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ON BELATED DIFFERENTIATION AND A CHARACTERIZATION OF HENSTOCK-KURZWEIL-ITO INTEGRABLE PROCESSES

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Abstract. The Henstock-Kurzweil approach, also known as the generalized Riemann approach, has been successful in giving an alternative definition to the classical Itô integral. The Riemann approach is well-known for its directness in defining integrals. In this note we will prove the Fundamental Theorem for the Henstock-Kurzweil-Itô integral, thereby providing a characterization of Henstock-Kurzweil-Itô integrable stochastic processes in terms of their primitive processes.

Keywords: belated differentiation, Henstock-Kurzweil-Itô integral, integrable processes

MSC 2000: 26A39, 60H05

1. Introduction

The generalized Riemann approach, more commonly known as the Henstock-Kurzweil approach, has been successful in giving an alternative definition to the classical Itô integral, see [1], [5], [7], [8], [9], [10]. The advantage of using the Henstock-Kurzweil approach has been its explicitness and intuitiveness in giving a direct definition of the integral rather than the classical non-explicit L^2-procedure.

It is also well-known from the classical non-stochastic integration theory that all integrable functions can be characterized in terms of their primitives, that is, a function f is Lebesgue (Henstock-Kurzweil) integrable on a compact interval $[a,b]$ if and only if there exists a function F which is absolutely continuous (respectively, generalized absolutely continuous) there such that $F' = f$ a.e. on $[a,b]$, where F' is the usual derivative of F, see for example [4].

In this paper, we will define the "belated derivative" of a stochastic process and thereby characterize the class of all Henstock-Kurzweil-Itô integrable processes on $[a,b]$ by its primitive process.

2. Setting

Let Ω denote the set of all real-valued continuous functions on $[a,b]$ and $\mathbb{R}$ the set of all real numbers.

The class of all Borel cylindrical sets B in Ω, denoted by $\mathscr{C}$, is a collection of all sets B in Ω of the form

$$B = \{w\colon\ (w(t_1), w(t_2), \ldots, w(t_n)) \in E\}$$

where $0 \leqslant t_1 < t_2 < \ldots < t_n \leqslant 1$ and E is any Borel set in $\mathbb{R}^n$ (n is not fixed). The Borel σ-field of $\mathscr{C}$ is denoted by $\mathscr{F}$, i.e., it is the smallest σ-field which contains $\mathscr{C}$. Let P be the Wiener measure defined on $(\Omega, \mathscr{F})$. Then $(\Omega, \mathscr{F}, P)$ is a probability space, that is, a measure space with $P(\Omega) = 1$.

A stochastic process $\{\varphi(t,\omega)\colon\ t \in [a,b]\}$ on $(\Omega, \mathscr{F}, P)$ is a family of $\mathscr{F}$-measurable functions (which are called random variables) on $(\Omega, \mathscr{F}, P)$. Very often, $\varphi(t,\omega)$ is denoted by $\varphi_t(\omega)$. Now we shall consider a very special and important process, namely, the Brownian motion denoted by W.

Let $W = \{W_t(\omega)\}_{a \leqslant t \leqslant b}$ be a canonical Brownian motion, that is, it possesses the following properties:

1. $W_a(\omega) = 0$ for all $\omega \in \Omega$;
2. it has *Normal Increments*, that is, $W_t - W_s$ has a normal distribution with mean 0 and variance $t - s$ for all $t > s$ (which naturally implies that W_t has a normal distribution with mean 0 and variance t);
3. it has *Independent Increments*, that is, $W_t - W_s$ is independent of its past, that is, of W_u, $0 \leqslant u < s < t$; and
4. its sample paths are continuous, i.e., for each $\omega \in \Omega$, $W_t(\omega)$ as a function of t is continuous on $[a,b]$.

A stochastic process $\{\varphi_t(\omega)\colon\ t \in [a,b]\}$ is said to be *adapted* to the standard filtering space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}, P)$ if φ_t is $\mathscr{F}_t$-measurable for each $t \in [a,b]$. We always assume that $W = \{W_t(\omega)\}$ is adapted to $\{\mathscr{F}_t\}$. For example, if $\{\mathscr{F}_t\}$ is the smallest σ-field generated by $\{W_s(\omega)\colon\ s \leqslant t\}$, then $W = \{W_t(\omega)\}$ is adapted to $\{\mathscr{F}_t\}$.

