{n+d} = {} & 4G_{n-\rho_2} + \sum\nolimits^{(1)} G_{n+d-j_1-j_2} + \sum\nolimits^{(2)} G_{n+d-1-j_1-\cdots-j_{d-2}} \\
& + 2\sum\nolimits^{(3)} G_{n-1-j_1-\cdots-j_{\rho_2-1}} + \sum\nolimits^{(4)} G_{n+\rho_2-2-j_1-\cdots-j_{2\rho_2-2}}
\end{aligned}
\tag{13}
$$

for the sequence $(G_n)_{n\geq 0}$. Let χ_G be the characteristic polynomial of this recurrence.

Now we will set up the recurrence for the desired sequence T_n. If we take absolute values in (12), we get

$$
\begin{aligned}
\tilde{T}_{n+d} := {} & \big|E_{n+d-1,d}(0) + E_{n+d,2}(0) + E_{n+d,2}(0)\cdot E_{n,2}(0) + 1\big|\, \tilde{T}_{n-\rho_2} \\
& + \sum\nolimits^{(1)} \tilde{T}_{n+d-j_1-j_2} + \sum\nolimits^{(2)} \tilde{T}_{n+d-1-j-j_1-\cdots-j_M} \\
& + 2\sum\nolimits^{(3)} \tilde{T}_{n-1-j_1-\cdots-j_{\rho_2-1}} + \sum\nolimits^{(4)} \tilde{T}_{n+\rho_2-2-j_1-\cdots-j_{2\rho_2-2}}\,.
\end{aligned}
\tag{14}
$$

Clearly, $\tilde{T}_n$ fulfills $|S_n| \leq \tilde{T}_n \leq G_n$. To get rid of the dependence on n and y of the coefficient of $\tilde{T}_{n-\rho_2}$ in (14), we observe that it has again the form

$$
|e(\varphi_1) + e(\varphi_2) + e(\varphi_3) + 1|
$$

with

$$\begin{aligned}\varphi_1 &= \frac{r}{s}\sum_{q=1}^{d-1} a_q + y(G_{n+d-1} - G_{n-1}),\\ \varphi_2 &= \frac{r}{s} + yG_{n+d-1},\\ \varphi_3 &= 2\frac{r}{s} + y(G_{n+d-1} + G_{n-1}).\end{aligned}$$

Thus, in this case we end up with

$$\varphi_1 + \varphi_3 - 2\varphi_2 = \frac{r}{s}(a_1 + \cdots + a_{d-1}) = \frac{r}{s}(a_1 + \cdots + a_d - 1),$$

and by arguing along the same lines as in the case $a_1 > 1$, the assumptions $\gcd(a_1 + \cdots + a_d - 1, s) = 1$ and $r \not\equiv 0\ (s)$ guarantee that $\varphi = \max_j\{\|\varphi_j\|\} \geq 1/(4s)$ and therefore

$$\left|E_{n+d-1,d}(0) + E_{n+d,\rho_2}(0) + E_{n+d,\rho_2}(0)\cdot E_{n,\rho_2}(0) + 1\right| \leq 4 - \frac{1}{2}\left(\frac{\pi}{4s}\right)^2.$$

In an analogous way as before we now define T_n via

$$\begin{aligned} T_{n+d} = &\left(4 - \frac{1}{2}\left(\frac{\pi}{4s}\right)^2\right)T_{n-\rho_2}\\ &+ \sum\nolimits^{(1)} T_{n+d-j_1-j_2} + \sum\nolimits^{(2)} T_{n+d-1-j-j_1-\cdots-j_M}\\ &+ 2\sum\nolimits^{(3)} T_{n-1-j_1-\cdots-j_{\rho_2-1}} + \sum\nolimits^{(4)} T_{n+\rho_2-2-j_1-\cdots-j_{2\rho_2-2}}\end{aligned} \tag{15}$$

and denote by χ_T the characteristic polynomial of this recurrence.

Summing up what we proved until so far, we get the following result.

PROPOSITION 3.1. *Let $(G_n)_{n\geq 0}$, r and s be defined as in Theorem* 1.1 *and let $(T_n)_{n\geq 0}$ be either given by* (9) *if $a_1 > 1$, or by* (15) *in the case $a_1 = 1$. Then there exist recurrences*

$$G_{n+D} = \sum_{j=1}^{D} b_j G_{n+D-j} \qquad \text{and} \qquad T_{n+D} = \sum_{j=1}^{D} b'_j T_{n+D-j}$$

for the sequences G_n and T_n and an index $j_0 \in \{1,\dots,D\}$ such that

(i) $b'_j = b_j$ *for* $j \neq j_0$,

(ii) $b'_{j_0} = b_{j_0} - \frac{1}{2}\left(\frac{\pi}{s(\Delta+2)}\right)^2$,

(iii) $|S_n| < T_n$ *for all* $y \in \mathbb{R}$,

where $\Delta := \max\{a_d, 2\}$.

