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THE INTERVAL FUNCTION OF A CONNECTED GRAPH AND
A CHARACTERIZATION OF GEODETIC GRAPHS

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Abstract. The interval function (in the sense of H. M. Mulder) is an important tool for studying those properties of a connected graph that depend on the distance between vertices. An axiomatic characterization of the interval function of a connected graph was published by Nebeský in 1994. In Section 2 of the present paper, a simpler and shorter proof of that characterization will be given. In Section 3, a characterization of geodetic graphs will be established; this characterization will utilize properties of the interval function.

Keywords: graphs, distance, interval function, geodetic graphs

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By a graph we mean here a finite undirected graph (without loops and multiple edges). If G is a graph, then $V(G)$ and $E(G)$ denote its vertex set and its edge set, respectively. Moreover, if G is connected and $u, v \in V(G)$, then $d_G(u, v)$ denotes the distance between u and v in G .

The letters f, g, \dots, n will be used for denoting integers.

1. Let G be a connected graph. Put $W = V(G)$. Following Mulder's book [4], by the *interval function* of G we mean the mapping I_G of $W \times W$ into 2^W (i.e. into the set of all subsets of W) defined as follows:

$$I_G(r, s) = \{t \in W; d_G(r, s) = d_G(r, t) + d_G(t, s)\}$$

for each ordered pair $r, s \in W$.

Lemma 1. *Let G be a connected graph. Put $W = V(G)$ and $J = I_G$. Then J satisfies the following axioms (i1)–(i7):*

(i1) $J(v, u) = J(u, v)$ for all $u, v \in W$;

- (i2) $u \in J(u, v)$ for all $u, v \in W$;
- (i3) if $x \in J(u, v)$, then $|J(u, x) \cap J(x, v)| = 1$ for all $u, v, x \in W$;
- (i4) if $x \in J(u, v)$ and $y \in J(x, v)$, then $y \in J(u, v)$ for all $u, v, x, y \in W$;
- (i5) if $x \in J(u, v)$ and $y \in J(x, v)$, then $x \in J(u, y)$ for all $u, v, x, y \in W$;
- (i6) if $|J(u, x)| = 2 = |J(v, y)|$, $x, y \in J(u, v)$ and $u \in J(x, y)$, then $v \in J(x, y)$ for all $u, v, x, y \in W$;
- (i7) if $|J(u, x)| = 2 = |J(v, y)|$, $x \in J(u, v)$, $x \notin J(u, y)$ and $y \notin J(u, v)$, then $v \in J(x, y)$ for all $u, v, x, y \in W$.

Proof. Axioms (i1)–(i5) can be verified easily. It is also not difficult to verify axioms (i6) and (i7); their verification can be found in [6]. □

Remark 1. Properties of the interval function of a connected graph that are very similar to axioms (i1)–(i5) were presented in [4, 1.1.2.].

Let W be a finite nonempty set, and let J be a mapping of $W \times W$ into 2^W . We denote by \mathbb{G}_J the graph H with $V(H) = W$ and

$$E(H) = \{rs; r, s \in W, r \neq s \text{ and } J(r, s) = \{r, s\} = J(s, r)\}.$$

If \mathbb{G}_J is connected and $n \geq 0$, then we denote by J_n the mapping of

$$Z_n \stackrel{\text{df}}{=} \{(u, v) \in W \times W; d_{\mathbb{G}_J}(u, v) = n\}$$

into 2^W such that $J_n(x, y) = J(x, y)$ for each $(x, y) \in Z_n$.

Lemma 2. *If G is a connected graph and $J = I_G$, then $G = \mathbb{G}_J$.*

Proof is obvious. □

In Lemmas 3–5 and in Corollary 1 we will assume that a finite nonempty set W and a mapping J of $W \times W$ into 2^W are given.

Lemma 3. *Assume that J satisfies axioms (i1), (i2) and (i3). Let $u_0, \dots, u_n \in W$, where $n \geq 1$, and let*

$$(1) \quad |J(u_0, u_1)| = \dots = |J(u_{n-1}, u_n)| = 2.$$

Then $u_0 u_1, \dots, u_{n-1} u_n \in E(\mathbb{G}_J)$.

