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A MULTIDIMENSIONAL INTEGRATION BY PARTS FORMULA FOR THE HENSTOCK-KURZWEIL INTEGRAL

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Abstract. It is shown that if g is of bounded variation in the sense of Hardy-Krause on $\prod_{i=1}^{m}[a_i,b_i]$, then $g\chi_{\prod_{i=1}^{m}(a_i,b_i)}$ is of bounded variation there. As a result, we obtain a simple proof of Kurzweil's multidimensional integration by parts formula.

Keywords: Henstock-Kurzweil integral, bounded variation in the sense of Hardy-Krause, integration by parts

MSC 2000: 26A39

1. Introduction

It is well known that if f is Henstock-Kurzweil integrable on a compact interval $[a,b]\subset\mathbb{R}$ and g is of bounded variation there, then fg is Henstock-Kurzweil integrable there and the integration by parts formula holds; see, for example, [12] and references therein. Although higher-dimensional analogues of the above-mentioned result have been studied by various authors ([1], [2], [3], [6], [7], [10], [14], [17], [18]), a simpler proof of Kurzweil's mutidimensional integration by parts formula for the Henstock-Kurzweil integral [1, Theorem 2.10] remained elusive. The purpose of this paper is to give a simpler proof of this result.

2. Functions of bounded variation

Let $m\geqslant 1$ be an integer and let $\mathbb{R}^m$ be the m-dimensional Euclidean space equipped with the maximum norm. An *interval* in $\mathbb{R}^m$ is a set of the form $\prod_{i=1}^{m}[u_i,v_i]$, where $u_i, v_i \in \mathbb{R}$ and $u_i \leqslant v_i$ for $i = 1, \ldots, m$. Let $[\mathbf{a}, \mathbf{b}] := \prod_{i=1}^{m} [a_i, b_i]$ be a fixed non-degenerate compact interval in $\mathbb{R}^m$, where $\mathbf{a} = (a_1, \ldots, a_m)$ and $\mathbf{b} = (b_1, \ldots, b_m)$, and let $\mathcal{I}_m([\mathbf{a}, \mathbf{b}])$ denote the family of all non-degenerate subintervals of $[\mathbf{a}, \mathbf{b}]$. For each $\prod_{i=1}^{m} [u_i, v_i] \in \mathcal{I}_m([\mathbf{a}, \mathbf{b}])$, we set $[\mathbf{u}, \mathbf{v}] := \prod_{i=1}^{m} [u_i, v_i]$ and $(\mathbf{u}, \mathbf{v}) := \prod_{i=1}^{m} (u_i, v_i)$, where $\mathbf{u} = (u_1, \ldots, u_m)$ and $\mathbf{v} = (v_1, \ldots, v_m)$.

A *division* of $[\mathbf{a}, \mathbf{b}]$ is a finite collection $\{I_1, \ldots, I_p\}$ of non-overlapping intervals such that $\bigcup_{i=1}^{p} I_i = [\mathbf{a}, \mathbf{b}]$. For any given real-valued function g defined on $[\mathbf{a}, \mathbf{b}]$, the total variation of g over $[\mathbf{a}, \mathbf{b}]$ is defined by

$$\operatorname{Var}(g, [\mathbf{a}, \mathbf{b}]) := \sup\left\{ \sum_{[\mathbf{u},\mathbf{v}] \in P} |\, \Delta_g([\mathbf{u}, \mathbf{v}])| :\ P \text{ is a division of } [\mathbf{a}, \mathbf{b}] \right\},$$

where

$$\Delta_g([\mathbf{u}, \mathbf{v}]) := \sum_{\substack{\mathbf{t} \in [\mathbf{u}, \mathbf{v}] \\ t_i \in \{u_i, v_i\}\ \forall\ i \in \{1, \ldots, m\}}} g(\mathbf{t}) \prod_{i=1}^{m} \operatorname{sgn}\left(t_i - \frac{u_i + v_i}{2}\right)$$

for each $[\mathbf{u}, \mathbf{v}] \in \mathcal{I}_m([\mathbf{a}, \mathbf{b}])$.

Definition 2.1. A function $g\colon [\mathbf{a}, \mathbf{b}] \longrightarrow \mathbb{R}$ is said to be of bounded variation (in the sense of Vitali) on $[\mathbf{a}, \mathbf{b}]$ if $\operatorname{Var}(g, [\mathbf{a}, \mathbf{b}])$ is finite.

The space of functions of bounded variation (in the sense of Vitali) on $[\mathbf{a}, \mathbf{b}]$ is denoted by $\mathrm{BV}[\mathbf{a}, \mathbf{b}]$. Set

$$\mathrm{BV}_0[\mathbf{a}, \mathbf{b}] := \{g \in \mathrm{BV}[\mathbf{a}, \mathbf{b}]\colon\ g(\mathbf{x}) = 0 \text{ whenever } \mathbf{x} \in [\mathbf{a}, \mathbf{b}] \setminus (\mathbf{a}, \mathbf{b}]\},$$

where $(\mathbf{a}, \mathbf{b}] := \prod_{i=1}^{m} (a_i, b_i]$. The next theorem is an m-dimensional analogue of [16, Theorem 1].

