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ON THE σ-FINITENESS OF A VARIATIONAL MEASURE

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Abstract. The σ-finiteness of a variational measure, generated by a real valued function, is proved whenever it is σ-finite on all Borel sets that are negligible with respect to a σ-finite variational measure generated by a continuous function.

Keywords: variational measure, H-differentiable, H-density

MSC 2000: 26A39, 26A24

1. Introduction

In 1994, a question was posed by W. Pfeffer (see [13]) whether the absolute continuity of a variational measure, generated by a real valued function, with respect to the Lebesgue measure would imply its σ-finiteness. The affirmative answer was first given in [2], providing a full descriptive characterization of the Henstock-Kurzweil integral (see also [14], and [4], [5], [6], [8] for higher dimensional results). Then in [18], strengthening the result presented in [2], the author proved that a variational measure is σ-finite whenever it is σ-finite on all subsets of zero Lebesgue measure (see also [3] for a variational measure related to a certain class of differentiation bases). In this paper we show that the same result holds if the Lebesgue measure is replaced by a suitable variational measure. Namely, the variational measure V_*F, generated by a function $F\colon [a,b] \to \mathbb{R}$, is σ-finite on $[a,b]$ whenever it is σ-finite on all subsets having measure zero with respect to a σ-finite variational measure V_*U generated by a continuous function $U\colon [a,b] \to \mathbb{R}$. We derive some results on the differentiability of the function F with respect to U, and a representation theorem for the variational measure V_*F in terms of the Lebesgue integral.

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2. Preliminaries

If $E \subset \mathbb{R}$, then $|E|$ and $\mathrm{int}E$ denote the outer Lebesgue measure and the interior of E, respectively. All functions we consider are real-valued. By $(\mathcal{L})\int$ we denote the Lebesgue integral. We always consider nondegenerate subintervals of $\mathbb{R}$. For $c, d \in \mathbb{R}$ with $c < d$, we denote by $[c, d]$ the compact subinterval of $\mathbb{R}$ with endpoints c and d, and by (c, d) the open one. A collection of intervals is called *nonoverlapping* whenever their interiors are disjoint. Throughout this note $[a, b]$ will be a fixed interval. A *partition in* $[a, b]$ is a collection $P = \{([a_1, b_1], x_1), \dots, ([a_p, b_p], x_p)\}$ where $[a_1, b_1], \dots, [a_p, b_p]$ are nonoverlapping subintervals of $[a, b]$ and $x_i \in [a_i, b_i]$ for $i = 1, \dots, p$. A positive function δ on $E \subset [a, b]$ is called a *gauge* on E. Given a gauge δ on $[a, b]$, a *partition* $P = \{([a_1, b_1], x_1), \dots, ([a_p, b_p], x_p)\}$ in $[a, b]$ is called

(i) *δ-fine* if $b_i - a_i < \delta(x_i)$, $i = 1, \dots, p$;

(ii) *of* $[a, b]$ if $\bigcup_{i=1}^{p} [a_i, b_i] = [a, b]$;

(iii) *anchored in* E if $x_i \in E \subset [a, b]$ for each $i = 1, \dots, p$.

Let $H\colon [a, b] \to \mathbb{R}$ be a given function. The *variational measure* of H (see [17] and [2]) is the metric outer measure defined for each $E \subset [a, b]$ by

$$V_*H(E) = \inf_{\delta} \sup_{P} \sum_{i=1}^{p} |H(b_i) - H(a_i)|$$

where the infimum is taken over all gauges δ on E, and the supremum over all δ-fine partitions $P = \{([a_1, b_1], x_1), \dots, ([a_p, b_p], x_p)\}$ anchored in E.

If $V_*H(N) = 0$, then the set $N \subset [a, b]$ is called *H-negligible*. For details on metric outer measure we refer to [15] and [17]. We recall that H-negligible sets are V_*H-measurable, and any set that differs from a V_*H-measurable one by an H-negligible set is itself V_*H-measurable. We also recall that the restriction of a metric outer measure to the Borel sets is a measure.