Proof is very easy. □

Lemma 4. *Assume that J satisfies axioms (i1)–(i4). Consider arbitrary distinct $u, v \in W$. Then (a) u and v belong to the same component of \mathbb{G}_J and (b) there exists $w \in J(u, v)$ such that $|J(u, w)| = 2$.*

Proof. We proceed by induction on $|J(u, v)|$. By (i1) and (i2), $|J(u, v)| \geq 2$. If $|J(u, v)| = 2$, then, by virtue of (i3), u and v are adjacent and thus they belong to the same component; we put $w = v$. Now, let $|J(u, v)| > 2$. There exists $x \in J(u, v)$, $u \neq x \neq v$. By (i1) and (i2), $u \in J(u, x)$ and $v \in J(x, v)$. According to (i1) and (i4), $J(u, x), J(x, v) \subseteq J(u, v)$. By virtue of (i3), $|J(u, x)|, |J(x, v)| < |J(u, v)|$. By the induction hypothesis, (a') x belongs to the same component as u and to the same component as v , and (b') there exists $w \in J(u, x)$ such that $|J(u, w)| = 2$. Obviously, u and v belong to the same component. Combining (i1) and (i4), we get $w \in J(u, v)$. Hence (a) and (b) hold. \square

Corollary 1. *If J satisfies axioms (i1)–(i4), then \mathbb{G}_J is connected.*

Lemma 5. *Assume that J satisfies axioms (i2), (i4) and (i5). Let $u_0, \dots, u_n, v \in W$, where $n \geq 1$, and let*

$$(2_i) \quad u_{i+1} \in J(u_i, v)$$

for each i , $0 \leq i \leq n - 1$. Then

$$(3_j) \quad u_j \in J(u_0, v) \text{ and } u_{j-1} \in J(u_0, u_j)$$

for each j , $1 \leq j \leq n$.

Proof. We proceed by induction on j . The case $j = 1$ is trivial. Let $j \geq 2$. By the induction hypothesis, $u_{j-1} \in J(u_0, v)$. By (2_{j-1}), $u_j \in J(u_{j-1}, v)$. As follows from (i4), $u_j \in J(u_0, v)$. As follows from (i5), $u_{j-1} \in J(u_0, u_j)$. \square

2. The interval function of a connected graph G plays a very important role in studying those structural properties of G that depend on distance between vertices. Cf. Mulder [4] or, for example, Bandelt and Mulder [1] and [2], and Bandelt, Mulder and Wilkeit [3].

However, the concept of the interval function of a connected graph is not only well-motivated; it is also transparently characterizable. Nebeský [6] proved a theorem which can be reformulated as follows: If W is a finite nonempty set, J is a mapping of $W \times W$ into 2^W and \mathbb{G}_J is connected, then J is the interval function of \mathbb{G}_J if and only if J satisfies axioms (i1)–(i7).

The proof given in [6] was unnecessarily complicated. A new proof will be presented here. It will utilize some ideas of the original proof but it will be shorter and

significantly simpler. We will formulate a theorem slightly stronger than the one mentioned above:

Theorem 1. *Let W be a finite nonempty set, and let J be a mapping of $W \times W$ into 2^W . Then J is the interval function of a connected graph if and only if J satisfies axioms (i1)–(i7).*

Proof. If J is the interval function of a connected graph G , then, by virtue of Lemma 2, $V(G) = W$ and thus, by Lemma 1, J satisfies axioms (i1)–(i7).

Conversely, let J satisfy axioms (i1)–(i7). Put $G = \mathbb{G}_J$. By Corollary 1, G is connected. Put $d = d_G$ and $I = I_G$. We will prove that $J = I$.

Suppose, to the contrary, that $I \neq J$. Then there exists $n \geq 0$ such that $J_n \neq I_n$ and

$$(4) \quad J_f = I_f \text{ for all } f, 0 \leq f < n.$$

It is easy to see that $n \geq 2$. We distinguish two cases.

