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Mathematica Slovaca, Vol. 54 (2004), No. 3, 281--289

Persistent URL: http://dml.cz/dmlcz/133337

Math. Slovaca, **54** (2004), No. 3, 281–289

Mathematica Slovaca

STATISTICALLY σ-MULTIPLICATIVE MATRICES AND SOME INEQUALITIES

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(*Communicated by Ľubica Holá*)

ABSTRACT. In this paper we define and characterize the statistically σ-multiplicative matrices using the concepts of statistical convergence and invariant means. We further use these matrices to establish some inequalities involving sublinear functionals.

1. Invariant mean

Let ℓ_∞ and c denote the Banach spaces of bounded and convergent sequences $\mathbf{x} = (x_k)_{k=1}^{\infty}$ respectively. Let σ be an injection of the set of positive integers $\mathbb{N}$ into itself having no finite orbits, and T be the operator defined on ℓ_∞ by $T\big((x_n)_{n=1}^{\infty}\big) = \big(x_{\sigma(n)}\big)_{n=1}^{\infty}$.

A positive linear functional ϕ, with $\|\phi\| = 1$, is called a *σ-mean* or an *invariant mean* if $\phi(\mathbf{x}) = \phi(T\mathbf{x})$ for all $\mathbf{x} \in \ell_\infty$.

A sequence $\mathbf{x}$ is said to be *σ-convergent*, denoted by $\mathbf{x} \in V_\sigma$, if $\phi(\mathbf{x})$ takes the same value, called σ-$\lim \mathbf{x}$, for all σ-means ϕ. We have (see Schaefer [16])

$$V_\sigma := \Big\{\mathbf{x} \in \ell_\infty : \lim_{p\to\infty} t_{pn}(\mathbf{x}) = L \text{ uniformly in } n\,,\ L = \sigma\text{-}\lim \mathbf{x}\Big\}\,,$$

where for $p \ge 0$, $n > 0$

$$t_{pn}(\mathbf{x}) = \frac{x_n + x_{\sigma(n)} + \cdots + x_{\sigma^p(n)}}{p+1}\,, \qquad \text{and} \qquad t_{-1,n} = 0\,. \tag{1}$$

Throughout this paper we assume that $\sigma^j(n) \ne n$ for all $n \ge 0$, $j \ge 1$, where $\sigma^p(n)$ denotes the pth iterate of σ at n. In particular, if σ is the translation, a

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2000 Mathematics Subject Classification: Primary 40F05, 40G99, 40H05.
Keywords: Knopp core, Banach core, σ-core, invariant mean, statistical convergence, multiplicative matrix.

σ-mean is often called a *Banach limit* and V_σ reduces to f, the set of *almost-convergent sequences* (see Lorentz [10]).

An infinite matrix $\mathbf{A} = (a_{nk})_{n,k=1}^{\infty}$ is said to be *σ-regular* if $\mathbf{Ax} \in V_\sigma$ for all $\mathbf{x} \in c$ and $\sigma\text{-}\lim \mathbf{Ax} = \lim \mathbf{x}$. The matrix $\mathbf{A}$ is σ-regular if and only if (see Schaefer [16])

(1.1) $\|\mathbf{A}\| = \sup\limits_{n\in\mathbb{N}} \sum\limits_{k=1}^{\infty} |a_{nk}| < \infty$,

(1.2) $\mathbf{a}_{(k)} = (a_{nk})_{n=1}^{\infty} \in V_\sigma$ with σ-limit zero for each $k \in \mathbb{N}$,

and

(1.3) $\mathbf{a} = \left(\sum\limits_{k=1}^{\infty} a_{nk}\right)_{n=1}^{\infty} \in V_\sigma$ with σ-limit 1.

A matrix $\mathbf{A}$ is called *σ-coercive* (see [16]) if $\mathbf{Ax} \in V_\sigma$ for all $\mathbf{x} \in \ell_\infty$. The matrix $\mathbf{A}$ is σ-coercive if and only if (1.1), (1.4) and (1.5) hold, where