V_*H is said to be *σ-finite* on $E \subset [a, b]$ if the set E is the union of sets E_n, $n = 1, 2, \dots$, satisfying $V_*H(E_n) < \infty$. A variational measure V_*F is said to be *absolutely continuous* with respect to V_*H if $V_*F(N)=0$ for any H-negligible set $N \subset [a, b]$.

Remark 2.1. (i) Let $x \in [a, b]$. Then H is continuous at x if and only if $V_*H(\{x\}) = 0$.

(ii) If H is a continuous monotone function, then V_*H is the Lebesgue-Stieltjes measure associated with H, in which case

(a) $V_*H([c, d]) = H(d) - H(c)$ for any subinterval $[c, d] \subset [a, b]$;

(b) V_*H is G_δ-regular, i.e. for every $E \subset [a,b]$ there is a V_*H-measurable G_δ set $Y \subset [a,b]$ containing E for which $V_*H(E) = V_*H(Y)$ (see [17, p. 62]).

According to [10, p. 416] a set $E \subset [a,b]$ is said to be H-*null* if it is the union of a countable set and an H-negligible set. A property is said to hold H-*almost everywhere* (abbreviated as H-*a.e.*) if the set of points where it fails to hold is H-null. However, if H is a continuous function, by Remark 2.1(i) we have that a set is H-null if and only if it is H-negligible.

Let F and H be any two functions on $[a,b]$. We need some definitions and results on the differentiability of the function F with respect to H. The *lower* and *upper derivative* of F with respect to H,

$$\underline{D}_H F(x) = \liminf_{y \to x} \frac{F(y) - F(x)}{H(y) - H(x)} \quad \text{and} \quad \overline{D}_H F(x) = \limsup_{y \to x} \frac{F(y) - F(x)}{H(y) - H(x)},$$

are defined for all $x \in [a,b]$ for which $H(y) \neq H(x)$ in a neighborhood of x.

If $\underline{D}_H F(x) = \overline{D}_H F(x) \neq \pm\infty$ this common value is denoted by F'_H and F is said to be H-differentiable at x. Moreover, set

$$|\overline{D}|_H F(x) = \limsup_{y \to x} \frac{|F(y) - F(x)|}{|H(y) - H(x)|}.$$

The following result on H-differentiability will be useful. We point out that in [10] a function F is said to be VBG^o if V_*F is σ-finite on $[a,b]$.

Lemma 2.2 [10, Proposition 3.10]**.** *Let F, $H\colon [a,b] \to \mathbb{R}$ be given. If the variational measures V_*F and V_*H are σ-finite on $[a,b]$, then F is H-differentiable H-a.e. in $[a,b]$.*

The following lemma can be proved by standard arguments (cf. for example [12, Proposition 5.3.3]).

Lemma 2.3. *Let $F\colon [a,b] \to \mathbb{R}$ be given. If $H\colon [a,b] \to \mathbb{R}$ is a strictly increasing function, then for each $x \in [a,b]$ we have*

$$\overline{D}_H F(x) = \inf_{\delta} \sup_{[c,d]} \frac{F(d) - F(c)}{H(d) - H(c)} \tag{1}$$

where δ is a positive number and the supremum is taken over all subintervals $[c,d]$ of $[a,b]$ with $x \in [c,d]$ and $d - c < \delta$. If in addition H and F are continuous at x, then the supremum in (1) *can be taken over all subintervals $[c,d]$ of $[a,b]$ with $x \in (c,d)$ and $d - c < \delta$.*

Lemma 2.4. *Let $F\colon [a,b] \to \mathbb{R}$ be a continuous function. If $H\colon [a,b] \to \mathbb{R}$ is a continuous strictly increasing function, then $\overline{D}_H F$ is Borel-measurable.*

Proof. In view of Lemma 2.3, $\overline{D}_H F(x)$ can be written as in (1) where the supremum is taken over all subintervals $[c,d]$ of $[a,b]$ with $x \in (c,d)$ and $d - c < \delta$. Then by standard arguments (see for example [17, Theorem 4.2]), the upper derivative $\overline{D}_H F$ is Borel-measurable. □

Clearly the same considerations of Lemma 2.3 and Lemma 2.4 apply to $\underline{D}_H F(x)$ and $|\overline{D}|_H F(x)$.