Theorem 2.2. *Let $g\colon [\mathbf{a}, \mathbf{b}] \longrightarrow \mathbb{R}$. Then $g \in \mathrm{BV}_0[\mathbf{a}, \mathbf{b}]$ if and only if there exists a sequence $\{\varphi_n\}_{n=1}^{\infty}$ in $L^1[\mathbf{a}, \mathbf{b}]$ such that $\sup_{n \in \mathbb{N}} \|\varphi_n\|_{L^1[\mathbf{a},\mathbf{b}]}$ is finite and $\lim_{n\to\infty} \int_{[\mathbf{a},\mathbf{x}]} \varphi_n(\mathbf{t})\, \mathrm{d}\mathbf{t} = g(\mathbf{x})$ for each $\mathbf{x} \in [\mathbf{a}, \mathbf{b}]$.*

The following result of Young [20] is also useful.

Theorem 2.3. *Let* $\mathbf{x} \in [\mathbf{a}, \mathbf{b}]$ *and let* $\{\mathbf{x}_n\}_{n=1}^{\infty}$ *be a sequence in* $[\mathbf{a}, \mathbf{b}]$ *such that* $\operatorname{sgn}(x_{k,i} - x_k) = \operatorname{sgn}(x_{k,j} - x_k)$ *for all* $i, j \in \mathbb{N}$ *and* $k \in \{1, \ldots, m\}$*. If* $g \in \mathrm{BV}_0[\mathbf{a}, \mathbf{b}]$ *and* $\lim_{n\to\infty} \mathbf{x}_n = \mathbf{x}$*, then the limit* $\lim_{n\to\infty} g(\mathbf{x}_n)$ *exists. In particular,* g *is continuous everywhere on* $[\mathbf{a}, \mathbf{b}]$ *except for a countable number of hyperplanes parallel to the coordinate axes.*

New proofs of Theorems 2.2 and 2.3 are given in [13].

3. The m-dimensional Riemann-Stieltjes integral

The purpose of this section is to recall some useful facts concerning the m-dimensional Riemann-Stieltjes integral. In particular, we obtain a useful result (Theorem 3.4) which plays an important role in the proof of Theorem 4.10.

Definition 3.1. Let F and H be two real-valued functions defined on $[\mathbf{a}, \mathbf{b}]$. F is said to be Riemann-Stieltjes integrable with respect to H on $[\mathbf{a}, \mathbf{b}]$ if there exists $A \in \mathbb{R}$ with the following property: for each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\left| \sum_{i=1}^{p} F(x_i)\Delta_H(I_i) - A \right| < \varepsilon$$

for each division $\{I_1, \ldots, I_p\}$ of $[\mathbf{a}, \mathbf{b}]$ such that $x_i \in I_i$ and the diameter of I_i is less than δ for $i = 1, \ldots, p$. In this case, the value of A is uniquely determined and we write A as $\int_{[\mathbf{a},\mathbf{b}]} F(\mathbf{x})\,\mathrm{d}H(\mathbf{x})$.

It is well known that if $F \in C[\mathbf{a}, \mathbf{b}]$ and $H \in \mathrm{BV}[\mathbf{a}, \mathbf{b}]$, then the Riemann-Stieltjes integral $\int_{[\mathbf{a},\mathbf{b}]} F(\mathbf{x})\,\mathrm{d}H(\mathbf{x})$ exists; in particular, we have the following result.

Theorem 3.2. *If* $F \in C[\mathbf{a}, \mathbf{b}]$*,* $h \in L^1[\mathbf{a}, \mathbf{b}]$ *and* $H(\mathbf{x}) = \int_{[\mathbf{a},\mathbf{x}]} h(\mathbf{t})\,\mathrm{d}\mathbf{t}$ *for each* $\mathbf{x} \in [\mathbf{a}, \mathbf{b}]$*, then the Riemann-Stieltjes integral* $\int_{[\mathbf{a},\mathbf{b}]} F(\mathbf{x})\,\mathrm{d}H(\mathbf{x})$ *exists,* $Fh \in L^1[\mathbf{a}, \mathbf{b}]$ *and*

$$\int_{[\mathbf{a},\mathbf{b}]} F(\mathbf{x})\,\mathrm{d}H(\mathbf{x}) = \int_{[\mathbf{a},\mathbf{b}]} F(\mathbf{x})h(\mathbf{x})\,\mathrm{d}\mathbf{x}.$$

The following convergence theorem is also well known.

Theorem 3.3. *Let* $F \in C[\mathbf{a}, \mathbf{b}]$ *and suppose that the following assertions hold:*

(i) $\{g_n\}_{n=1}^{\infty} \subset \mathrm{BV}[\mathbf{a}, \mathbf{b}]$ *so that* $\sup_{n\in\mathbb{N}} \operatorname{Var}(g_n, [\mathbf{a}, \mathbf{b}])$ *is finite.*

(ii) $g_n \to g$ *pointwise on* $[\mathbf{a}, \mathbf{b}]$*.*

Then the Riemann-Stieltjes integral $\int_{[\mathbf{a},\mathbf{b}]} F(\mathbf{x})\,\mathrm{d}g(\mathbf{x})$ *exists. Moreover, the limit* $\lim\limits_{n\to\infty} \int_{[\mathbf{a},\mathbf{b}]} F(\mathbf{x})\,\mathrm{d}g_n(\mathbf{x})$ *exists and*

$$\lim_{n\to\infty} \int_{[\mathbf{a},\mathbf{b}]} F(\mathbf{x})\,\mathrm{d}g_n(\mathbf{x}) = \int_{[\mathbf{a},\mathbf{b}]} F(\mathbf{x})\,\mathrm{d}g(\mathbf{x}).$$

Using Theorems 2.2, 3.2 and 3.3, we obtain the following result.