A stochastic process $X = \{X_t(\omega)\colon\ t \in [a,b]\}$ on the standard filtering space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}, P)$ is called a *martingale* if

1. X is adapted to $\{\mathscr{F}_t\}$, that is, X_t is $\mathscr{F}_t$-measurable for each $t \in [a,b]$;
2. $\int_\Omega |X_t| \, \mathrm{d}P$ is finite for almost all $t \in [a,b]$; and
3. $E(X_t|\mathscr{F}_s) = X_s$ for all $t \geqslant s$, where $E(X_t|\mathscr{F}_s)$ is the conditional expectation of X_t given $\mathscr{F}_s$.

If in addition

$$\sup_{t\in[a,b]}\int_\Omega |X_t|^2\,\mathrm{d}P$$

is finite, we say that X is an L_2-*martingale.*

In the following we define $E(f)$ to be $\int_\Omega f\,\mathrm{d}P$ for any random variable f.

It is well-known, see for example [6, P239], that the following assertions hold. The details are given for the convenience of readers who are not familiar with stochastic analysis.

(i)

$$E[X_s]=E\left[E[X_t|\mathscr{F}_s]\right]=E[X_t]$$

for any $t\geqslant s$, that is, $E[X_s]$ is a constant for all $s\in[a,b]$.

(ii) For any $a\leqslant u<v\leqslant s<t\leqslant b$, we have

$$E[(X_t-X_s)(X_v-X_u)]=0,$$

that is, a martingale has orthogonal increments.

(iii) From (ii) we get

$$E\Big|(D)\sum(X_v-X_u)\Big|^2=(D)\sum E(X_v-X_u)^2$$

for any partial partition $D=\{[u,v]\}$ of $[a,b]$.

(iv) For any $u<v$ we have

$$E[X_vX_u]=E[E[X_vX_u|\mathscr{F}_u]]=E[X_uE[X_v|\mathscr{F}_u]]=E[X_u^2]$$

and hence

$$E(X_v-X_u)^2=E(X_v^2-X_u^2).$$

It is also well-known, see for example [6, P28], that a canonical Brownian motion is a martingale. In fact, it is an L_2-martingale with $E(W_t^2)=t$, see property 2 of a Brownian motion.

3. Differentiation

In this section we define our belated derivative and state its basic properties.

Definition 1. Let $F = \{F_t\colon\ t \in [a,b]\}$ be an L^2-martingale. A stochastic process F is said to be *belated differentiable* at $t \in [a,b)$ if there exists a random variable f_t such that for any $\varepsilon > 0$, there exists a positive number $\delta(t)$ on $[a,b]$ such that whenever $[t,v] \subset [t, t+\delta(t))$, we have

$$E(|f_t(W_v - W_t) - (F_v - F_t)|^2) \leqslant \varepsilon E(W_v - W_t)^2 = \varepsilon |v - t|.$$

The random variable f_t is called the *belated derivative* of F at the point t. We will denote f_t by $D_\beta F_t$ in our subsequent presentation. It is also easily checked that the belated derivative of F is defined uniquely up to a set of probability measure zero. The proof is omitted.

The L^2-martingale F is said to be belated differentiable at $t \in [a,b)$ if f_t in the above definition exists.

Remark. The word belated is used in the above definition because the point of differentiation t is always the left end point of the interval $[t,v]$. This is motivated by the use of belated division in the definition of Henstock-Kurzweil-Itô integrals, see [1].

Next we shall state the standard properties of belated differentiation.

Theorem 2. *Let X and Y be two L^2-martingales which are belated differentiable at $t \in [a,b)$ and let $\alpha \in \mathbb{R}$. Then*
(a) *$X + Y$ is belated differentiable at t and*

$$D_\beta(X+Y)_t = (D_\beta X)_t + (D_\beta Y)_t,$$

(b) *αX is belated differentiable at t and*

$$(D_\beta(\alpha X))_t = \alpha (D_\beta X)_t.$$

Proof. The proof of Theorem 2 is straightforward and hence omitted. □

Example 3. Let $X = \{X_t\colon\ t \in [0,1]\}$ be the stochastic process $X_t = \frac{1}{2}W_t^2 - \frac{1}{2}t$, where W is the Brownian motion, over the standard filtering space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}, P)$. Then it is easy to verify that X is in fact an L^2-martingale with respect to the standard filtering space. Furthermore, it can be proved that

$$D_\beta X_t = W_t$$

for all $t \in [a,b]$.