(1.4) $\mathbf{a}_{(k)} \in V_\sigma$ for each $k \in \mathbb{N}$,

(1.5) $\lim\limits_{p\to\infty} \sum\limits_{k=1}^{\infty} |t(n,k,p) - u_k| = 0$ uniformly in n where $u_k = \sigma\text{-}\lim \mathbf{a}_{(k)}$, and

$$t(n,k,p) = \frac{1}{p+1} \sum_{j=0}^{p} a_{\sigma^j(n),k}\,.$$

A matrix $\mathbf{A}$ is said to be *σ-multiplicative* if $\mathbf{Ax} \in V_\sigma$ for all $\mathbf{x} \in c$ and $\sigma\text{-}\lim \mathbf{Ax} = s \lim \mathbf{x}$, where s is any non-negative real number. We denote this by $\mathbf{A} \in (c, V_\sigma)_s$. The matrix $\mathbf{A}$ is σ-multiplicative if and only if (see [1]) (1.1), (1.2) and (1.6) hold, where

(1.6) $\mathbf{a} = \left(\sum\limits_{k=1}^{\infty} a_{nk}\right)_{n=1}^{\infty} \in V_\sigma$ with σ-limit s.

A sublinear functional Q *generates σ-means* if ϕ is a continuous linear functional on ℓ_∞ and $\phi < Q$ implies ϕ is a σ-mean. We say that Q *dominates σ-means* if every σ-mean ϕ is less than Q. If a sublinear functional Q both generates and dominates σ-means, then we define the *σ-core* of $\mathbf{x}$ as $[-Q(-\mathbf{x}), Q(\mathbf{x})]$ (see [11] and [13]).

Let $Q\colon \ell_\infty \to \mathbb{R}$ be defined by

$$Q(\mathbf{x}) = \limsup_{p\to\infty} \sup_{n\in\mathbb{N}} t_{pn}(\mathbf{x})\,, \tag{2}$$

where $t_{pn}(\mathbf{x})$ is defined by (1). Then Q generates and dominates σ-means.

If σ is a translation, then σ-core is reduced to the Banach core (or B-core) (see Orhan [14]). Recall that the Knopp core (or K-core) for real $\mathbf{x}$ is the closed interval $[\ell(\mathbf{x}), L(\mathbf{x})]$, where

$$\ell(\mathbf{x}) = \liminf \mathbf{x} \qquad \text{and} \qquad L(\mathbf{x}) = \limsup \mathbf{x}\,.$$

It can be noted that a σ-mean extends the limit functional on c in the sense that $\phi(\mathbf{x}) = \lim \mathbf{x}$ for all $\mathbf{x} \in c$ if and only if σ has no finite orbits, that is $\sigma^j(n) \neq n$ for all $n \geq 0$, $j \geq 1$ (see Mursaleen [12]). Consequently $c \subset V_\sigma$, and it follows that $\sigma\text{-}\operatorname{core}\{\mathbf{x}\} \subseteq \text{K-}\operatorname{core}\{\mathbf{x}\}$.

2. Statistical core

The notion of statistical convergence was first introduced by Fast [5] and further studied by Šalát [15], Fridy [6], Connor [2], Kolk [9], Fridy and Orhan [7], [8] and many others.

Let $\mathbb{N}$ be the set of natural numbers and $E \subseteq \mathbb{N}$. Then the *natural density* of E is denoted by

$$\delta(E) := \lim_{n\to\infty} n^{-1}|\{k \leq n : \ k \in E\}|,$$

where the vertical bars denote the cardinality of the enclosed set.

The sequence $\mathbf{x}$ is said to be *statistically convergent* to the number L, if for every $\varepsilon > 0$, the set $\{k : \ |x_k - L| \geq \varepsilon\}$ has natural density zero, and we write $L = \operatorname{st-lim} \mathbf{x}$. By st we will denote the set of all statistically convergent sequences.

For a real number sequence $\mathbf{x}$, let

$$B_{\mathbf{x}} := \{b \in \mathbb{R} : \ \delta(\{k : x_k > b\}) \neq 0\}$$

and

$$A_{\mathbf{x}} := \{a \in \mathbb{R} : \ \delta(\{k : \ x_k < a\}) \neq 0\}.$$

Then

$$\operatorname{st-lim\,sup} \mathbf{x} := \begin{cases} \sup B_{\mathbf{x}} & \text{if } B_{\mathbf{x}} \neq \emptyset, \\ -\infty & \text{if } B_{\mathbf{x}} = \emptyset. \end{cases}$$

$$\operatorname{st-lim\,inf} \mathbf{x} := \begin{cases} \inf A_{\mathbf{x}} & \text{if } A_{\mathbf{x}} \neq \emptyset, \\ +\infty & \text{if } A_{\mathbf{x}} = \emptyset. \end{cases}$$