Theorem 3.4. *Let* $F \in C[\mathbf{a},\mathbf{b}]$ *and let* $g \in \mathrm{BV}[\mathbf{a},\mathbf{b}]$. *If* $g(\mathbf{x}) = 0$ *for all* $\mathbf{x} \in [\mathbf{a},\mathbf{b}] \setminus (\mathbf{a},\mathbf{b})$ *and there exists* $k \in \{1,\ldots,m\}$ *such that* F *is independent of* x_k, *then*

$$\int_{[\mathbf{a},\mathbf{b}]} F(\mathbf{x})\,\mathrm{d}g(\mathbf{x}) = 0.$$

P r o o f. We may assume that $k = 1$ and $m \geqslant 2$. According to Theorem 2.2, there exists a sequence $\{\varphi_n\}_{n=1}^{\infty}$ in $L^1[\mathbf{a},\mathbf{b}]$ such that

$$(1) \qquad \sup_{n\in\mathbb{N}} \|\varphi_n\|_{L^1[\mathbf{a},\mathbf{b}]} < \infty$$

and

$$(2) \qquad \lim_{n\to\infty} \int_{[\mathbf{a},\mathbf{x}]} \varphi_n(\mathbf{t})\,\mathrm{d}\mathbf{t} = g(\mathbf{x}) \quad \text{for each } \mathbf{x} \in [\mathbf{a},\mathbf{b}].$$

As a consequence of (1), (2), Theorems 3.3 and 3.2, we conclude that

$$(3) \qquad \int_{[\mathbf{a},\mathbf{b}]} F(\mathbf{x})\,\mathrm{d}g(\mathbf{x}) = \lim_{n\to\infty} \int_{[\mathbf{a},\mathbf{b}]} F(\mathbf{x})\varphi_n(\mathbf{x})\,\mathrm{d}\mathbf{x}.$$

Moreover, it follows from Fubini's theorem and our assumptions that

$$(4) \qquad \int_{[a,b]} F(\mathbf{x})\varphi_n(\mathbf{x})\,\mathrm{d}\mathbf{x} = \int_{\prod\limits_{i=2}^{m}[a_i,b_i]} F(\mathbf{x}) \Big\{ \int_{[a_1,b_1]} \varphi_n(\mathbf{x})\,\mathrm{d}x_1 \Big\}\,\mathrm{d}(x_2,\ldots,x_m)$$

for $n = 1, 2, \ldots$. In view of (3) and (4), it suffices to prove that

$$(5) \qquad \lim_{n\to\infty} \int_{\prod\limits_{i=2}^{m}[a_i,b_i]} F(\mathbf{x}) \Big\{ \int_{[a_1,b_1]} \varphi_n(\mathbf{x})\,\mathrm{d}x_1 \Big\}\,\mathrm{d}(x_2,\ldots,x_m) = 0.$$

From (1), we get

$$(6) \qquad \sup_{n\in\mathbb{N}} \int_{\prod\limits_{i=2}^{m}[a_i,b_i]} \Big| \int_{[a_1,b_1]} \varphi_n(\mathbf{x})\,\mathrm{d}x_1 \Big|\,\mathrm{d}(x_2,\ldots,x_m) \leqslant \sup_{n\in\mathbb{N}} \int_{[\mathbf{a},\mathbf{b}]} |\varphi_n(\mathbf{x})|\,\mathrm{d}\mathbf{x} < \infty.$$

For each $(b_1, x_2, \ldots, x_m) \in [\mathbf{a}, \mathbf{b}]$, Fubini's theorem, (2) and our choice of g yield

$$(7)\qquad \lim_{n\to\infty} \int_{\prod_{i=2}^{m}[a_i,x_i]} \left\{ \int_{[a_1,b_1]} \varphi_n(\mathbf{t})\,\mathrm{d}t_1 \right\} \mathrm{d}(t_2,\ldots,t_m) = g(b_1, x_2, \ldots, x_m) = 0.$$

Using an $(m-1)$-dimensional analogue of Theorem 3.2, (6), (7) and an $(m-1)$-dimensional analogue of Theorem 3.3, we get (5). The proof is complete.

4. A new proof of Kurzweil's multidimensional integration by parts formula

The aim of this section is to give a new proof of the multidimensional integration by parts formula for the Henstock-Kurzweil integral; see Theorem 4.10 for details. Unlike the original proof of [1, Theorem 2.10], our method of proof depends on our simple Theorems 4.8 and 4.5. For the definition, properties and recent results concerning the Henstock-Kurzweil integral, consult for instance [4], [5], [6], [7], [8], [9].

Set $\Phi_{[a,b],k}(X_k) := \prod_{i=1}^{m} W_i$ where $W_k = X_k$ and $W_i = [a_i, b_i]$ for all $i \in \{1, \ldots, m\} \setminus \{k\}$.

Definition 4.1. A function $g\colon [\mathbf{a}, \mathbf{b}] \longrightarrow \mathbb{R}$ is said to be of bounded variation (in the sense of Hardy-Krause) on $[\mathbf{a}, \mathbf{b}]$ if $g \in \mathrm{BV}[\mathbf{a}, \mathbf{b}]$ and, for each non-empty set $\Gamma \subset \{1, \ldots, m\}$,

$$g\Big|_{\bigcap_{k=1,\, k\notin\Gamma}^{m} \Phi_{[a,b],k}(\{a_k\})} \in \mathrm{BV}\Bigg(\prod_{\substack{k=1\\ k\in\Gamma}}^{m} [a_k, b_k]\Bigg).$$

The class of functions of bounded variation (in the sense of Hardy-Krause) on $[\mathbf{a}, \mathbf{b}]$ will be denoted by $\mathrm{BV}_{\mathrm{HK}}[\mathbf{a}, \mathbf{b}]$. As an immediate consequence of Definition 4.1, we have

Theorem 4.2. $\mathrm{BV}_0[\mathbf{a}, \mathbf{b}] \subset \mathrm{BV}_{\mathrm{HK}}[\mathbf{a}, \mathbf{b}]$.

