B. Bongiorno; Luisa Di Piazza; Kazimierz Musiał

Kurzweil-Henstock and Kurzweil-Henstock-Pettis integrability of strongly measurable functions

Mathematica Bohemica, Vol. 131 (2006), No. 2, 211–223

Persistent URL: http://dml.cz/dmlcz/134086

KURZWEIL-HENSTOCK AND KURZWEIL-HENSTOCK-PETTIS INTEGRABILITY OF STRONGLY MEASURABLE FUNCTIONS

B. Bongiorno, Palermo, L. Di Piazza, Palermo, K. Musiał, Wrocław

(Received November 16, 2005)

Dedicated to Prof. J. Kurzweil on the occasion of his 80th birthday

Abstract. We study the integrability of Banach valued strongly measurable functions defined on $[0,1]$. In case of functions f given by $\sum\limits_{n=1}^{\infty} x_n\chi_{E_n}$, where x_n belong to a Banach space and the sets E_n are Lebesgue measurable and pairwise disjoint subsets of $[0,1]$, there are well known characterizations for the Bochner and for the Pettis integrability of f (cf Musial (1991)). In this paper we give some conditions for the Kurzweil-Henstock and the Kurzweil-Henstock-Pettis integrability of such functions.

Keywords: Kurzweil-Henstock integral, Kurzweil-Henstock-Pettis integral, Pettis integral

MSC 2000: 26A42, 26A39, 26A45

1. Introduction

In this paper we study the Kurzweil-Henstock and the Kurzweil-Henstock-Pettis integrability of strongly measurable functions. It is well known (cf [7, Lemma 5.1]) that each strongly measurable Banach valued function, defined on a measurable space, can be written as $f = g + \sum\limits_{n=1}^{\infty} x_n\chi_{E_n}$, where g is a bounded strongly measurable function, x_n are vectors of the given Banach space and E_n are measurable and pairwise disjoint sets. As each bounded strongly measurable function is Bochner integrable, it is enough to study the integrability only for functions of the form $\sum\limits_{n=1}^{\infty} x_n\chi_{E_n}$. In the case of the Bochner and Pettis integrals, a necessary and sufficient condition for the integrability of a function given by $\sum\limits_{n=1}^{\infty} x_n\chi_{E_n}$ is, respectively, the absolute and the unconditional convergence of the series $\sum_{n=1}^{\infty} x_n|E_n|$ (see Theorem A). In the case of the Kurzweil-Henstock or of the Kurzweil-Henstock-Pettis integrability, in general the series $\sum_{n=1}^{\infty} x_n|E_n|$ is only conditionally convergent. So the conditions for the integrability depend on the order of the terms $x_n|E_n|$. We present one sufficient condition for the Kurzweil-Henstock and the Kurzweil-Henstock-Pettis integrability of such functions.

2. Basic facts

Let $[0,1]$ be the unit interval of the real line equipped with the usual topology and the Lebesgue measure. If a set $E \subset [0,1]$ is Lebesgue measurable, then $|E|$ denotes its Lebesgue measure. $\mathcal{I}$ denotes the family of all closed subintervals of $[0,1]$. A *partition in* $[0,1]$ is a finite collection of pairs $\mathcal{P} = \{(I_1,t_1),\dots,(I_p,t_p)\}$, where $I_1,\dots,I_p$ are nonoverlapping subintervals of $[0,1]$ and $t_i \in I_i$, $i=1,\dots,p$. If $\bigcup_{i=1}^{p} I_i = [0,1]$ we say that $\mathcal{P}$ is a *partition of* $[0,1]$. Given a subset E of $[0,1]$, we say that the partition $\mathcal{P}$ is *anchored on* E if $t_i \in E$ for each $i=1,\dots,p$. A *gauge* on $E \subset [0,1]$ is a positive function on E. For a given gauge δ, we say that a partition $\{(I_1,t_1),\dots,(I_p,t_p)\}$ is δ*-fine* if $I_i \subset (t_i-\delta(x_i), t_i+\delta(x_i))$, $i=1,\dots,p$.

Throughout this paper X is a Banach space with the dual X^*. The closed unit ball of X^* is denoted by $\mathcal{B}(X^*)$.