Case 1. Let $I_n \setminus J_n \neq \emptyset$. There exist $u, v, w \in W$ such that $d(u, v) = n$ and $w \in I(u, v) \setminus J(u, v)$. Thus, there exist $v_0, \dots, v_n \in W$ and $g, 0 < g < n$, such that $v_0 = v, v_n = u, v_g = w$ and (v_n, \dots, v_0) is a path from u to v of length $d(u, v)$ in G . Let $v_{n-1} \in J(u, v)$; clearly, $d(v_{n-1}, v) = n - 1$ and $v_g \in I(v_{n-1}, v)$; by (4), $v_g \in J(v_{n-1}, v)$ and by (i4), $v_g \in J(u, v)$; a contradiction. Hence $v_{n-1} \notin J(u, v)$. By (i1) and (i2), $v \in J(u, v)$. Recall that $d(u, v) = n$. Lemmas 3 and 4 imply that there exist $u_0, \dots, u_n \in W$ such that $u_0 = u$, (1) holds and (2_i) holds for each $i, 0 \leq i \leq n - 1$. By Lemma 5, (3_n) holds. As follows from Lemma 3,

$$(5) \quad d(u_j, v_j) \leq n \text{ for each } j, 0 \leq j \leq n.$$

Put $u_{-1} = v_{n-1}$. The following three statements hold for $i = 0$:

$$(6_i) \quad d(u_i, v_i) = n,$$

$$(7_i) \quad v \in J(u_i, v_i),$$

$$(8_i) \quad u_{i-1} \notin J(u_i, v_i).$$

By (3_n) and (i1), $u_{n-1} \in J(u_n, u_0)$. Since $v_n = u_0$, (8_n) does not hold.

There exists $h, 0 \leq h \leq n - 1$, such that $(6_h), (7_h)$ and (8_h) hold but at least one of $(6_{h+1}), (7_{h+1})$ and (8_{h+1}) does not. Combining $(2_h), (7_h)$ and (i1) with (i4) and (i5), we get

$$(9) \quad u_{h+1} \in J(u_h, v_h),$$

$$(10) \quad v \in J(u_{h+1}, v_h).$$

As follows from (6_h),

$$(11) \quad d(u_h, v_{h+1}) = n - 1.$$

Clearly, $u_{h-1} \in I(u_h, v_{h+1})$. By (11) and (4), $u_{h-1} \in J(u_h, v_{h+1})$. Let $v_{h+1} \in J(u_h, v_h)$; by (i1) and (i4) we get $u_{h-1} \in J(u_h, v_h)$, which contradicts (8_h). Hence

$$(12) \quad v_{h+1} \notin J(u_h, v_h).$$

Let $u_{h+1} \in J(u_h, v_{h+1})$. By (11) and (4), $u_{h+1} \in I(u_h, v_{h+1})$. Thus $d(u_{h+1}, v_{h+1}) = n - 2$. As follows from (6_h), $d(u_{h+1}, v_h) = n - 1$ and $v_{h+1} \in I(u_{h+1}, v_h)$. By (4), $v_{h+1} \in J(u_{h+1}, v_h)$. Combining (9) and (i4), we see that $v_{h+1} \in J(u_h, v_h)$, which contradicts (12). Hence $u_{h+1} \notin J(u_h, v_{h+1})$. Thus, combining (9), (12) and (i7), we get

$$(13) \quad v_h \in J(u_{h+1}, v_{h+1}).$$

Let $d(u_{h+1}, v_{h+1}) < n$. By (13) and (4), $v_h \in I(u_{h+1}, v_{h+1})$. Therefore, $d(u_{h+1}, v_h) < n - 1$. This means that $d(u_h, v_h) < n$, which contradicts (6_h). Thus, by virtue of (5), we get (6_{h+1}).

Combining (10), (13), (i1) and (i4), we get (7_{h+1}).

Assume that $u_h \in J(u_{h+1}, v_{h+1})$. Combining (9), (13) and (i6), we see that $v_{h+1} \in J(u_h, v_h)$, which contradicts (12). We get (8_{h+1}), which is a contradiction with the definition of h .

Case 2. Let $I_n \subseteq J_n$. Then $J_n \setminus I_n \neq \emptyset$. There exist $u, v, z \in W$ such that $d(u, v) = n$ and $z \in J(u, v) \setminus I(u, v)$. By (i2), $z \neq u$. By Lemma 3, there exists $t \in J(u, z)$ such that $|J(u, t)| = 2$. By (i1), (i4) and (i5), $t \in J(u, v)$ and $z \in J(t, v)$. If $d(t, v) < n$, then $d(t, v) = n - 1$, $t \in I(u, v)$ and, by (4), $z \in I(t, v) \subseteq I(u, v)$; a contradiction. Hence $d(t, v) \geq n$. Lemmas 3 and 4 imply that there exist $u_0, \dots, u_n \in W$ such that $u_0 = u$, $u_1 = t$, (1) holds and (2_i) holds for each $i, 0 \leq i \leq n - 1$. Since $d(u_1, v) \geq n$ and $u_1 u_2, \dots, u_{n-1} u_n \in E(G)$, we have