Let χ_Y denote the characteristic function of a set Y. In order to prove a crucial result for $\mathrm{BV}_{\mathrm{HK}}$ functions (cf. Theorem 4.5), we need the following lemmas.

Lemma 4.3. *Let $g \in \mathrm{BV}_{\mathrm{HK}}[\mathbf{a}, \mathbf{b}]$. If $\mathcal{T} \subset \{1, \ldots, m\}$ is non-empty and $c_k \in \{a_k, b_k\}$ for all $k \in \{1, \ldots, m\} \setminus \mathcal{T}$, then*

$$g\Big|_{\bigcap_{k=1,\, k\notin\mathcal{T}}^{m} \Phi_{[a,b],k}(\{c_k\})} \in \mathrm{BV}\Bigg(\prod_{\substack{k=1\\ k\in\mathcal{T}}}^{m} [a_k, b_k]\Bigg).$$

P r o o f. This is an immediate consequence of Definition 4.1.

Let

$$\mathcal{P}_m := \left\{ \prod_{k=1}^{m} Y_k \colon\ Y_k \in \{\{a_k\}, \{b_k\}, [a_k, b_k]\} \text{ for each } k \in \{1, \ldots, m\} \right\}$$

and for $\prod\limits_{k=1}^{m} Y_k \in \mathcal{P}_m$, let

$$\Gamma\biggl(\prod_{k=1}^{m} Y_k\biggr) = \{i \in \{1, \ldots, m\} \colon\ Y_i = [a_i, b_i]\}.$$

Lemma 4.4. *If $g \in \mathrm{BV}_{\mathrm{HK}}[\mathbf{a}, \mathbf{b}]$ and $Y \in \mathcal{P}_m$, then $g\chi_Y \in \mathrm{BV}[\mathbf{a}, \mathbf{b}]$.*

P r o o f. Let $g \in \mathrm{BV}_{\mathrm{HK}}[\mathbf{a}, \mathbf{b}]$. If $Y \in \mathcal{P}_m$ and $\Gamma(Y)$ is empty, then it is clear that $g\chi_Y \in \mathrm{BV}[\mathbf{a}, \mathbf{b}]$. On the other hand, for any $Y \in \mathcal{P}_m$ satisfying $\Gamma(Y) \neq \emptyset$, it follows from Lemma 4.3 that $g\chi_Y \in \mathrm{BV}[\mathbf{a}, \mathbf{b}]$. □

Let μ_0 denote the counting measure.

Theorem 4.5. *If $g \in \mathrm{BV}_{\mathrm{HK}}[\mathbf{a}, \mathbf{b}]$, then $g\chi_{(\mathbf{a},\mathbf{b})} \in \mathrm{BV}_0[\mathbf{a}, \mathbf{b}]$ and*

$$g\chi_{(\mathbf{a},\mathbf{b})} = \sum_{Y \in \mathcal{P}_m} (-1)^{m-\mu_0(\Gamma(Y))} g\chi_Y. \tag{8}$$

P r o o f. It is clear that (8) holds for any real-valued function g defined on $[\mathbf{a}, \mathbf{b}]$. It remains to prove that $g\chi_{(\mathbf{a},\mathbf{b})} \in \mathrm{BV}_0[\mathbf{a}, \mathbf{b}]$ whenever $g \in \mathrm{BV}_{\mathrm{HK}}[\mathbf{a}, \mathbf{b}]$. But this is an immediate consequence of (8) and Lemma 4.4. The proof is complete. □

Our next step is to prove Theorem 4.8, which is a special case of Theorem 4.10. We need the following theorems.

Theorem 4.6. *If $f \in L^1[\mathbf{a}, \mathbf{b}]$ and $g \in \mathrm{BV}_0[\mathbf{a}, \mathbf{b}]$, then $fg \in L^1[\mathbf{a}, \mathbf{b}]$ and*

$$\int_{[\mathbf{a},\mathbf{b}]} f(\mathbf{x})g(\mathbf{x})\,\mathrm{d}\mathbf{x} = \int_{[\mathbf{a},\mathbf{b}]} \biggl\{ \int_{[\mathbf{x},\mathbf{b}]} f(\mathbf{t})\,\mathrm{d}\mathbf{t} \biggr\}\,\mathrm{d}g(\mathbf{x}).$$

P r o o f. Let $\{\varphi_n\}_{n=1}^{\infty}$ be given as in Theorem 2.2. For each $n \in \mathbb{N}$ we have, by Fubini's theorem and Theorem 3.2,

$$\begin{aligned}\int_{[\mathbf{a},\mathbf{b}]} f(\mathbf{x})\biggl\{ \int_{[\mathbf{a},\mathbf{x}]} \varphi_n(\mathbf{t})\,\mathrm{d}\mathbf{t} \biggr\}\,\mathrm{d}\mathbf{x} &= \int_{[\mathbf{a},\mathbf{b}]} \biggl\{ \int_{[\mathbf{t},\mathbf{b}]} f(\mathbf{x})\,\mathrm{d}\mathbf{x} \biggr\}\varphi_n(\mathbf{t})\,\mathrm{d}\mathbf{t} \\ &= \int_{[\mathbf{a},\mathbf{b}]} \biggl\{ \int_{[\mathbf{t},\mathbf{b}]} f(\mathbf{x})\,\mathrm{d}\mathbf{x} \biggr\}\,\mathrm{d}g_n(\mathbf{t}),\end{aligned}$$

where $g_n(\mathbf{t}) := \int_{[\mathbf{a},\mathbf{t}]} \varphi_n(\mathbf{x})\,\mathrm{d}\mathbf{x}$. Therefore Lebesgue's dominated convergence theorem and Theorem 3.3 yield the desired result.