Definition 1. A function $f\colon [0,1]\to X$ is said to be *Kurzweil-Henstock integrable*, or simply KH-*integrable*, on $[0,1]$ if there exists $w\in X$ with the following property: for every $\varepsilon>0$ there exists a gauge δ on $[0,1]$ such that

$$\Bigl\|\sum_{i=1}^{p} f(t_i)|I_i| - w\Bigr\| < \varepsilon$$

for each δ-fine partition $\{(I_1,t_1),\dots,(I_p,t_p)\}$ of $[0,1]$. We set $w =: (\mathrm{KH})\int_0^1 f$.

We denote the set of all KH-integrable functions $f\colon [0,1]\to X$ by $\mathrm{KH}([0,1],X)$.

The space $\mathrm{KH}([0,1],X)$ is endowed with the Alexiewicz norm (cf. [1])

$$\|f\|_A = \sup_{0<\alpha\leqslant 1}\Bigl\|(\mathrm{KH})\int_0^{\alpha} f(t)\,\mathrm{d}t\Bigr\|.$$

A family $\mathcal{A}\subset \mathrm{KH}([0,1],X)$ is said to be *Kurzweil-Henstock equiintegrable*, or simply KH-*equiintegrable*, on $[0,1]$ if in Definition 1, for every $\varepsilon>0$ there exists a gauge δ on $[0,1]$ which works for all the functions in $\mathcal{A}$.

A function $f\colon [0,1] \to X$ is said to be *scalarly Kurzweil-Henstock integrable*, or simply *scalarly* KH-*integrable*, if for each $x^* \in X^*$, the function x^*f is Kurzweil-Henstock integrable on $[0,1]$.

Definition 2. A scalarly KH-integrable function $f\colon [0,1] \to X$ is said to be *Kurzweil-Henstock-Dunford* integrable or simply KHD-*integrable*, if, for each non-empty interval $[a,b] \subset [0,1]$, there exists a vector $w_{ab} \in X^{**}$ such that for every $x^* \in X^*$

$$\langle x^*, w_{ab}\rangle = (\mathrm{KH}) \int_a^b x^* f(t)\,\mathrm{d}t. \tag{1}$$

It follows from [5, Theorem 3] that a function $f\colon [0,1] \to X$ is KHD-integrable if and only if f is scalarly KH-integrable.

The generalization of the Pettis integral obtained by replacing the Lebesgue integrability of the functions by the Kurzweil-Henstock integrability produces the Kurzweil-Henstock-Pettis integral (for the definition of the Pettis integral see [3]).

Definition 3. If a function $f\colon [0,1] \to X$ is scalarly KH-integrable and for each subinterval $[a,b]$ of $[0,1]$ and for each $x^* \in X^*$ there exists a vector $w_{[a,b]} \in X$ such that $x^* w_{[a,b]} = (\mathrm{KH}) \int_a^b \langle x^*, f\rangle$, then f is said to be *Kurzweil-Henstock-Pettis integrable*, or simply KHP-*integrable*, on $[0,1]$ and we set $w_{[a,b]} =: (\mathrm{KHP}) \int_a^b f$.

We recall that a function $f\colon [0,1] \to X$ is said to be *strongly measurable* if there is a sequence of simple functions f_n with $\lim\limits_n \|f_n(t) - f(t)\| = 0$ for almost all $t \in [0,1]$.

3. Integration of strongly measurable function

The aim of this section is to give conditions for the Kurzweil-Henstock or the Kurzweil-Henstock-Pettis integrability of strongly measurable functions.

We start by recalling the following simple lemma (cf. Lemma 5.1 of [7]).

Lemma 1. *If $f\colon [0,1] \to X$ is strongly measurable, then there exists a bounded strongly measurable function $g\colon [0,1] \to X$ such that $f = g + \sum\limits_{n=1}^{\infty} x_n \chi_{E_n}$ where $x_n \in X$ and the sets E_n are Lebesgue measurable and pairwise disjoint.*

Since each bounded strongly measurable function is Bochner integrable, and then Kurzweil-Henstock (and Kurzweil-Henstock-Pettis) integrable (see e.g. [4]), it is enough to give criteria of integrability only for functions of the form $\sum\limits_{n=1}^{\infty} x_n \chi_{E_n}$, where $x_n \in X$ and the sets E_n are measurable and pairwise disjoint.

For the Bochner and Pettis integrals we have the following classical result:

Theorem A (cf [7]). *Let* $f = \sum_{n=1}^{\infty} x_n \chi_{E_n}$, *where* $x_n \in X$ *and the sets* E_n *are Lebesgue measurable and pairwise disjoint subsets of* $[0,1]$. *Then*

(i) f *is Pettis integrable if and only if the series* $\sum_{n=1}^{\infty} x_n |E_n|$ *is unconditionally convergent;*

(ii) f *is Bochner integrable if and only if the series* $\sum_{n=1}^{\infty} x_n |E_n|$ *is absolutely convergent.*

In both cases $\int_E f = \sum_{n=1}^{\infty} x_n |E_n \cap E|$ *for every measurable set E.*

Here we would like to give similar conditions for the Kurzweil-Henstock and Kurzweil-Henstock-Pettis integrals.