3. The variational measure

In order to study the properties of a variational measure, we introduce the following notion of H-density.

Definition 3.1. Let $H\colon [a,b] \to \mathbb{R}$ and let E be a subset of $[a,b]$. We say that a point $x \in [a,b]$ is a *point of H-density* for E if

$$\lim_{r \to 0^+} \frac{V_* H(E \cap [x-r, x+r])}{V_* H([x-r, x+r])} = 1.$$

The following lemma is a particular case of [11, Corollary 2.14].

Lemma 3.2. *Let $H\colon [a,b] \to \mathbb{R}$ be a continuous and strictly increasing function. Let E be a V_*H-measurable subset of $[a,b]$. Then H-almost all points of E are H-density points for E.*

In view of Remark 2.1 (ii) we have that if $H\colon [a,b] \to \mathbb{R}$ is a continuous and strictly increasing function, then V_*H is the corresponding Lebesgue-Stieltjes measure. Now we point out (see for example [7]) that the Vitali covering theorem holds for V_*H. Precisely, if a class of closed intervals covers a subset $A \subset [a,b]$ in the sense of Vitali, then there is a countable disjoint sequence of those intervals whose union differs from A by at most an H-negligible subset. In the following proposition we prove a result on the σ-finiteness of a variational measure by a technique similar to that used in [3, Theorem 3.1].

Proposition 3.3. *Let $F\colon [a,b] \to \mathbb{R}$ be given and let $H\colon [a,b] \to \mathbb{R}$ be a continuous and strictly increasing function. If V_*F is σ-finite on all H-negligible Borel subsets of $[a,b]$, then V_*F is σ-finite on $[a,b]$.*

P r o o f. Let Q be the set of all points $x \in [a,b]$ for which V_*F is not σ-finite on any open interval (c,d) of $[a,b]$ containing x. Clearly Q is closed and has no isolated points. Thus Q is a perfect set.

Now for any given interval $I \subset [a,b]$, let $\{I_j\}$ denote the sequence of intervals complementary to Q in I. Then a compactness argument shows that V_*F is σ-finite on I_j for each j. In particular, V_*F is σ-finite on the complement of Q in $[a,b]$. Therefore if $V_*H(Q) = 0$, by the hypothesis it follows that V_*F is σ-finite on $[a,b]$.

Assume by contradiction that $V_*H(Q) > 0$ and let K_Q be the set of all points of Q which are H-density points for Q. By Lemma 3.2, $V_*H(Q \setminus K_Q) = 0$. Let K denote the set of all $x \in K_Q$ for which the following condition holds: if $I \subset [a,b]$ is any interval containing x, then $V_*H(K_Q \cap \mathrm{int} I) > 0$. We claim that $V_*H(K_Q \setminus K) = 0$. The family $\mathcal{B}$ of all intervals $I \subset [a,b]$ for which $V_*H(K_Q \cap \mathrm{int} I) = 0$ is a Vitali cover of the set $K_Q \setminus K$. By the Vitali covering theorem for Lebesgue-Stieltjes measures there is a disjoint sequence $\{I_{x_i}\}$ in $\mathcal{B}$ with $x_i \in (K_Q \setminus K) \cap I_{x_i}$, such that

$$V_*H\biggl((K_Q \setminus K) \setminus \Bigl(\bigcup_i I_{x_i}\Bigr)\biggr) = 0. \tag{2}$$

For each i we have $V_*H(K_Q \cap \mathrm{int} I_{x_i}) = 0$, which together with the continuity of H implies $V_*H(K_Q \cap I_{x_i}) = 0$. Then we have

$$V_*H\biggl(K_Q \cap \Bigl(\bigcup_i I_{x_i}\Bigr)\biggr) = 0. \tag{3}$$

Thus by (2) and (3) we have

$$V_*H(K_Q \setminus K) = V_*H\biggl((K_Q \setminus K) \setminus \Bigl(\bigcup_i I_{x_i}\Bigr)\biggr) + V_*H\biggl((K_Q \setminus K) \cap \Bigl(\bigcup_i I_{x_i}\Bigr)\biggr) = 0.$$