P r o o f . For $a_1 > 1$ the result follows by comparing (6) and (9), while comparison of (13) and (15) yields the result for $a_1 = 1$. □

Remark. As mentioned in the introduction, Theorem 1.1 was proved for the special case of q-ary expansions in G e l' f o n d [12; p. 260f]. If $a_1 \geq 2$ it is possible to obtain a result which is equivalent to Proposition 3.1 by imitating G e l' f o n d's proof. In this case one has to deal with $(d+1)$-fold products of the shape

$$|A_{n+d,1}| \cdot |A_{n+d-1,1}| \cdots |A_{n,1}| = \left| \prod_{l=0}^{d} \frac{\sin\left(\pi a_1 \left(\frac{r}{s} + y\, G_{n+d-l}\right)\right)}{\sin\left(\pi \left(\frac{r}{s} + y\, G_{n+d-l}\right)\right)} \right| .$$

If $a_1 = 1$ we are forced to use the alternative method proposed in the present section.

4. Proofs of the results

4.1. Proof of Theorem 1.1.

In order to establish an estimate for the asymptotics of S_n we want to apply Lemma 2.1. Unfortunately the condition on the gcd of the indices can be violated by the iteration process introduced in the previous section.

Remark. An example for the described situation is the recursion $G_{n+3} = 2G_{n+2} + G_n$. After manipulating the recurrence to allow the nontrivial estimate, we get $G_{n+8} = 2G_{n+6} + 12G_{n+4} + 9G_{n+2} + 2G_n$ which produces the same sequence $(G_n)_{n\geq 0}$, but has $2 = \gcd(i \mid b_i \neq 0)$.

In what follows, we set $g = \gcd(i \mid b_i \neq 0) = \gcd(i \mid b'_i \neq 0)$. Then, with $D' := D/g$, one can write the recurrences defined in Proposition 3.1 in the form

$$G_{n+gD'} = b_g G_{n+g(D'-1)} + \cdots + b_{gD'} G_n$$

and

$$T_{n+gD'} = b_g T_{n+g(D'-1)} + \cdots + b_{gD'} T_n ,$$

where the initial conditions on $G_0, \ldots, G_{D-1}$, respectively $T_0, \ldots, T_{D-1}$, are induced by the original recurrence $(G_n)_{n\geq 0}$.

We are subdividing these recurrences into residue classes modulo g. Let therefore $0 \leq h \leq g-1$ and define

$$G^{(h)}_{n+D'} = \sum_{j=1}^{D'} b_{gj} G^{(h)}_{n+D'-j} \qquad \text{where} \quad G_0^{(h)} = G_h, \dots, G^{(h)}_{D'-1} = G_{h+(D'-1)g}$$

and

$$T^{(h)}_{n+D'} = \sum_{j=1}^{D'} b'_{gj} T^{(h)}_{n+D'-j} \qquad \text{where} \quad T_0^{(h)} = T_h, \dots, T^{(h)}_{D'-1} = T_{h+(D'-1)g} \,.$$

Thus for every $k \geq 0$

$$G_{gk+h} = G_k^{(h)}\,, \quad T_{gk+h} = T_k^{(h)} \qquad (0 \leq h \leq g-1) \tag{16}$$

holds.

Since $\gcd(j \mid b_{gj} \neq 0) = 1$, we can apply Lemma 2.1 to the recurrences for the $G_k^{(h)}$ $(0 \leq h \leq g-1)$. This guarantees the existence of a unique real dominating root $\tilde{\alpha} > 1$ such that by (16)

$$G_{kg+h} = G_k^{(h)} = C_h \tilde{\alpha}^k + \mathcal{O}\big(\tilde{\alpha}^{(1-\delta_h)k}\big) \qquad (0 \leq h \leq g-1)$$

with $\delta_h > 0$. Thus, for $n \equiv h \ (g)$

$$G_n = C'_h \tilde{\alpha}^{n/g} + \mathcal{O}\big(\tilde{\alpha}^{(1-\delta_h)n/g}\big)\,.$$

On the other hand the original recursion for $(G_n)_{n\geq 0}$ obeys the conditions of Lemma 2.1 which leads to $G_n = C\alpha^n + \mathcal{O}\big(\alpha^{(1-\delta)n}\big)$. Thus we have $\alpha = +\sqrt[g]{\tilde{\alpha}}$.