Let $|I| := \mu_m(I)$ $(I \in \mathcal{I}_m([\mathbf{a},\mathbf{b}])$, where μ_m denotes the m-dimensional Lebesgue measure.

Theorem 4.7. *If $f \in \mathrm{HK}[\mathbf{a},\mathbf{b}]$ and $g \in \mathrm{BV}_0[\mathbf{a},\mathbf{b}]$, then*

$$\begin{aligned}&\Bigl|\sum_{i=1}^{p}\Bigl\{f(\xi_i)g(\xi_i)\,|I_i| - \int_{[\mathbf{a},\mathbf{b}]}\Bigl((\mathrm{HK})\int_{[\mathbf{x},\mathbf{b}]} f(\mathbf{t})\chi_{I_i}(\mathbf{t})\,\mathrm{d}\mathbf{t}\Bigr)\,\mathrm{d}g(\mathbf{x})\Bigr\}\Bigr|\\ &\leqslant \sum_{i=1}^{p}|f(\xi_i)|\int_{I_i}|g(\xi_i)-g(\mathbf{t})|\,\mathrm{d}\mathbf{t}\\ &\quad+\sup_{\mathbf{x}\in[\mathbf{a},\mathbf{b}]}\Bigl|(\mathrm{HK})\int_{[\mathbf{x},\mathbf{b}]}\sum_{i=1}^{p}\Bigl\{f(\xi_i)\chi_{I_i}(\mathbf{t}) - f(\mathbf{t})\chi_{I_i}(\mathbf{t})\Bigr\}\,\mathrm{d}\mathbf{t}\Bigr|(\mathrm{Var}(g,[\mathbf{a},\mathbf{b}]))\end{aligned}$$

for each partial partition $\{(I_i,\xi_1),\dots,(I_p,\xi_p)\}$ of $[\mathbf{a},\mathbf{b}]$.

P r o o f. By the triangle inequality,

$$\begin{aligned}&\Bigl|\sum_{i=1}^{p}\Bigl\{f(\xi_i)g(\xi_i)\,|I_i| - \int_{[\mathbf{a},\mathbf{b}]}\Bigl((\mathrm{HK})\int_{[\mathbf{x},\mathbf{b}]} f(\mathbf{t})\chi_{I_i}(\mathbf{t})\,\mathrm{d}\mathbf{t}\Bigr)\,\mathrm{d}g(\mathbf{x})\Bigr\}\Bigr|\\ &\leqslant \sum_{i=1}^{p}|f(\xi_i)|\Bigl|g(\xi_i)\,|I_i| - \int_{I_i} g(\mathbf{t})\,\mathrm{d}\mathbf{t}\Bigr|\\ &\quad+\Bigl|\sum_{i=1}^{p}\Bigl\{f(\xi_i)\int_{I_i} g(\mathbf{t})\,\mathrm{d}\mathbf{t} - \int_{[\mathbf{a},\mathbf{b}]}\Bigl((\mathrm{HK})\int_{[\mathbf{x},\mathbf{b}]} f(\mathbf{t})\chi_{I_i}(\mathbf{t})\,\mathrm{d}\mathbf{t}\Bigr)\,\mathrm{d}g(\mathbf{x})\Bigr\}\Bigr|.\end{aligned}$$

It is evident that

$$\sum_{i=1}^{p}|f(\xi_i)|\Bigl|g(\xi_i)\,|I_i| - \int_{I_i} g(\mathbf{t})\,\mathrm{d}\mathbf{t}\Bigr| \leqslant \sum_{i=1}^{p}|f(\xi_i)|\int_{I_i}|g(\xi_i)-g(\mathbf{t})|\,\mathrm{d}\mathbf{t}$$

and, by Theorem 4.6,

$$\int_{I_i} g(\mathbf{t})\,\mathrm{d}\mathbf{t} = \int_{[\mathbf{a},\mathbf{b}]}\Bigl(\int_{[\mathbf{x},\mathbf{b}]}\chi_{I_i}(\mathbf{t})\,\mathrm{d}\mathbf{t}\Bigr)\,\mathrm{d}g(\mathbf{x}),$$