Theorem 1. *Let* $f\colon [0,1] \to X$ *be defined by* $f = \sum_{n=1}^{\infty} x_n \chi_{E_n}$, *where* $x_n \in X$ *and the sets* E_n *are Lebesgue measurable and pairwise disjoint. Assume that the following condition is satisfied:*

(A) *for every* $\varepsilon > 0$ *there exist a gauge* δ *and* $k_0 \in \mathbb{N}$ *such that given a* δ*-fine partition* $\{(I_1,t_1),\dots,(I_p,t_p)\}$ *of* $[0,1]$ *and given* $s > r > k_0$ *we have*

$$\Big\| \sum_{k=r}^{s} x_k \Big| \bigcup_{t_j \in E_k} I_j \Big| \Big\| < \varepsilon. \tag{2}$$

Then f *is Kurzweil-Henstock integrable with*

$$(\mathrm{KH}) \int_I f(t)\,\mathrm{d}t = \sum_{n=1}^{\infty} x_n |E_n \cap I| \tag{3}$$

for every interval $I \in \mathcal{I}$.

P r o o f. Let $\varepsilon > 0$ be arbitrary and let δ and k_0 be respectively a gauge and a natural number such that inequality (2) is satisfied. We are going to prove first that the series $\sum_{n=1}^{\infty} x_n |E_n \cap I|$ is convergent for every $I \in \mathcal{I}$. To this purpose we set $f_m = \sum_{n=1}^{m} x_n \chi_{E_n}$ for every $m \in \mathbb{N}$, and observe that each function f_m is Bochner, and hence also Kurzweil-Henstock integrable, with its integral equal to $\sum_{k=1}^{m} x_k |E_k|$. Now let $s > r > k_0$ and, according to Definition 1, let δ_{r-1} and δ_s be two gauges related to f_{r-1} and f_s respectively, such that

$$\Big\| \sum_{k=1}^{r-1} x_k |E_k| - \sum_{i=1}^{p} f_{r-1}(t_i)|I_i| \Big\| < \varepsilon$$

and

$$\left\| \sum_{k=1}^{s} x_k |E_k| - \sum_{i=1}^{p} f_s(t_i)|I_i| \right\| < \varepsilon$$

for each partition $\{(I_1,t_1),\ldots,(I_p,t_p)\}$ of $[0,1]$ which is both δ_{r-1}-fine and δ_s-fine. Now define $\delta^*(t) = \min\{\delta(t),\delta_{r-1}(t),\delta_s(t)\}$ and take any δ^*-fine partition $\{(I_1,t_1),\ldots,(I_p,t_p)\}$ of $[0,1]$.

Then

$$\begin{aligned}(4)\quad \left\| \sum_{k=r}^{s} x_k|E_k| \right\| &= \left\| \sum_{k=1}^{s} x_k|E_k| - \sum_{k=1}^{r-1} x_k|E_k| \right\| \\ &\leqslant \left\| \sum_{k=1}^{s} x_k|E_k| - \sum_{i=1}^{p} f_s(t_i)|I_i| \right\| \\ &\quad + \left\| \sum_{k=1}^{r-1} x_k|E_k| - \sum_{i=1}^{p} f_{r-1}(t_i)|I_i| \right\| + \left\| \sum_{k=r}^{s} x_k \Big| \bigcup_{t_i\in E_k} I_i \Big| \right\| < 3\varepsilon.\end{aligned}$$

This proves that the series $\sum\limits_{n=1}^{\infty} x_n|E_n|$ is norm convergent.

Now for each $i\in\mathbb{N}$ let K_i be a closed set and U_i an open set such that:

(s1) $K_i \subseteq E_i \subseteq U_i$;

(s2) $|U_i\setminus K_i| < 2^{-i}\varepsilon/(\|x_i\|+1)$;

(s3) if $j<i$, then $U_i\cap K_j=\emptyset$;

(s4) if $i\leqslant k_0$, then

$$U_i \cap \bigcup_{k_0\geqslant j\neq i} K_j = \emptyset.$$

It follows from (s_3)–(s_4) that if $\{(I_1,t_1),\ldots,(I_p,t_p)\}$ is a δ-fine partition of $[0,1]$, then $K_n \subset \bigcup\limits_{t_i\in E_n} I_i$ for every $n\leqslant k_0$.