$$(14_i) \quad d(u_{i+1}, v) \geq n - i$$

for each $i, 0 \leq i \leq n - 1$. Thus $u_n \neq v$. By Lemma 5, (3_n) holds. Since $d(u, v) = n$, there exist $v_0, \dots, v_n \in W$ such that $v_0 = v, v_n = u$ and (v_n, \dots, v_0) is a path from u to v of length $d(u, v)$ in G . Thus

$$(15_i) \quad d(v, v_i) = i$$

for each $i, 0 \leq i \leq n - 1$. Moreover, (5) holds.

Obviously, both (6₀) and (7₀) hold. By (3_n), $u_n \in J(v_n, v)$. Since $u_n \neq v$, (i1), (i2) and (i3) imply that (7_n) does not hold.

There exists h , $0 \leq h \leq n - 1$, such that (6_h) and (7_h) hold but at least one of (6_{h+1}) and (7_{h+1}) does not. Similarly as in Case 1, we get (9), (10) and (11).

Let $d(u_{h+1}, v_h) < n$. Combining (4) and (10), we get $v \in I(u_{h+1}, v_h)$. By virtue of (14_h), we have $d(v, v_h) < h$, which contradicts (15_h). Hence $d(u_{h+1}, v_h) \geq n$.

Let $d(u_{h+1}, v_{h+1}) < n$. Since $d(u_{h+1}, v_h) \geq n$, we have $d(u_{h+1}, v_h) = n$ and $d(u_{h+1}, v_{h+1}) = n - 1$. Therefore, $v_{h+1} \in I(u_{h+1}, v_h)$. Since $I_n \subseteq J_n$, we get $v_{h+1} \in J(u_{h+1}, v_h)$. Thus, combining (9) and (i5), we see that $u_{h+1} \in J(u_h, v_{h+1})$. By (11) and (4), $u_{h+1} \in I(u_h, v_{h+1})$ and therefore, $d(u_{h+1}, v_{h+1}) = n - 2$; a contradiction. Thus, by virtue of (5), (6_{h+1}) holds.

By virtue of (6_h), $v_{h+1} \in I(u_h, v_h)$; by (6_{h+1}), $u_h \in I(u_{h+1}, v_{h+1})$. Recall that $I_n \subseteq J_n$. We have $v_{h+1} \in J(u_h, v_h)$ and $u_h \in J(u_{h+1}, v_{h+1})$. Thus, (9) and (i6) imply (13). Similarly as in Case 1, we get (7_{h+1}), which is a contradiction with the definition of h .

Thus $J = I$, which completes the proof. □

Remark 2. An extension of Theorem 1 (with a different and rather long proof) was presented in Nebeský [8].

3. A graph G is said to be *geodetic* if it is connected and for each pair $r, s \in V(G)$, there exists exactly one path from r to s of length $d_G(r, s)$. The problem to characterize geodetic graphs was stated in Ore's book [10].

The next theorem gives a characterization of geodetic graphs based on properties of the interval function.

Theorem 2. *Let G be a graph. Put $W = V(G)$. Then G is geodetic if and only if there exists a mapping J of $W \times W$ into 2^W such that $G = \mathbb{G}_J$ and J satisfies axioms (i1), (i2), (i3) and the following axioms (g1) and (g2):*

(g1) *if $x \in J(u, v)$, then $J(u, v) = J(u, x) \cup J(x, v)$ for all $u, v, x \in W$;*

(g2) *if $|J(u, x)| = 2 = |J(v, y)|$ and $x \in J(u, v)$, then $x \in J(u, y)$ or $v \in J(x, y)$ for all $u, v, x, y \in W$.*

Proof. I. Assume that G is geodetic. Then G is connected. Let J denote its interval function. By Lemma 1, J satisfies (i1), (i2) and (i3). As immediately follows from the definition of a geodetic graph, J satisfies (g1). We will show that J satisfies (g2).