The real number sequence $\mathbf{x}$ is said to be *statistically bounded* if there is a constant M such that

$$\delta\Big(\{k : \ |x_k| > M\}\Big) = 0.$$

If $\mathbf{x}$ is a statistically bounded sequence, then the *statistical core* of $\mathbf{x}$ is the closed interval $[\operatorname{st-lim\,inf} \mathbf{x}, \operatorname{st-lim\,sup} \mathbf{x}]$. It is noted that

$$\liminf \mathbf{x} \leq \operatorname{st-lim\,inf} \mathbf{x} \leq \operatorname{st-lim\,sup} \mathbf{x} \leq \limsup \mathbf{x}$$

and so

$$\operatorname{st-core}\{\mathbf{x}\} \subseteq \text{K-}\operatorname{core}\{\mathbf{x}\}.$$

For an arbitrary index set $E \subseteq \mathbb{N}$ the sequence $\mathbf{x}^{[E]} = (y_n)_{n=1}^{\infty}$, where

$$y_n = \begin{cases} x_n, & n \in E, \\ 0, & \text{otherwise} \end{cases}$$

is called the *E-section* of $\mathbf{x}$. By $\mathbf{A}^{[E]}$ we denote the *E-column section* of the matrix $\mathbf{A} = (a_{nk})_{n,k=1}^{\infty}$, i.e. $\mathbf{A}^{[E]} = (d_{nk})_{n,k=1}^{\infty}$, where

$$d_{nk} = \begin{cases} a_{nk} & \text{if } k \in E, \\ 0 & \text{otherwise}. \end{cases}$$

3. Statistically σ-multiplicative matrices

DEFINITION 3.1. Let $s > 0$ and $\mathbf{A} = (a_{nk})_{n,k=1}^{\infty}$ be an infinite matrix. $\mathbf{A}$ is said to be *statistically σ-multiplicative* if $\mathbf{A}\mathbf{x} \in V_\sigma$ for $\mathbf{x} \in \mathrm{st} \cap \ell_\infty$ with $\sigma\text{-}\lim \mathbf{A}\mathbf{x} = s(\mathrm{st}\text{-}\lim \mathbf{x})$. We denote the class of such matrices by $(\mathrm{st} \cap \ell_\infty, V_\sigma)_s$. For $\sigma(n) = n + 1$, this class is reduced to $(\mathrm{st} \cap \ell_\infty, f)_s$ of *statistically almost multiplicative matrices.*

For typographical convenience, we write $t(n,k,p)$ for $\frac{1}{p+1} \sum_{j=0}^{p} a_{\sigma^j(n),k}$.

The following theorem characterizes the class $(\mathrm{st} \cap \ell_\infty, V_\sigma)_s$.

THEOREM 3.1. $\mathbf{A} \in (\mathrm{st} \cap \ell_\infty, V_\sigma)_s$ *if and only if*

(i) $\mathbf{A} \in (c, V_\sigma)_s$,

and

(ii) $\lim_{p\to\infty} \sum_{k\in E} |t(n,k,p)| = 0$ *uniformly in n for every $E \subseteq \mathbb{N}$ with $\delta(E) = 0$.*

Proof.

Necessity. Let $\mathbf{A} \in (\mathrm{st} \cap \ell_\infty, V_\sigma)_s$ and

$$s(\mathrm{st}\text{-}\lim \mathbf{x}) = \sigma\text{-}\lim \mathbf{A}\mathbf{x} = \ell,$$

say. Since $c \subset \mathrm{st}$, we have $\mathbf{A} \in (c, V_\sigma)_s$, i.e. (i) holds.

Let $E \subseteq \mathbb{N}$ with $\delta(E) = 0$ and let $\mathbf{x} \in \ell_\infty$. Then the E-section $\mathbf{y}$ of $\mathbf{x}$ converges statistically to zero, and $\mathbf{y} \in \ell_\infty$. Hence $\mathbf{y} \in \mathrm{st} \cap \ell_\infty$ and so $\mathbf{A}\mathbf{y} \in V_\sigma$ with $s(\mathrm{st}\text{-}\lim \mathbf{y}) = 0 = \sigma\text{-}\lim \mathbf{A}\mathbf{y}$. Also

$$\mathbf{A}_n^{[E]}(\mathbf{x}) = \mathbf{A}_n(\mathbf{y}), \qquad n = 1, 2, \dots,$$

which implies that $\mathbf{A}^{[E]}(\mathbf{x}) = \left(\mathbf{A}_n^{[E]}(\mathbf{x})\right)_{n=1}^{\infty} \in V_\sigma$ and $\sigma\text{-}\lim \mathbf{A}^{[E]}(\mathbf{x}) = 0$. Thus $\mathbf{A}^{[E]} \in (\ell_\infty, V_\sigma)$ for every index set E with $\delta(E) = 0$ and so, by condition (1.5) with $u_k = 0$ (for each k), we must have (ii).