The functions $f_1,f_2,\ldots,f_{k_0}$ are Bochner, and hence also Kurzweil-Henstock integrable on $[0,1]$. Let $\gamma(t)$ be a gauge on $[0,1]$ such that

$$\left\| \sum_{k=1}^{m} x_k|E_k| - \sum_{i=1}^{p} f_m(t_i)|I_i| \right\| < \varepsilon$$

for each γ-fine partition $\{(I_1,t_1),\ldots,(I_p,t_p)\}$ of $[0,1]$ and for $m=1,2,\ldots,k_0$.

Define $\delta_0(t) = \min\{\operatorname{dist}(t, U_i^c), \delta(t), \gamma(t)\}$ if $t \in E_i$ and let $\{(I_1, t_1), \ldots, (I_p, t_p)\}$ be a δ_0-fine partition of $[0,1]$. For each $m > k_0$, by (2)–(4) we get

$$\begin{aligned}\Big\| \sum_{k=1}^{m} x_k |E_k| - \sum_{i=1}^{p} f_m(t_i)|I_i| \Big\| &\leqslant \Big\| \sum_{k=1}^{k_0} x_k \Big(|E_k| - \sum_{t_i \in E_k} |I_i|\Big) \Big\| \\ &\quad + \Big\| \sum_{k=k_0+1}^{m} x_k \Big| \bigcup_{t_i \in E_k} I_i \Big| \Big\| + \Big\| \sum_{k=k_0+1}^{m} x_k |E_k| \Big\| \\ &< \sum_{k=1}^{k_0} \|x_k\| \Big| |E_k| - \Big| \bigcup_{t_i \in E_k} I_i \Big| \Big| + 4\varepsilon \leqslant \sum_{k=1}^{k_0} \|x_k\| \, |U_k \setminus K_k| + 4\varepsilon \\ &< 2\varepsilon \sum_{k=1}^{k_0} \frac{\|x_k\|}{2^k(\|x_k\| + 1)} + 4\varepsilon < 6\varepsilon.\end{aligned}$$

Thus, the sequence (f_m) is Kurzweil-Henstock equiintegrable. Moreover, since $\lim\limits_{m\to\infty} f_m(t) = f(t)$ in $[0,1]$, by [8, Theorem 1] f is Kurzweil-Henstock integrable and (f_m) converges to f in the Alexiewicz topology. So, in particular, for each $I \in \mathcal{I}$ we have

$$(\mathrm{KH}) \int_I f(t) \, \mathrm{d}t = \lim_n (\mathrm{KH}) \int_I f_n(t) \, \mathrm{d}t,$$

and the assertion follows. □

R e m a r k 1. Within the proof of the previous theorem it is also showed that condition (A) implies the Kurzweil-Henstock equiintegrability of the sequence (f_m). It is easy to check that the same proof can be used to prove the reverse implication. So we have also:

If the sequence $(f_m = \sum\limits_{k=1}^{m} x_k \chi_{E_k})$ is Kurzweil-Henstock equiintegrable, then the function $f = \sum\limits_{n=1}^{\infty} x_n \chi_{E_n}$ is Kurzweil-Henstock integrable and

$$(\mathrm{KH}) \int_I f(t) \, \mathrm{d}t = \sum_{n=1}^{\infty} x_n |E_n \cap I|$$

for every interval $I \in \mathcal{I}$.

R e m a r k 2. There exist points $x_n \in X$ and pairwise disjoint Lebesgue measurable sets E_n, $n = 1, 2, \ldots$, such that the series $\sum\limits_{n=1}^{\infty} x_n |E_n \cap I|$ is convergent for every $I \in \mathcal{I}$, the function $f = \sum\limits_{n=1}^{\infty} x_n \chi_{E_n}$ is Kurzweil-Henstock integrable and

$$(\mathrm{KH}) \int_0^1 f(t) \, \mathrm{d}t \neq \sum_{n=1}^{\infty} x_n |E_n|.$$