so that

$$\begin{aligned}&\Bigl|\sum_{i=1}^{p}\Bigl\{f(\xi_i)\int_{I_i} g(\mathbf{t})\,\mathrm{d}\mathbf{t} - \int_{[\mathbf{a},\mathbf{b}]}\Bigl((\mathrm{HK})\int_{[\mathbf{x},\mathbf{b}]} f(\mathbf{t})\chi_{I_i}(\mathbf{t})\,\mathrm{d}\mathbf{t}\Bigr)\,\mathrm{d}g(\mathbf{x})\Bigr\}\Bigr|\\ &=\Bigl|\int_{[\mathbf{a},\mathbf{b}]}\Bigl((\mathrm{HK})\int_{[\mathbf{x},\mathbf{b}]}\sum_{i=1}^{p}\Bigl\{f(\xi_i)\chi_{I_i}(\mathbf{t}) - f(\mathbf{t})\chi_{I_i}(\mathbf{t})\Bigr\}\,\mathrm{d}\mathbf{t}\Bigr)\,\mathrm{d}g(\mathbf{x})\Bigr|\\ &\leqslant \sup_{\mathbf{x}\in\,[\mathbf{a},\mathbf{b}]}\Bigl|(\mathrm{HK})\int_{[\mathbf{x},\mathbf{b}]}\sum_{i=1}^{p}\Bigl\{f(\xi_i)\chi_{I_i}(\mathbf{t}) - f(\mathbf{t})\chi_{I_i}(\mathbf{t})\Bigr\}\,\mathrm{d}\mathbf{t}\Bigr|(\mathrm{Var}(g,[\mathbf{a},\mathbf{b}])).\end{aligned}$$

Combining the above estimates proves the theorem.

Theorem 4.8. *If* $f \in \mathrm{HK}[\mathbf{a},\mathbf{b}]$ *and* $g \in \mathrm{BV}_0[\mathbf{a},\mathbf{b}]$, *then* $fg \in \mathrm{HK}[\mathbf{a},\mathbf{b}]$ *and*

$$(9)\qquad (\mathrm{HK})\int_{[\mathbf{a},\mathbf{b}]} f(\mathbf{x})g(\mathbf{x})\,\mathrm{d}\mathbf{x} = \int_{[\mathbf{a},\mathbf{b}]} \left\{(\mathrm{HK})\int_{[\mathbf{x},\mathbf{b}]} f(\mathbf{t})\,\mathrm{d}\mathbf{t}\right\}\mathrm{d}g(\mathbf{x}).$$

P r o o f. We may assume that $\mathrm{Var}(g,[\mathbf{a},\mathbf{b}]) < 1$. According to the Saks-Henstock Lemma, given $\varepsilon > 0$ there exists a gauge δ_1 on $[\mathbf{a},\mathbf{b}]$ such that

$$(10)\qquad \sum_{i=1}^{q}\left|f(\zeta_i)\,|J_i| - (\mathrm{HK})\int_{J_i} f(\mathbf{x})\,\mathrm{d}\mathbf{x}\right| < \frac{\varepsilon}{2^m+2}$$

for each δ_1-fine partial partition $\{(J_1,\zeta_1),\ldots,(J_q,\zeta_q)\}$ of $[\mathbf{a},\mathbf{b}]$. For each $\mathbf{x}\in[\mathbf{a},\mathbf{b}]$, it follows from (10) that

$$(11)\qquad \left|\sum_{i=1}^{q}\left\{f(\zeta_i)\mu_m([\mathbf{x},\mathbf{b}]\cap J_i) - (\mathrm{HK})\int_{[\mathbf{a},\mathbf{b}]} f(\mathbf{t})\chi_{[\mathbf{x},\mathbf{b}]\,\cap\, J_i}(\mathbf{t})\,\mathrm{d}\mathbf{t}\right\}\right| < \frac{2^m\varepsilon}{2^m+2}$$

for each δ_1-fine partial partition $\{(J_1,\zeta_1),\ldots,(J_q,\zeta_q)\}$ of $[\mathbf{a},\mathbf{b}]$.

As $f \in \mathrm{BV}_0[\mathbf{a},\mathbf{b}]$, it follows from Theorem 2.3 that there exists a gauge δ_2 on $[\mathbf{a},\mathbf{b}]$ such that

$$\sum_{j=1}^{r}|f(\mathbf{z}_j)|\int_{K_i}|g(\mathbf{z}_j)-g(\mathbf{t})|\;\mathrm{d}\mathbf{t} < \frac{\varepsilon}{2^m+2}$$

for each δ_2-fine McShane partial partition $\{(K_1,\mathbf{z}_1),\ldots,(K_r,\mathbf{z}_r)\}$ of $[\mathbf{a},\mathbf{b}]$.

Define a gauge δ on $[\mathbf{a},\mathbf{b}]$ by $\delta(\mathbf{x}) = \min\{\delta_1(\mathbf{x}),\delta_2(\mathbf{x})\}$. For each δ-fine partition $\{(I_1,\xi_1),\ldots,(I_p,\xi_p)\}$ of $[\mathbf{a},\mathbf{b}]$, we infer from Theorem 4.7 and the above estimates that

$$\begin{aligned}
&\left|\sum_{i=1}^{p} f(\xi_i)g(\xi_i)\,|I_i| - \int_{[\mathbf{a},\mathbf{b}]}\left((\mathrm{HK})\int_{[\mathbf{x},\mathbf{b}]} f(\mathbf{t})\,\mathrm{d}\mathbf{t}\right)\mathrm{d}g(\mathbf{x})\right| \\
&= \left|\sum_{i=1}^{p}\left\{f(\xi_i)g(\xi_i)\,|I_i| - \int_{[\mathbf{a},\mathbf{b}]}\left((\mathrm{HK})\int_{[\mathbf{x},\mathbf{b}]} f(\mathbf{t})\chi_{I_i}(\mathbf{t})\,\mathrm{d}\mathbf{t}\right)\mathrm{d}g(\mathbf{x})\right\}\right| \\
&\leqslant \sum_{i=1}^{p}|f(\xi_i)|\int_{I_i}|\,g(\xi_i)-g(\mathbf{t})|\;\mathrm{d}\mathbf{t} \\
&\quad + \sup_{x\in[a,b]}\left|(\mathrm{HK})\int_{[\mathbf{x},\mathbf{b}]}\sum_{i=1}^{p}\left\{f(\xi_i)\chi_{I_i}(\mathbf{t}) - f(\mathbf{t})\chi_{I_i}(\mathbf{t})\right\}\mathrm{d}\mathbf{t}\right|(\mathrm{Var}(g,[\mathbf{a},\mathbf{b}])) \\
&< \varepsilon,
\end{aligned}$$

thereby completing the proof of the theorem.