We show now that V_*F is not σ-finite on $K \cap I$, whenever I is an interval of $[a,b]$ which intersects K. As before let $\{I_j\}$ denote the sequence of intervals complementary to Q in I. Write

$$I = (K \cap I) \cup ((Q \setminus K) \cap I) \cup \Bigl(\bigcup_j I_j\Bigr),$$

and by Remark 2.1 (ii)(b) find an H-negligible G_δ set $Y \subset [a,b]$ containing $Q \setminus K$. Then we get

$$V_*F(I) \leqslant V_*F(K \cap I) + V_*F(Y \cap I) + V_*F\Bigl(\bigcup_j I_j\Bigr).$$

By the hypothesis V_*F is σ-finite on $Y \cap I$, and we have shown that it is σ-finite on $\bigcup_j I_j$. Hence the σ-finiteness of V_*F on $K \cap I$ would imply its σ-finiteness on I, which is not the case. This implies that for any gauge δ we have

$$\sup_P \sum_{i=1}^{p} |F(b_i) - F(a_i)| = \infty \tag{4}$$

where $P = \{([a_1, b_1], x_1), \dots, ([a_p, b_p], x_p)\}$ runs over all δ-fine partitions anchored in $K \cap I$.

Fix an open interval (c, d) containing a point of K. In view of Remark 2.1 (ii)(a), we may assume that $V_*H((c, d)) < 1/2$. Using (4) we can choose a finite collection $\{[a_i^{(1)}, b_i^{(1)}],\ i = 1, \dots p_1\}$ of intervals contained in (c, d), such that

$$\sum_{i=1}^{p_1} |F(b_i^{(1)}) - F(a_i^{(1)})| > 2.$$

We may assume that the family consists of at least two intervals. Also we have that the interior of each $[a_i^{(1)}, b_i^{(1)}]$ intersects K. Clearly,

$$\sum_{i=1}^{p_1} V_*H([a_i^{(1)}, b_i^{(1)}]) < 1/2.$$

Thinking of $[a, b]$ as $[a_1^{(0)}, b_1^{(0)}]$, we construct inductively finite collections $\{[a_i^{(k)}, b_i^{(k)}],\ i = 1, \dots, p_k\}$ such that the following conditions are satisfied for $k = 1, 2, \dots$:

(i) $K \cap (a_i^{(k)}, b_i^{(k)}) \neq \emptyset$ for $i = 1, \dots p_k$;

(ii) each $[a_i^{(k)}, b_i^{(k)}]$ is contained in some $[a_j^{(k-1)}, b_j^{(k-1)}]$;

(iii) each $[a_j^{(k-1)}, b_j^{(k-1)}]$ contains at least two intervals $[a_i^{(k)}, b_i^{(k)}]$;

(iv) $\sum_{i=1}^{p_k} V_*H([a_i^{(k)}, b_i^{(k)}]) < 2^{-k}$;

(v) $\sum_{i:\ [a_i^{(k)}, b_i^{(k)}] \subset [a_j^{(k-1)}, b_j^{(k-1)}]} |F(b_i^{(k)}) - F(a_i^{(k)})| > 2^k$ for each $j = 1, \dots p_{k-1}$.