P r o o f. Let π be a permutation of $\mathbb{N}$ such that the series

$$\sum_{n=1}^{\infty}(-1)^{\pi(n)}[\pi(n)+1]\Big(\frac{1}{\pi(n)}-\frac{1}{\pi(n)+1}\Big)=\sum_{n=1}^{\infty}\frac{(-1)^{\pi(n)}}{\pi(n)}$$

is convergent but

$$\sum_{n=1}^{\infty}\frac{(-1)^{\pi(n)}}{\pi(n)}\neq\sum_{n=1}^{\infty}\frac{(-1)^{n}}{n}.$$

Let

$$f=\sum_{n=1}^{\infty}(-1)^{\pi(n)}[\pi(n)+1]\chi_{[1/[\pi(n)+1],1/\pi(n))}. \tag{5}$$

Remark that the function f can be written also in the form

$$f=\sum_{n=1}^{\infty}(-1)^{n}(n+1)\chi_{[1/(n+1),1/n)}.$$

Since the function f is Riemann improper integrable on $[0,1]$, then it is Kurzweil-Henstock integrable, with

$$(\mathrm{KH})\int_0^1 f(t)\,\mathrm{d}t=\sum_{n=1}^{\infty}\frac{(-1)^{n}}{n}.$$

Note that the series

$$\sum_{n=1}^{\infty}(-1)^{\pi(n)}[\pi(n)+1]\Big|I\cap\Big[\frac{1}{\pi(n)+1},\frac{1}{\pi(n)}\Big)\Big|$$

is convergent, but

$$(\mathrm{KH})\int_0^1 f(t)\,\mathrm{d}t\neq\sum_{n=1}^{\infty}\frac{(-1)^{\pi(n)}}{\pi(n)}.$$

□

R e m a r k 3. It follows from Remark 2 that condition (A) of Theorem 1 is not necessary for the KH-integrability of a function f given by $\sum_{n=1}^{\infty} x_n\chi_{E_n}$, with $x_n\in X$ and E_n pairwise disjoint Lebesgue measurable sets.

However, there are cases in which the convergence of the series $\sum_{n=1}^{\infty} x_n|E_n|$ implies the Kurzweil-Henstock integrability of the function $f=\sum_{n=1}^{\infty} x_n\chi_{E_n}$ and the equality (3) holds.

Proposition 1. *Let (a_n) be a decreasing sequence converging to zero such that $a_1 = 1$, and let $f\colon [0,1] \to X$ be defined by $f = \sum\limits_{n=1}^{\infty} x_n \chi_{E_n}$, where $x_n \in X$ and $E_n = [a_{n+1}, a_n)$. If the series $\sum\limits_{n=1}^{\infty} x_n |E_n|$ is convergent, then condition* (A) *of Theorem* 1 *is satisfied.*

Proof. Let $\varepsilon > 0$ be given, and let k_0 be a natural number such that

$$\left\| \sum_{n=r}^{\infty} x_n |E_n| \right\| < \frac{\varepsilon}{5} \quad \text{and} \quad \|x_r\| |E_r| < \frac{\varepsilon}{5}, \tag{6}$$

for every $r > k_0$. Now we define a gauge δ on $[0,1]$ in the following way:

$\delta(0) = a_{k_0}$,

$\delta(x) = \operatorname{dist}(x, E_n^c)$ if $x \in E_n^0$ (the interior of E_n),

$\delta(a_n) = \frac{1}{5} \min\{2^{-n}\varepsilon/(\|x_{n-1}\| + \|x_n\| + 1),\, \operatorname{dist}(a_n, 0)\}$ for $n = 1, 2, \ldots$

Now let $\{(I_1, t_1), \ldots, (I_p, t_p)\}$ be any δ-fine partition of $[0,1]$. By the definition of δ, the point 0 has to be the tag of one of the pairs of the partition; we may assume that $t_1 = 0$. Let $\bar{n}$ be the first natural number such that $a_{\bar{n}+1} \in I_1$. By the definition of $\delta(0)$ it follows that $a_{\bar{n}+1} < a_{k_0}$, hence $\bar{n} \geqslant k_0$. Then

$$\left| \bigcup_{t_j \in E_{\bar{n}}} I_j \right| \leqslant |E_{\bar{n}}| \tag{7}$$

and

$$\left| \bigcup_{t_j \in E_n} I_j \right| = 0 \quad \text{for } n > \bar{n}. \tag{8}$$

Hence

$$\left\| \sum_{n>r} x_n \left| \bigcup_{t_j \in E_n} I_j \right| \right\| = 0$$

for each $r \geqslant \bar{n}$.