The same considerations give

$$T_n = c'_h \tilde{\beta}_h^{n/g} + \mathcal{O}\big(\tilde{\beta}^{(1-\delta_h)n/g}\big) \qquad (0 \leq h \leq g-1)$$

and we define $\beta := +\sqrt[g]{\tilde{\beta}}$.

Denote by $\tilde{\chi}_G(z)$ and $\tilde{\chi}_T(z)$ the accompanying characteristic polynomials of $G_n^{(h)}$ and $T_n^{(h)}$, respectively.

By Proposition 3.1 we see that

$$\tilde{\chi}_T(\alpha^g) = \frac{1}{2}\left(\frac{\pi}{s(\Delta+2)}\right)^2 \alpha^{j_0}\,.$$

Setting

$$F := \max_{x\in[0,\alpha^g]} (\tilde{\chi}_T)'(x)$$

we arrive at

$$\tilde{\beta} \leq \alpha^g - \frac{\pi^2 \alpha^{j_0}}{2F\big(s(\Delta+2)\big)^2}$$

and thus

$$\beta \le \alpha - \frac{\pi^2 \alpha^{j_0}}{2gF\big(s(\Delta+2)\big)^2}\,.$$

This proves that $|S_n| \le T_n \le \alpha^{\tilde{\lambda} n}$ with

$$\tilde{\lambda} := \frac{\log\left(\alpha - \frac{\pi^2\alpha^{j_0}}{2j_1(s(\Delta+2)^2)}\right)}{\log \alpha}$$

and $j_1 := gF$. Theorem 1.1 now follows by an application of Lemma 2.3.

4.2. Proofs of the distribution results.

Proof of Theorem 1.2. For convenience, we set

$$V_{r,s,a,m}(N) := \left|\{n \in \mathcal{U}_{r,s}(N) :\ n \equiv a\ (m)\}\right|.$$

Following the lines of the proof in Gel'fond [12] we derive

$$\begin{aligned}
V_{r,s,a,m}(N) &= \frac{1}{ms}\sum_{n<N}\sum_{k=0}^{m-1}\sum_{\ell=0}^{s-1} e\left(\frac{S_G(n)-r}{s}\ell + \frac{n-a}{m}k\right)\\
&= \frac{1}{ms}\sum_{k=0}^{m-1}\sum_{\ell=0}^{s-1} e\left(-\frac{r\ell}{s} - \frac{ak}{m}\right)\sum_{n<N} e\left(\frac{S_G(n)\ell}{s} + \frac{nk}{m}\right)\\
&= \frac{N}{ms} + \frac{1}{ms}\sum_{k=1}^{m-1}\sum_{n<N} e\left(\frac{n-a}{m}k\right)\\
&\qquad + \frac{1}{ms}\sum_{k=0}^{m-1}\sum_{\ell=1}^{s-1} e\left(-\frac{r\ell}{s} - \frac{ak}{m}\right)\sum_{n<N} e\left(\frac{S_G(n)\ell}{s} + \frac{nk}{m}\right).
\end{aligned} \tag{17}$$

By a standard argument we know

$$\left|\sum_{k=1}^{m-1}\sum_{n<N} e\left(\frac{n-a}{m}k\right)\right| < 2m$$

and, by Theorem 1.1,

$$\left|\sum_{n<N} e\left(\frac{S_G(n)\ell}{s} + \frac{nk}{m}\right)\right| \ll N^{\lambda}.$$

Taking absolute values in (17) together with the above estimates yields the result. □

Proof of Corollary 1.1. This can be proved in the same way as Gel'fond [12; Théorème II] with our Theorem 1.2 playing the part of Théorème I of Gel'fond's paper. □

Proof of Corollary 1.2. Here we can follow the same lines as in the proof of Mauduit—Sárközy [21; Theorem 2]. □

Proof of Corollary 1.3. The proof of this statement is the same as the proof of Mauduit—Sárközy [21; Theorem 1] with Theorem 1.2 playing the part of the original result of Gel'fond. Note that in the proof of this statement the uniformity in y of the bound in Theorem 1.1 plays an important rôle (cf. [21; p. 30]). □

Proof of Corollary 1.4. Theorem 1.2 ensures that

$$\{n \in \mathcal{U}_{r,s}(N) : \ n \equiv a \ (m)\}$$

fulfills Criterion U of Hooley [17]. Thus the result follows from [17; Theorem 1]. □

Acknowledgements

The authors are indebted to Prof. Peter J. Grabner and Prof. Robert F. Tichy for their encouragement as well as for valuable discussions. We also want to thank Prof. Michael Drmota and Wolfgang Steiner for useful comments.

REFERENCES

[1] BRAUER, A.: *On algebraic equations with all but one root in the interior of the unit circle*, Math. Nachr. **4** (1951), 250–257.