Now we define $N = \bigcap_{k=1}^{\infty} \bigcup_{i=1}^{p_k} [a_i^{(k)}, b_i^{(k)}]$. From conditions (i)–(iv) it follows that N is a perfect H-negligible set. As V_*F is σ-finite on N, we can write $N = \bigcup_{s=1}^{\infty} N_s$, where N_s are disjoint V_*F-measurable subsets of finite V_*F-measure. Choose a gauge δ on N such that for every integer $s \geqslant 1$

$$\sup_P \sum_{i=1}^{p} |F(b_i) - F(a_i)| < \infty$$

where $P = \{([a_1, b_1], x_1), \dots, ([a_p, b_p], x_p)\}$ runs over all δ-fine partitions anchored in N_s. Let $L_m = \{x \in N \colon \delta(x) > 1/m\}$ for $m = 1, 2, \dots$. Since $N = \bigcup_{m,s} (L_m \cap N_s)$, using the Baire category theorem we conclude that there exist integers m and s and an interval I with $N \cap I \neq \emptyset$ such that $L_m \cap N_s$ is a dense subset of $N \cap I$. We may assume $|I| < 1/m$. By the choice of δ we have

$$\sup_P \sum_{i=1}^{p} |F(b_i) - F(a_i)| < \infty \tag{5}$$

where $P = \{([a_1, b_1], x_1), \dots, ([a_p, b_p], x_p)\}$ runs over all δ-fine partitions anchored in $L_m \cap N_s$. Since I intersects N, then for all sufficiently large k there is some j such that $[a_j^{(k-1)}, b_j^{(k-1)}] \subset I$. Each interval $[a_i^{(k)}, b_i^{(k)}] \subset [a_j^{(k-1)}, b_j^{(k-1)}]$ contains a point of N and consequently a point, say x_{ik}, of $L_m \cap N_s$. Then $\{([a_i^{(k)}, b_i^{(k)}], x_{ik}) \colon [a_i^{(k)}, b_i^{(k)}] \subset [a_j^{(k-1)}, b_j^{(k-1)}]\}$ is a δ-fine partition anchored in $L_m \cap N_p$. Condition (v) implies

$$\sum_{i \colon [a_i^{(k)}, b_i^{(k)}] \subset [a_j^{(k-1)}, b_j^{(k-1)}]} |F(b_i^{(k)}) - F(a_i^{(k)})| > 2^k.$$

For a sufficiently large k, the last inequality contradicts (5), and the proposition is proved. □

Theorem 3.4. *Let $F \colon [a, b] \to \mathbb{R}$ be given and let $U \colon [a, b] \to \mathbb{R}$ be a continuous function such that V_*U is σ-finite on $[a, b]$. If V_*F is σ-finite on all U-negligible Borel subsets of $[a, b]$, then V_*F is σ-finite on $[a, b]$.*

Proof. Since U is continuous we observe that V_*U coincides with the full variational measure ΔU^* introduced by Thomson in [17]. Then by [17, Theorem 7.8] the function U is VBG_* in the sense of Saks and by a theorem of Ward (see [16, p. 237]) there exists a continuous strictly increasing function H such that $|\overline{D}|_H U(x)$ is finite at every $x \in [a, b]$. Therefore by [10, Lemma 3.8], V_*U is absolutely continuous with respect to V_*H. This last property and the hypothesis imply that V_*F is σ-finite on all H-negligible Borel subsets of $[a, b]$. By Proposition 3.3, the σ-finiteness of V_*F on $[a, b]$ follows. □

Corollary 3.5. *Let $F \colon [a, b] \to \mathbb{R}$ be given and let $U \colon [a, b] \to \mathbb{R}$ be a continuous function such that V_*U is σ-finite on $[a, b]$. If V_*F is σ-finite on all U-negligible Borel subsets of $[a, b]$, then F is U-differentiable U-a.e. in $[a, b]$.*

Proof. By Theorem 3.4, V_*F is σ-finite on $[a, b]$. Then the corollary follows from Lemma 2.2. □

As a corollary of Theorem 3.4, we obtain a recently published result of V. Ene [9, Theorem 3.2]. We whish to point out that this result allows one to furnish a full descriptive characterization of the Henstock-Stieltjes integral introduced by Faure in [10] (see [9, Theorem 5.1 (iii)]).