Our next aim is to deduce Kurzweil's multidimensional integration by parts formula [1, Theorem 2.10]. For $\mathbf{s},\mathbf{t}\in[\mathbf{a},\mathbf{b}]$, we set

$$\langle\mathbf{s},\mathbf{t}\rangle:=\{(x_1,\dots,x_m)\colon\ \min\{s_i,t_i\}\leqslant x_i\leqslant\max\{s_i,t_i\}\text{ for each } i=1,\dots,m\}.$$

For each $f\in\mathrm{HK}[\mathbf{a},\mathbf{b}]$ and $\alpha\in[\mathbf{a},\mathbf{b}]$, we define a function $\widetilde{F}_\alpha$ on $[\mathbf{a},\mathbf{b}]$ by

$$\widetilde{F}_\alpha(\mathbf{x})=\Big\{(\mathrm{HK})\int_{\langle\alpha,\mathbf{x}\rangle}f(\mathbf{t})\,\mathrm{d}\mathbf{t}\Big\}\prod_{i=1}^{m}\operatorname{sgn}(x_i-\alpha_i).$$

It is well known that $\widetilde{F}_\alpha\in C[\mathbf{a},\mathbf{b}]$. The next theorem gives our multidimensional integration by parts formula.

Theorem 4.9. *If $f\in\mathrm{HK}[\mathbf{a},\mathbf{b}]$, $\alpha\in[\mathbf{a},\mathbf{b}]$ and $g\in\mathrm{BV}_{\mathrm{HK}}[\mathbf{a},\mathbf{b}]$, then $fg\in\mathrm{HK}[\mathbf{a},\mathbf{b}]$ and*

$$(12)\qquad (\mathrm{HK})\int_{[\mathbf{a},\mathbf{b}]}f(\mathbf{x})g(\mathbf{x})\,\mathrm{d}\mathbf{x}=\sum_{Y\in\mathcal{P}_m}(-1)^{\mu_0(\Gamma(Y))}\Big\{\int_{[\mathbf{a},\mathbf{b}]}\widetilde{F}_\alpha\,\mathrm{d}(g\chi_Y)\Big\}.$$

P r o o f. Let $g_0=g\chi_{(\mathbf{a},\mathbf{b})}$. By Theorems 4.5 and 4.8, $fg_0\in\mathrm{HK}[\mathbf{a},\mathbf{b}]$. As $g=g_0$ μ_m-almost everywhere on $[\mathbf{a},\mathbf{b}]$, we see that $fg\in\mathrm{HK}[\mathbf{a},\mathbf{b}]$ and

$$(\mathrm{HK})\int_{[\mathbf{a},\mathbf{b}]}f(\mathbf{x})g(\mathbf{x})\,\mathrm{d}\mathbf{x}=(\mathrm{HK})\int_{[\mathbf{a},\mathbf{b}]}f(\mathbf{x})g_0(\mathbf{x})\,\mathrm{d}\mathbf{x}.$$

By Theorem 4.8 again,

$$(\mathrm{HK})\int_{[\mathbf{a},\mathbf{b}]}f(\mathbf{x})g_0(\mathbf{x})\,\mathrm{d}\mathbf{x}=\int_{[\mathbf{a},\mathbf{b}]}\Big\{(\mathrm{HK})\int_{[\mathbf{x},\mathbf{b}]}f(\mathbf{t})\,\mathrm{d}\mathbf{t}\Big\}\,\mathrm{d}g_0(\mathbf{x}).$$

Using the additivity of the indefinite HK-integral of f over $[\mathbf{a},\mathbf{b}]$ and [1, Lemma 1.3], we see that

$$(\mathrm{HK})\int_{[\mathbf{x},\mathbf{b}]}f(\mathbf{t})\,\mathrm{d}\mathbf{t}=\Delta_{\widetilde{F}_\alpha}([\mathbf{x},\mathbf{b}])$$

for each $\mathbf{x}\in[\mathbf{a},\mathbf{b}]$. Thus it follows from [1, (1.9), (1.8)], the linearity of the Riemann-Stieltjes integral and Theorem 3.4 that

$$\int_{[\mathbf{a},\mathbf{b}]}\Delta_{\widetilde{F}_\alpha}([\mathbf{x},\mathbf{b}])\,\mathrm{d}g_0(\mathbf{x})=\int_{[\mathbf{a},\mathbf{b}]}(-1)^m\widetilde{F}_\alpha(\mathbf{x})\,\mathrm{d}g_0(\mathbf{x}).$$

Now the linearity of the Riemann-Stieltjes integral and Theorem 4.5 imply that

$$\int_{[\mathbf{a},\mathbf{b}]} (-1)^m \widetilde{F}_\alpha(\mathbf{x})\,\mathrm{d}g_0(\mathbf{x}) = \sum_{Y\in\mathcal{P}_m} (-1)^{\mu_0(\Gamma(Y))}\Big\{\int_{[\mathbf{a},\mathbf{b}]} \widetilde{F}_\alpha\,\mathrm{d}(g\chi_Y)\Big\}.$$

Combining the above equalities yields (12). The proof is complete.