Sufficiency. Suppose that conditions (i) and (ii) hold and $\mathbf{x} \in \mathrm{st} \cap \ell_\infty$ with $\mathrm{st}\text{-}\lim \mathbf{x} = \ell$. Write $E := \{k : |x_k - \ell| \geq \varepsilon\}$ for a given $\varepsilon > 0$, so that $\delta(E) = 0$. Since $\mathbf{A} \in (c, V_\sigma)_s$, $\sigma\text{-}\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{nk} = s$. We have

$$\begin{aligned}\sigma\text{-}\lim \mathbf{A}\mathbf{x} &= \sigma\text{-}\lim_{n\to\infty}\left(\sum_{k=1}^{\infty} a_{nk}(x_k - \ell) + \ell \sum_{k=1}^{\infty} a_{nk}\right) \\ &= \sigma\text{-}\lim_{n\to\infty}\sum_{k=1}^{\infty} a_{nk}(x_k - \ell) + \ell s\,.\end{aligned} \tag{$*$}$$

Since

$$\left|\sum_{k=1}^{\infty} a_{nk}(x_k - \ell)\right| \leq \|\mathbf{x}\| \sum_{k\in E} |a_{nk}| + \varepsilon \|\mathbf{A}\|\,,$$

applying condition (ii), we have

$$\sigma\text{-}\lim_{n\to\infty}\sum_{k=1}^{\infty} a_{nk}(x_k - \ell) = 0\,.$$

Hence, by $(*)$, $\sigma\text{-}\lim \mathbf{A}\mathbf{x} = \ell s = s(\mathrm{st}\text{-}\lim \mathbf{x})$; i.e. $\mathbf{A} \in (\mathrm{st}\cap \ell_\infty, V_\sigma)_s$.

This completes the proof of the theorem. □

For $\sigma(n) = n + 1$, we have the following:

COROLLARY 3.2. $\mathbf{A} \in (\mathrm{st}\cap\ell_\infty, f)_s$ *if and only if*

(i) $A \in (c, f)_s$,

and

(ii) $\lim_{p\to\infty} \sum_{k\in E} \frac{1}{p+1}\left|\sum_{j=0}^{p} a_{j+n,k}\right| = 0$ *uniformly in* n *for every* $E \subseteq \mathbb{N}$ *with* $\delta(E) = 0$.

Remark 1. For $\sigma(n) = n + 1$, and $s = 1$, the class $(\mathrm{st}\cap\ell_\infty, V_\sigma)_s$ is reduced to the class of statistically almost regular matrices, which we denote by $(\mathrm{st}\cap\ell_\infty, f)_{\mathrm{reg}}$ (see [4]).

4. Main result

THEOREM 4.1. $Q(\mathbf{A}\mathbf{x}) \leq sS(\mathbf{x})$ *for all* $\mathbf{x} \in \ell_\infty$ *if and only if*

(4.1.1) $\mathbf{A} \in (\mathrm{st}\cap\ell_\infty, V_\sigma)_s$,

and

(4.1.2) $\limsup_{p\to\infty} \sup_{n\in\mathbb{N}} \sum_{k=1}^{\infty} |t(n,k,p)| = s$,

where $S(\mathbf{x}) = \mathrm{st}\text{-}\limsup \mathbf{x}$, *and* Q *is defined by* (2).

Proof.

Necessity. Suppose that $Q(\mathbf{A}\boldsymbol{x}) \le sS(\boldsymbol{x})$ for all $\boldsymbol{x} \in \ell_\infty$. Then

$$s(-S(-\boldsymbol{x})) \le -Q(-\mathbf{A}\boldsymbol{x}) \le Q(\mathbf{A}\boldsymbol{x}) \le s\, S(\boldsymbol{x}),$$

or

$$s(\operatorname{st-lim\,inf} \boldsymbol{x}) \le -Q(-\mathbf{A}\boldsymbol{x}) \le Q(\mathbf{A}\boldsymbol{x}) \le s(\operatorname{st-lim\,sup} \boldsymbol{x}). \tag{4.1.3}$$

If $\boldsymbol{x} \in \mathrm{st} \cap \ell_\infty$, then we have (see Fridy and Orhan [7])

$$\operatorname{st-lim\,inf} \boldsymbol{x} = \operatorname{st-lim\,sup} \boldsymbol{x} = \operatorname{st-lim} \boldsymbol{x}.$$