Besides, for $1 < n < \bar{n}$ we have

$$\left| \bigcup_{t_j \in E_n} I_j \right| = |E_n| + \varepsilon_n' \delta(a_{n+1}) - \varepsilon_n'' \delta(a_n) \tag{9}$$

for suitable $\varepsilon_n', \varepsilon_n'' \in [0,1]$.

So, for $\bar{n} > r > k_0$, by (6), (7), (8) and (9) we obtain

$$\begin{aligned}\Big\|\sum_{n>r} x_n\Big|\bigcup_{t_j\in E_n} I_j\Big|\Big\| &= \Big\|x_{\bar{n}}\Big|\bigcup_{t_j\in E_{\bar{n}}} I_j\Big| + \sum_{\bar{n}>n>r} x_n(|E_n| + \varepsilon_n'\delta(a_{n+1}) - \varepsilon_n''\delta(a_n))\Big\| \\ &\leqslant \|x_{\bar{n}}\||E_{\bar{n}}| + \Big\|\sum_{\bar{n}>n>r} x_n|E_n|\Big\| + \sum_{\bar{n}>n>r}\|x_n\|\delta(a_{n-1}) + \sum_{\bar{n}>n>r}\|x_n\|\delta(a_n) \\ &\leqslant \frac{\varepsilon}{5} + 2\frac{\varepsilon}{5} + \frac{1}{5}\sum_{n>r}\frac{\varepsilon}{2^n} + \frac{1}{5}\sum_{n>r}\frac{\varepsilon}{2^n} < \varepsilon.\end{aligned}$$

This completes the proof. □

Theorem 1 and Proposition 1 yield

Theorem 2. *Let $f\colon [0,1] \to X$ be defined by $f = \sum\limits_{n=1}^{\infty} x_n\chi_{E_n}$, where $x_n \in X$, $E_n = [a_{n+1}, a_n)$ and (a_n) is a decreasing sequence converging to zero such that $a_1 = 1$. If the series $\sum\limits_{n=1}^{\infty} x_n|E_n|$ is convergent, then f is Kurzweil-Henstock integrable and*

$$\text{(KH)}\int_I f(t)\,\mathrm{d}t = \sum_{n=1}^{\infty} x_n|E_n \cap I|$$

for every interval $I \in \mathcal{I}$.

Open Problem. Suppose that f is defined like in Theorem 1 and it is KH-integrable. Does there exist a permutation π of $\mathbb{N}$ such that

$$\text{(KH)}\int_I f(t)\,\mathrm{d}t = \sum_{n=1}^{\infty} x_{\pi(n)}|E_{\pi(n)} \cap I|$$

for each $I \in \mathcal{I}$?

By applying Theorem 1 the following two results follow:

Theorem 3. *Let $f\colon [0,1] \to X$ be defined by $f = \sum\limits_{n=1}^{\infty} x_n\chi_{E_n}$, where $x_n \in X$ and the sets E_n are Lebesgue measurable and pairwise disjoint. We assume also that the following conditions are satisfied:*

(a) *$\sum\limits_{n=1}^{\infty} x_n|E_n|$ is weakly convergent;*

(b) *for every $\varepsilon > 0$ and every $x^* \in X^*$ there exist a gauge δ and $k_0 \in \mathbb{N}$ such that for each δ-fine partition $\{(I_1,t_1),\dots,(I_p,t_p)\}$ of $[0,1]$ and each $n > m > k_0$ we have*

$$\Big|\sum_{k=m}^{n} x^*(x_k)\Big|\bigcup_{t_j\in E_k} I_j\Big|\Big| < \varepsilon.$$

Then f is Kurzweil-Henstock-Pettis integrable and for every $I \in \mathcal{I}$ and $x^* \in X^*$ we have

$$\text{(10)} \qquad (\mathrm{KH}) \int_I \langle x^*, f(t) \rangle \, \mathrm{d}t = \sum_{n=1}^{\infty} x^*(x_n) |E_n \cap I|.$$

Theorem 4. *Let $f\colon [0,1] \to X$ be defined by $f = \sum\limits_{n=1}^{\infty} x_n \chi_{E_n}$, where $x_n \in X$ and the sets E_n are Lebesgue measurable and pairwise disjoint. We assume also that the following conditions are satisfied:*

(a) $\sum\limits_{n=1}^{\infty} x_n |E_n|$ *is weakly Cauchy;*

(b) *for every $\varepsilon > 0$ and every $x^* \in X^*$ there exist a gauge δ and a natural number k_0 such that*

$$\left| \sum_{k=m}^{n} x^*(x_k) \left| \bigcup_{t_j \in E_k} I_j \right| \right| < \varepsilon$$

for each δ-fine partition $\{(I_1, t_1), \dots, (I_p, t_p)\}$ of $[0,1]$ and each $n > m > k_0$.