Corollary 3.6. *Let $F\colon [a,b]\to\mathbb{R}$ be given and let $U\colon [a,b]\to\mathbb{R}$ be a continuous function such that V_*U is σ-finite on $[a,b]$. If V_*F is absolutely continuous with respect to V_*U, then V_*F is σ-finite on $[a,b]$.*

The following proposition allows us to represent V_*F on Borel sets in terms of the Lebesgue integral with respect to a σ-finite variational measure. It is based on a result of B. Bongiorno [1, Theorem 1] where a finite measure is considered.

Proposition 3.7. *Let $F\colon [a,b]\to\mathbb{R}$ be given and let $U\colon [a,b]\to\mathbb{R}$ be a continuous function such that V_*U is σ-finite on $[a,b]$. If V_*F is absolutely continuous with respect to V_*U, then*

$$V_*F(E) = (\mathcal{L})\int_E |F'_U|\,\mathrm{d}V_*U \tag{6}$$

for every Borel set $E\subset[a,b]$.

Proof. In view of Corollary 3.5 the variational measure V_*F is σ-finite on $[a,b]$. Therefore by Lemma 2.2, F'_U exists U-a.e. We observe that by the absolute continuity of V_*F with respect to V_*U and Remark 2.1(i), the function F is continuous. Let $E\subset[a,b]$ be a Borel set.

Assume first that U is strictly increasing. Since the set of all $x\in[a,b]$ for which $F'_U(x)\neq\overline{D}_UF(x)$ is U-negligible and by Lemma 2.4 $\overline{D}_UF$ is Borel-measurable, we have that F'_U is V_*U-measurable. Thus the Lebesgue integral $(\mathcal{L})\int_E|F'_U|\,\mathrm{d}V_*U$ exists (possibly equal to $+\infty$). By Remark 2.1(ii), V_*U is the Lebesgue-Stieltjes measure generated by U and $V_*U([c,d]) = U(d)-U(c)$. Thus F'_U coincides with the derivative of the set function $[c,d]\to F(d)-F(c)$ with respect to the measure V_*U.

Hence (6) follows by [1, Theorem 1] (cf. also [14, Proposition 10]).

Assume now V_*U to be σ-finite and let H denote, as in the proof of Theorem 3.3, a continuous strictly increasing function on $[a,b]$ such that V_*U is absolutely continuous with respect to V_*H. Then by the first part of the proof we get

$$V_*U(E) = (\mathcal{L})\int_E |U'_H|\,\mathrm{d}V_*H. \tag{7}$$

The hypothesis implies that V_*F is absolutely continuous with respect to V_*H, hence we also have

$$V_*F(E) = (\mathcal{L})\int_E |F'_H|\,\mathrm{d}V_*H. \tag{8}$$

Let N_1 denote the H-negligible, and hence U-negligible, subset of $[a,b]$ such that F'_H and U'_H exist for each $x \in [a,b] \setminus N_1$. Now let $N_2 = \{x \in [a,b] \setminus N_1 \colon\ U'_H(x) = 0\}$. We observe that N_2 is V_*H-measurable. Choose an $\varepsilon > 0$. Given $x \in N_2$, find a $\delta(x) > 0$ such that

$$|U(d) - U(c)| < \varepsilon(H(d) - H(c))$$

for any subinterval $[c,d]$ of $[a,b]$ with $x \in [c,d]$ and $d - c < \delta$. If $P = \{([a_1,b_1],x_1),\dots,([a_p,b_p],x_p)\}$ is a δ-fine partition anchored in N_2, then

$$\sum_{i=1}^{p} |U(b_i) - U(a_i)| < \varepsilon(H(b) - H(a)).$$

As ε is arbitrary, the set N_2 is U-negligible. Then the set $N = N_1 \cup N_2$ is U-negligible, and for any $x \in [a,b] \setminus N$ we have

$$F'_U(x) = F'_H(x)(U'_H(x))^{-1}. \tag{9}$$

Since by (7), for every V_*H-measurable function $g\colon\ [a,b] \to [0,\infty]$ we have

$$(\mathcal{L})\int_E g \,\mathrm{d}V_*U = (\mathcal{L})\int_E |U'_H| g \,\mathrm{d}V_*H,$$

by virtue of (8) and (9) the theorem follows for $g = |F'_U|$. □

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