Thus (4.1.3) implies

$$-Q(-\mathbf{A}\boldsymbol{x}) = Q(\mathbf{A}\boldsymbol{x}) = s(\operatorname{st-lim} \boldsymbol{x})$$

or

$$\sigma\text{-}\lim \mathbf{A}\boldsymbol{x} = s(\operatorname{st-lim} \boldsymbol{x}).$$

Hence $\mathbf{A} \in (\mathrm{st} \cap \ell_\infty, V_\sigma)_s$, i.e. (4.1.1) holds.

Since, by (4.1.1), $\mathbf{A} \in (c, V_\sigma)_s$, we have

$$\limsup_{p\to\infty} \sup_{n\in\mathbb{N}} \sum_{k=1}^{\infty} |t(n,k,p)| \ge \limsup_{p\to\infty} \sup_{n\in\mathbb{N}} \sum_{k=1}^{\infty} t(n,k,p) = s.$$

Hence

$$\limsup_{p\to\infty} \sup_{n\in\mathbb{N}} \sum_{k=1}^{\infty} |t(n,k,p)| \ge s. \tag{4.1.4}$$

By Lemma 2 of Das [3], for $\boldsymbol{y} \in \ell_\infty$ with $\|\boldsymbol{y}\| \le 1$, we have

$$\limsup_{p\to\infty} \sup_{n\in\mathbb{N}} \sum_{k=1}^{\infty} t(n,k,p) y_k = \limsup_{p\to\infty} \sup_{n\in\mathbb{N}} \sum_{k=1}^{\infty} |t(n,k,p)|. \tag{4.1.5}$$

Also by the hypothesis

$$Q(\mathbf{A}\boldsymbol{y}) \le sS(\boldsymbol{y}) \le sL(\boldsymbol{y}) = s\|\boldsymbol{y}\| \le s,$$

that is

$$\limsup_{p\to\infty} \sup_{n\in\mathbb{N}} \sum_{k=1}^{\infty} t(n,k,p) y_k \le s.$$

Therefore by (4.1.5) we get

$$\limsup_{p\to\infty} \sup_{n\in\mathbb{N}} \sum_{k=1}^{\infty} |t(n,k,p)| \le s, \tag{4.1.6}$$

which together with (4.1.4) gives (4.1.2).

Sufficiency. Let the conditions (4.1.1) and (4.1.2) hold and $\mathbf{x} \in \ell_\infty$. Then $S(\mathbf{x})$ is finite. For given $\varepsilon > 0$, let $E := \{k : x_k > S(\mathbf{x}) + \varepsilon\}$. Thus $\delta(E) = 0$, and if $k \notin E$, then $x_k \le S(\mathbf{x}) + \varepsilon$. Now for a fixed positive integer m,

$$t_{pn}(\mathbf{Ax}) \le \|\mathbf{x}\| \sum_{k<m} |t(n,k,p)| + \big(S(\mathbf{x}) + \varepsilon\big) \sum_{k\ge m,\, k\notin E} |t(n,k,p)| \\ + \|\mathbf{x}\| \sum_{k\ge m} \big(|t(n,k,p)| - t(n,k,p)\big) + \|\mathbf{x}\| \sum_{k\ge m,\, k\in E} |t(n,k,p)| .$$

Applying (4.1.1) and (4.1.2) we have

$$\limsup_{p\to\infty} \sup_{n\in\mathbb{N}} t_{pn}(\mathbf{Ax}) \le sS(\mathbf{x}) + \varepsilon .$$

Since ε is arbitrary, we conclude that

$$Q(\mathbf{Ax}) \le sS(\mathbf{x})$$

for all $\mathbf{x} \in \ell_\infty$.

This completes the proof of the theorem. □

Remark. Similarly we can show that

$$s(\mathrm{st}\text{-}\liminf \mathbf{x}) = -sS(-\mathbf{x}) \le -Q(-\mathbf{Ax}) .$$

Hence, finally we have

$$\sigma\text{-}\operatorname{core}\{\mathbf{Ax}\} \subseteq \mathrm{st}\text{-}\operatorname{core}\{s\mathbf{x}\} \qquad \text{for all} \quad \mathbf{x} \in \ell_\infty$$

if and only if (4.1.1) and (4.1.2) hold.

5. Consequences of Theorem 4.1

From Theorem 4.1 we deduce the following results.