[2] COQUET, J.—RHIN, G.—TOFFIN, P.: *Représentation des entiers naturels et indépendence statistique* 2, Ann. Inst. Fourier (Grenoble) **31** (1981), 1–15.

[3] DELANGE, H.: *Sur la fonction sommatoire de la fonction 'somme des chiffres'*, Enseign. Math. (2) **21** (1975), 31–47.

[4] DRMOTA, M.—GAJDOSIK, J.: *The distribution of the sum-of-digits function*, J. Théor. Nombres Bordeaux **10** (1998), 17–32.

[5] DRMOTA, M.—GAJDOSIK, J.: *The parity of the sum-of-digits-function of generalized zeckendorf representations*, Fibonacci Quart **36** (1998), 3–19.

[6] DUMONT, J. M.—SIDOROV, N.—THOMAS, A.: *Number of representations related to a linear recurrent basis*, Acta Arith. **88** (1999), 371–396.

[7] DUMONT, J. M.—THOMAS, A.: *Digital sum moments and substitutions*, Acta Arith. **64** (1993), 205–225.

[8] DUMONT, J. M.—THOMAS, A.: *Gaussian asymptotic properties of the sum-of-digits function*, J. Number Theory **62** (1997), 19–38.

[9] FINE, N. J.: *The distribution of the sum of digits* (mod p), Bull. Amer. Math. Soc. **71** (1965), 651–652.

[10] FOUVRY, E.—MAUDUIT, C.: *Sommes des chiffres et nombres presque premiers*, Math. Ann. **305** (1996), 571–599.

[11] FROUGNY, C.: *Representations of numbers and finite automata*, Math. Systems Theory **25** (1992), 37–60.

[12] GELFOND, A. O.: *Sur les nombres qui ont des propriétés additives et multiplicatives données*, Acta Arith. **13** (1968), 259–265.

[13] GRABNER, P.—TICHY, R.: *Contributions to digit expansions with respect to linear recurrences*, J. Number Theory **36** (1990), 160–169.

[14] GRABNER, P. J.: *Ziffernentwicklungen bezüglich linearer Rekursionen*, Anz. Österreich. Akad. Wiss Math.-Natur. Kl. **1990** (1990), 1–21.

[15] GRABNER, P. J.—TICHY, R. F.: *α-expansions, linear recurrences, and the sum-of-digits function*, Manuscripta Math. **70** (1991), 311–324.

[16] HOIT, A.: *The distribution of generalized sum-of-digits functions in residue classes*, J. Number Theory **79** (1999), 194–216.

[17] HOOLEY, C.: *On the Barban-Davenport-Halberstam theorem III*, J. London Math. Soc. **10** (1975), 249–256.

[18] KIM, D.-H.: *On the joint distribution of q-additive functions in residue classes*, J. Number Theory **74** (1999), 307–336.

[19] KIRSCHENHOFER, P.—TICHY, R.: *On the distribution of digits in cantor representations of integers*, J. Number Theory **18** (1984), 121–134.

[20] LARCHER, G.: *Bemerkung zur Diskrepanz einer Klasse von Folgen*, Anz. Österreich. Akad. Wiss Math.-Natur. Kl. **1991** (1991), 1–4.

[21] MAUDUIT, C.—SÁRKÖZY, A.: *On the arithmetic structure of sets characterized by sum of digits properties*, J. Number Theory **61** (1996), 25–38.

[22] MAUDUIT, C.—SÁRKÖZY, A.: *On the arithmetic structure of the integers whose sum of digits is fixed*, Acta Arith. **81** (1997), 145–173.

[23] PARRY, W.: *On the β-expansion of real numbers*, Acta Math. Acad. Sci. Hung. **11** (1960), 401–416.

[24] PETHŐ, A.—TICHY, R.: *On digit expansions with respect to linear recurrences*, J. Number Theory **33** (1989), 243–256.

[25] STEINER, W.: *Digital expansions and the distribution of related functions*. Master's thesis, Technical University of Vienna, 2000.

[26] THUSWALDNER, J. M.: *The sum of digits function in number fields: Distribution in residue classes*, J. Number Theory **74** (1999), 111–125.

Received August 20, 2002

* *Institut für Mathematik*
Technische Universität Graz
Steyrergasse 30/III
A–8010 GRAZ
AUSTRIA
E-mail: Mario.Lamberger@tugraz.at

** *Institut für Mathematik*
und Angewandte Geometrie
Abteilung für Mathematik und Statistik
Montanuniversität Leoben
Franz-Josef-Strasse 18
A–8700 Leoben
AUSTRIA
E-mail: Joerg.Thuswaldner@unileoben.ac.at