Let $\sigma\Big(\prod\limits_{k=1}^m Y_k\Big) := \{i \in \{1,\dots,m\}\colon\ Y_i = \{a_i\}\}$. It remains to show that Theorem 4.9 is equivalent to the following Kurzweil's multidimensional integration by parts formula [1, Theorem 2.10].

Theorem 4.10. *If* $f \in \mathrm{HK}[\mathbf{a},\mathbf{b}]$, $g \in \mathrm{BV_{HK}}[\mathbf{a},\mathbf{b}]$ *and* $\alpha \in [\mathbf{a},\mathbf{b}]$, *then* $fg \in \mathrm{HK}[\mathbf{a},\mathbf{b}]$ *and*

$$\begin{aligned}(\mathrm{HK})\int_{[\mathbf{a},\mathbf{b}]} f(\mathbf{x})g(\mathbf{x})\,\mathrm{d}\mathbf{x} = &\sum_{\substack{\{\mathbf{c}\}\in\mathcal{P}_m\\ \mu_0(\Gamma(\{\mathbf{c}\}))=0}} (-1)^{\sigma(\{\mathbf{c}\})}\widetilde{F}_\alpha(\mathbf{c})g(\mathbf{c})\\ &+\sum_{k=1}^m \sum_{\substack{Y\in\mathcal{P}_m\\ \mu_0(\Gamma(Y))=k}} (-1)^k \int_{\prod\limits_{\substack{j=1\\ j\in\Gamma(Y)}}^m [a_j,b_j]} (-1)^{\sigma(Y)}\widetilde{F}_\alpha\big|_Y\,\mathrm{d}(g\big|_Y).\end{aligned}$$

P r o o f. If $Y\in\mathcal{P}_m$ and $\mu_0(\Gamma(Y)) = 0$, then there exists a vertex $\mathbf{c}$ of $[\mathbf{a},\mathbf{b}]$ such that

$$\int_{[\mathbf{a},\mathbf{b}]} \widetilde{F}_\alpha\,\mathrm{d}(g\chi_Y) = (-1)^{\sigma(\{\mathbf{c}\})}\widetilde{F}_\alpha(\mathbf{c})g(\mathbf{c}).$$

A similar argument shows that if $Y\in\mathcal{P}_m$ and $\mu_0(\Gamma(Y)) > 0$, then

$$\int_{[\mathbf{a},\mathbf{b}]} \widetilde{F}_\alpha\,\mathrm{d}(g\chi_Y) = \int_{\prod\limits_{\substack{j=1\\ j\in\Gamma(Y)}}^m [a_j,b_j]} (-1)^{\sigma(Y)}\widetilde{F}_\alpha\big|_Y\,\mathrm{d}(g\big|_Y).$$

Hence, as a consequence of Theorem 4.9, we get the desired result:

$$\begin{aligned}(\mathrm{HK})\int_{[\mathbf{a},\mathbf{b}]} f(\mathbf{x})g(\mathbf{x})\,\mathrm{d}\mathbf{x} &= \sum_{Y\in\mathcal{P}_m} (-1)^{\mu_0(\Gamma(Y))}\Big\{\int_{[\mathbf{a},\mathbf{b}]} \widetilde{F}_\alpha\,\mathrm{d}(g\chi_Y)\Big\}\\ &= \sum_{\substack{Y\in\mathcal{P}_m\\ \Gamma(Y)=\emptyset}} (-1)^{\mu_0(\Gamma(Y))}\Big\{\int_{[\mathbf{a},\mathbf{b}]} \widetilde{F}_\alpha\,\mathrm{d}(g\chi_Y)\Big\} + \sum_{k=1}^m \sum_{\substack{Y\in\mathcal{P}_m\\ \mu_0(\Gamma(Y))=k}} (-1)^k\Big\{\int_{[\mathbf{a},\mathbf{b}]} \widetilde{F}_\alpha\,\mathrm{d}(g\chi_Y)\Big\}\\ &= \sum_{\substack{\{\mathbf{c}\}\in\mathcal{P}_m\\ \mu_0(\Gamma(\{\mathbf{c}\}))=0}} (-1)^{\sigma(\{\mathbf{c}\})}\widetilde{F}_\alpha(\mathbf{c})g(\mathbf{c})\\ &\quad+\sum_{k=1}^m \sum_{\substack{Y\in\mathcal{P}_m\\ \mu_0(\Gamma(Y))=k}} (-1)^k \int_{\prod\limits_{\substack{j=1\\ j\in\Gamma(Y)}}^m [a_j,b_j]} (-1)^{\sigma(Y)}\widetilde{F}_\alpha\big|_Y\,\mathrm{d}(g\big|_Y).\end{aligned}$$

The proof of Theorem 4.10 depends heavily on (11), which is also true for some other generalized Riemann integrals; more precisely, we have

R e m a r k 4.11. Theorem 4.10 also holds if the Henstock-Kurzweil integral is replaced by any of the following generalized Riemann integrals:

(i) the Lebesgue integral (see also [19], [21]);
(ii) the Cauchy-Lebesgue integral;
(iii) the strong ϱ-integral in [6];
(iv) the $\mathcal{R}$-integral in [10].

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