For $s = 1$, we get:

THEOREM 5.1. *$Q(\mathbf{Ax}) \le S(\mathbf{x})$ for all $\mathbf{x} \in \ell_\infty$ if and only if*

(5.1.1) $\mathbf{A} \in \big(\mathrm{st} \cap \ell_\infty, V_\sigma\big)_{\mathrm{reg}}$,

and

(5.1.2) $\limsup\limits_{p\to\infty} \sup\limits_{n\in\mathbb{N}} \sum\limits_{k=1}^{\infty} |t(n,k,p)| = 1$.

For $\sigma(n) = n + 1$, we get:

THEOREM 5.2. *$L^*(\mathbf{A}x) \le sS(x)$ for all $x \in \ell_\infty$ if and only if*

(5.2.1) $\mathbf{A} \in (\mathrm{st} \cap \ell_\infty, f)_s$,

and

(5.2.2) $\lim_{p\to\infty}\sup \sup_{n\in\mathbb{N}} \sum_{k=1}^{\infty} \frac{1}{p+1} \left| \sum_{j=0}^{p} a_{j+n,k} \right| = s$.

For $\sigma(n) = n+1$ and $s = 1$, we get:

THEOREM 5.3. (see [4]) *$L^*(\mathbf{A}x) \le S(x)$ for all $x \in \ell_\infty$ if and only if*

(5.3.1) $\mathbf{A} \in (\mathrm{st} \cap \ell_\infty, f)_{\mathrm{reg}}$,

and

(5.3.2) $\lim_{p\to\infty}\sup \sup_{n\in\mathbb{N}} \sum_{k=1}^{\infty} \frac{1}{p+1} \left| \sum_{j=0}^{p} a_{j+n,k} \right| = 1$,

where

$$L^*(x) = \lim_{p\to\infty}\sup \sup_{n\in\mathbb{N}} \frac{1}{p+1} \sum_{j=0}^{p} x_{j+n}$$

is a bounded linear functional on ℓ_∞.

REFERENCES

[1] AHMAD, Z. U.—SARASWAT, S. K.—MURSALEEN: *Invariant means and multiplicative matrices*, Commun. Fac. Sci. Univ. Ank. Sér. A1 Math. Stat. **29** (1980), 55–61.

[2] CONNOR, J. S.: *The statistical and strong p-Cesáro convergence of sequences*, Analysis (Munich) **8** (1988), 47–63.

[3] DAS, G.: *Sublinear functionals and a class of conservative matrices*, Bull. Inst. Math. Acad. Sinica **15** (1987), 89–106.

[4] EDELY, OSAMA H. H.: *Statistical Convergence and Infinite Matrices.* Ph.D. Thesis, A. M. U., Aligarh, 2001.

[5] FAST, H.: *Sur la convergence statistique*, Colloq. Math. **2** (1951), 241–244.

[6] FRIDY, J. A.: *On statistical convergence*, Analysis (Munich) **5** (1985), 301–313.

[7] FRIDY, J. A.—ORHAN, C.: *Statistical limit superior and limit inferior*, Proc. Amer. Math. Soc. **125** (1997), 3625–3631.

[8] FRIDY, J. A.—ORHAN, C.: *Statistical core theorems*, J. Math. Anal. Appl. **208** (1997), 520–527.

[9] KOLK, E.: *Matrix summability of statistically convergent sequences*, Analysis (Munich) **13** (1993), 77–83.

[10] LORENTZ, G. G.: *A contribution to the theory of divergent sequences*, Acta Math. **80** (1948), 167–190.

[11] MISHRA, S. L.—SATAPATHY, B.—RATH, N.: *Invariant means and σ-core*, J. Indian Math. Soc. **60** (1984), 151–158.

[12] MURSALEEN: *On some new invariant matrix methods of summability*, Q. J. Math. **34** (1983), 77–86.

[13] MURSALEEN—GAUR, A. K.—CHISHTI, T. A.: *On some new sequence spaces of invariant means*, Acta Math. Hungar. **753** (1997), 209–214.

[14] ORHAN, C.: *Some inequalities involving sublinear functionals*, Comment. Math. Prace Mat. **31** (1991), 89–96.

[15] ŠALÁT, T.: *On statistically convergent sequences of real numbers*, Math. Slovaca **30** (1980), 139–150.

[16] SCHAEFER, P.: *Infinite matrices and invariant means*, Proc. Amer. Math. Soc. **36** (1972), 104–110.

Received October 23, 2001
Revised May 14, 2003

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