Then f is Kurzweil-Henstock-Dunford integrable and for every $I \in \mathcal{I}$ and $x^* \in X^*$ we have

$$\text{(11)} \qquad (\mathrm{KH}) \int_I \langle x^*, f(t) \rangle \, \mathrm{d}t = \sum_{n=1}^{\infty} x^*(x_n) |E_n \cap I|.$$

Theorem 5. *Let $f\colon [0,1] \to X$ be defined by $f = \sum\limits_{n=1}^{\infty} x_n \chi_{E_n}$, where $x_n \in X$, $E_n = [a_{n+1}, a_n)$ and (a_n) is a decreasing sequence converging to zero such that $a_1 = 1$. If the series $\sum\limits_{n=1}^{\infty} x_n |E_n|$ is weakly convergent, then f is Kurzweil-Henstock-Pettis integrable and*

$$(\mathrm{KH}) \int_I x^* f(t) \, \mathrm{d}t = \sum_{n=1}^{\infty} x^*(x_n) |E_n \cap I|$$

for every interval $I \in \mathcal{I}$ and every $x^* \in X^*$.

If the series $\sum\limits_{n=1}^{\infty} x_n |E_n|$ is weak-convergent, then f is Kurzweil-Henstock-Dunford integrable and*

$$(\mathrm{KH}) \int_I x^* f(t) \, \mathrm{d}t = \sum_{n=1}^{\infty} x^*(x_n) |E_n \cap I|$$

for every interval $I \in \mathcal{I}$ and every $x^* \in X^*$.

We recall that a function $G\colon [0,1] \to \mathbb{R}$ is a KH-primitive if and only if for each $N \subset [0,1]$ with $|N| = 0$ and for each $\varepsilon > 0$ there exists a gauge δ in N such that

$$\sum_{i=1}^{p} |G(I_i)| < \varepsilon$$

for each δ-fine partition $\{(I_1,t_1),\dots,(I_p,t_p)\}$ anchored on N. Here $G([a,b]) = G(b) - G(a)$. In this case the derivative of G exists almost everywhere in $[0,1]$ and G is the KH-primitive of G' (see e.g. [2] and the bibliography there).

Theorem 6. *Let $f\colon [0,1] \to X$ be defined by $f = \sum\limits_{n=1}^{\infty} x_n \chi_{E_n}$, where $x_n \in X$ and the sets E_n are pairwise disjoint intervals. Then f is Kurzweil-Henstock-Pettis integrable and*

$$\text{(KHP)} \int_I f(t)\,\mathrm{d}t = \sum_{n=1}^{\infty} x_n |E_n \cap I| \quad \textit{for every} \quad I \in \mathcal{I} \tag{12}$$

if and only if the following conditions are satisfied:

(a) $\sum\limits_{n=1}^{\infty} x_n |E_n \cap I|$ *is weakly convergent for each $I \in \mathcal{I}$;*

(b) *for every $\varepsilon > 0$, every $x^* \in X^*$ and every $N \subset [0,1]$ with $|N| = 0$, there exists a gauge δ in N such that*

$$\Bigl|\sum_{n=1}^{\infty} x^*(x_n)\Bigl|E_n \cap \bigcup_{i=1}^{p} I_i\Bigr|\Bigr| < \varepsilon \tag{13}$$

for each δ-fine partition $\{(I_1,t_1),\dots,(I_p,t_p)\}$ anchored on N.

Proof. If f is KH-integrable and condition (12) is satisfied, then condition (a) follows directly from (12). Now we prove condition (b). By hypothesis, for every $x^* \in X^*$ the scalar function $x^* f$ is KH-integrable. Set

$$\alpha(t) = \text{(KH)} \int_0^t \langle x^*, f(s)\rangle\,\mathrm{d}s = \sum_{n=1}^{\infty} x^*(x_n)|E_n \cap [0,t]|.$$

Since $\alpha(t)$ is the KH-primitive of $x^* f$, hence for each $N \subset [0,1]$ with $|N| = 0$ and for each $\varepsilon > 0$ there exists a gauge δ on N such that

$$\sum_{i=1}^{p} |\alpha(I_i)| < \varepsilon$$

for each δ-fine partition $\{(I_1,t_1),\dots,(I_p,t_p)\}$ anchored on N.