Consider arbitrary $u, v, x, y \in W$. Assume that $|J(u, x)| = 2 = |J(v, y)|$ and $x \in J(u, v)$. If $y = u$, then $x = v$ and thus $v \in J(x, y)$. Let $y \neq u$. Since G is geodetic, there exists exactly one path P from u to v of length $d_G(u, v)$ in G . If $y \in J(u, v)$, then y belongs to P and thus $x \in J(u, y)$. Let $y \notin J(u, v)$. Then

$d_G(x, v) \leq d_G(x, y) \leq d_G(x, v) + 1$. If $d_G(x, y) = d_G(x, v) + 1$, then $v \in J(x, y)$. If $d_G(x, y) = d_G(x, v)$, then $x \in J(u, y)$. Thus J satisfies (g2).

II. Conversely, assume that J satisfies (i1), (i2), (i3), (g1) and (g2). The fact that J satisfies (g2) implies that J satisfies (i7). First, we will show that J also satisfies (i4), (i5) and (i6).

Consider arbitrary $u, v, x, y \in W$.

Let $x \in J(u, v)$ and $y \in J(x, v)$. By (g1),

$$(16) \quad J(u, v) = J(u, x) \cup J(x, v) \text{ and } J(x, v) = J(x, y) \cup J(y, v).$$

Combining (i2) and (16), we get $y \in J(u, v)$. By (g1) again,

$$(17) \quad J(u, v) = J(u, y) \cup J(y, v).$$

If $x = y$, then (i1), (i2) and (17) imply that $x \in J(u, y)$. Let $x \neq y$. By (i2), $x \in J(x, y)$. Since $y \in J(x, v)$, it follows from (i3) that $J(x, y) \cap J(y, v) = \{y\}$, and therefore, $x \notin J(y, v)$. Since $x \in J(u, v)$, (17) implies that $x \in J(u, y)$. We see that J satisfies (i4) and (i5).

Let $|J(u, x)| = 2 = |J(v, y)|$ and let $x, y \in J(u, v)$. As follows from (i1), (i2) and (i3), we have $x \neq u \neq v \neq y$, $J(u, x) = \{u, x\}$, $J(v, y) = \{v, y\}$ and $u, v \in J(u, v)$. Hence $|J(u, v)| \geq 2$. First, let $|J(u, v)| = 2$. Then $u = y$ and $v = x$. We have $u, v \in J(u, v)$. Now, let $|J(u, v)| > 2$. Then $x \neq v$ and $y \neq u$. It follows from (i3) and (g1) that $|\{u, x\} \cap J(x, v)| = 1$ and $J(u, v) = J(u, x) \cup J(x, v)$. Thus $u \notin J(x, v)$. As follows from (i3) and (g1) again, $|J(x, y) \cap \{y, v\}| = 1$ and $J(x, v) = J(x, y) \cup \{y, v\}$. By (i1) and (i2), $y \in J(x, y)$. Thus $v \notin J(x, y)$. Since $u \notin J(x, v)$, we get $u \notin J(x, y)$. We see that J satisfies also (i6).

Put $G = \mathbb{G}_J$. By Corollary 1, G is connected. By Theorem 1 and Lemma 2, J is the interval function of G . Recall that J satisfies (g1). We will show that

$$(18) \quad \text{there exists exactly one path from } u \text{ to } v \text{ of length } d_G(u, v) \text{ in } G$$

for each pair of distinct $u, v \in W$.

Consider arbitrary distinct $u, v \in W$. Put $n = d_G(u, v)$. To prove (18), we will proceed by induction on n . Obviously, $n \geq 1$. There exist $u_0, u_1, \dots, u_n \in W$ such that $u_0 = u, u_n = v$ and (u_0, u_1, \dots, u_n) is a path in G . Since $d_G(u, v) = n$, we see that u_i and u_j are adjacent in G if and only if $|i - j| = 1$ for all i and j , $0 \leq i \leq n$ and $0 \leq j \leq n$. Clearly, if $n = 1$, then (18) holds. Let $n > 1$. Then $u_1 \neq v$. Since $d_G(u_1, v) = n - 1$, the induction hypothesis implies that (u_1, \dots, u_n) is the only path from u_1 to v of length $n - 1$ in G . Thus $J(u_1, v) = \{u_1, \dots, u_n\}$. By virtue of (g1), $J(u, v) = \{u_0, u_1, \dots, u_n\}$. This means that (u_0, u_1, \dots, u_n) is the only path from u to v of length n in G .

Thus G is geodetic. □

R e m a r k 3. A characterization of geodetic graphs utilizing properties of the set of all shortest paths was given in Nebeský [5] and [7]. A characterization of geodetic graphs based on a binary operation on the vertex set was given in Nebeský [9].

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