Therefore

$$\Big|\sum_{n=1}^{\infty} x^*(x_n)\Big|E_n \cap \bigcup_{i=1}^{p} I_i\Big|\Big| \leqslant \sum_{i=1}^{p}\Big|\sum_{n=1}^{\infty} x^*(x_n)|E_n \cap I_i|\Big| = \sum_{i=1}^{p}|\alpha(I_i)| < \varepsilon.$$

Assume now that conditions (a) and (b) are satisfied, and fix $x^* \in X^*$. By condition (b), for every $\varepsilon > 0$ and every $N \subset [0,1]$ with $|N| = 0$, there exists a gauge δ on N such that inequality (13) holds for each δ-fine partition $\{(I_1,t_1),\dots,(I_p,t_p)\}$ anchored on N. Set once again $\alpha(t) = \sum\limits_{n=1}^{\infty} x^*(x_n)|E_n \cap [0,t]|$. Then, by (a) and (13), we infer

$$\begin{aligned}\sum_{i=1}^{p}|\alpha(I_i)| &= \Big|\sum_{I^+}\alpha(I_i)\Big| + \Big|\sum_{I^-}\alpha(I_i)\Big| \\ &= \Big|\sum_{n=1}^{\infty} x^*(x_n)\Big|E_n \cap \bigcup_{I^+} I_i\Big|\Big| + \Big|\sum_{n=1}^{\infty} x^*(x_n)\Big|E_n \cap \bigcup_{I^-} I_i\Big|\Big| < 2\varepsilon,\end{aligned}$$

where I^+ and I^- denote the set of all indices $i = 1,\dots,p$ such that $\alpha(I_i)$ is positive or negative, respectively. Therefore $\alpha(t)$ is a KH-primitive. As the sets E_n are intervals, it follows easily that $\alpha'(t) = \sum\limits_{n=1}^{\infty} x^*(x_n)\chi_{E_n}$ almost everywhere in $[0,1]$. Then $\alpha(t)$ is the KH-primitive of the function $x^*f = \sum\limits_{n=1}^{\infty} x^*(x_n)\chi_{E_n}$. Consequently

$$(\mathrm{KH})\int_0^1 \langle x^*, f(t)\rangle \,\mathrm{d}t = \sum_{n=1}^{\infty} x^*(x_n)|E_n| = \Big\langle x^*, \sum_{n=1}^{\infty} x_n|E_n|\Big\rangle.$$

Hence the function f is KHP-integrable and (12) holds. □

O p e n p r o b l e m. Is Theorem 6 still true if the sets E_n are arbitrary pairwise disjoint Lebesgue measurable sets?

References

[1] *A. Alexiewicz*: Linear functionals on Denjoy integrable functions. Coll. Math. *1* (1948), 289–293. Zbl 0037.32302

[2] *B. Bongiorno*: The Henstock-Kurzweil integral. Handbook of Measure Theory I, E. Pap ed., Elsevier Amsterdam, 2002, pp. 587–615. Zbl 0998.28001

[3] *J. Diestel, J. J. Uhl*: Vector measures. Math. Surveys, vol. 15, AMS, Providence, R.I., 1977. Zbl 0369.46039

[4] *D. H. J. Fremlin*: The Henstock and McShane integrals of vector-valued functions. Illinois J. Math. *38* (1994), 471–479. Zbl 0797.28006

[5] *J. L. Gamez, J. Mendoza*: On Denjoy-Dunford and Denjoy-Pettis integrals. Studia Math. *130* (1998), 115–133. Zbl 0971.28009

[6] *P. Y. Lee*: Lanzhou Lectures on Henstock Integration. World Scientific, Singapore, 1989. Zbl 0699.26004

[7] *K. Musial*: Topics in the theory of Pettis integration. Rend. Istit. Mat. Univ. Trieste *23* (1991), 177–262. Zbl 0798.46042

[8] *C. Swartz*: Norm convergence and uniform integrability for the Kurzweil-Henstock integral. Real Anal. Exchange *24* (1998/99), 423–426. Zbl 0943.26022

Authors' addresses: *B. Bongiorno*, *L. Di Piazza*, Department of Mathematics and Applications, University of Palermo, via Archirafi 34, 90123 Palermo, Italy, e-mail: `bbongi@math.unipa.it`, `dipiazza@math.unipa.it`; *K. Musiał*, Wrocław University, Institute of Mathematics, Plac Grunwaldzki 2/4, 50-384 Wrocław, Poland, e-mail: `musial@math.uni.wroc.pl`.