Proof. That X is a martingale follows from the fact that

$$E(W_b^2 - W_a^2 | \mathscr{F}_a) = b - a$$

where $0 \leqslant a \leqslant b$. Furthermore,

$$E(|X_t|)^2 = \frac{1}{2}t^2 \leqslant \frac{1}{2}b^2$$

for all $t \in [a, b]$, thereby showing that X is in fact an L^2-martingale. We next show that $D_\beta X_t = W_t$ for all $t \in [a, b)$.

Given $\varepsilon > 0$, let $\delta(t) \leqslant 2\varepsilon$ for all $t \in [a, b)$. Consider a δ-fine interval-point pair $([t, v], t)$ such that $[t, v] \subset [t, t + \delta(t)]$ so that $|v - t| \leqslant 2\varepsilon$. Then

$$\begin{aligned}
E(W_t(W_v - W_t) - (X_v - X_t))^2 &= E\Big(W_t(W_v - W_t) - \frac{1}{2}(W_v^2 - W_t^2) - \frac{1}{2}(t - v)\Big)^2 \\
&= E\Big(\frac{1}{2}(W_v - W_t)^2 + \frac{1}{2}(t - v)\Big)^2 \\
&= \frac{1}{4}E((W_v - W_t)^2 - (v - t))^2 \\
&= \frac{1}{4}E[(W_v - W_t)^4 - 2(W_v - W_t)^2(v - t) + (v - t)^2] \\
&= \frac{1}{2}(v - t)^2 \leqslant \frac{1}{2} \cdot 2\varepsilon(v - t) = \varepsilon(v - t),
\end{aligned}$$

which completes our proof. □

By Definition 1, belated differentiation is defined for L^2-martingales in our context. If we were to allow the belated differentiation to be defined for more general stochastic processes, we could even have $D_\beta(\frac{1}{2}W_t^2) = W_t$. However, in this sense, the anti-derivative of W_t would not be uniquely defined. Hence we restrict ourselves to the belated differentiation of L^2-martingales.

Definition 4. A stochastic process $X = \{X_t \colon t \in [a, b]\}$ on $(\Omega, \mathscr{F}, P)$ is said to be AC^2 on $[a, b]$ if given any $\varepsilon > 0$, there exists $\eta > 0$ such that

$$E\Big(\sum_{i=1}^{n}(X_{v_i} - X_{u_i})^2\Big) \leqslant \varepsilon$$

for any finite collection $D = \{[u_i, v_i]\}_{i=1}^n$ of non-overlapping intervals for which $\sum\limits_{i=1}^{n} |v_i - u_i| \leqslant \eta$.

E x a m p l e 5. The stochastic process $X = \{X_t\colon t \in [a,b]\}$, where

$$X_t = \frac{1}{2}W_t^2 - \frac{1}{2}t$$

in Example 3, is AC^2 on $[a,b]$. The proof is easy and hence omitted.

4. Antiderivative and Henstock-Ito integral

In this section we will characterize the class of all Henstock-Itô adapted processes in terms of their derivatives.

Let δ be a positive function on $[a,b]$. A finite collection D of interval-point pairs $\{([\xi_i, v_i], \xi_i), i = 1,2,\dots,n\}$ is called a *δ-fine belated partial division* of $[a,b]$ if

1. $\{[\xi_i, v_i], i = 1,2,\dots,n\}$ is a collection of non-overlapping subintervals of $[a,b]$; and
2. $[\xi_i, v_i] \subset [\xi_i, \xi_i + \delta(\xi_i)]$ for each $i = 1,2,3,\dots,n$.

In the sequel we will denote $\{([\xi_i, v_i], \xi_i), i = 1,2,3,\dots,n\}$ by $\{([\xi, v], \xi)\}$.

Definition 6 (See [1, Definition 2]). Let $f = \{f_t\colon t \in [a,b]\}$ be an adapted process on the standard filtering space $(\Omega, \mathscr{F}, \{\mathscr{F}\}_t, P)$. Then f is said to be Henstock-Kurzweil-Itô integrable on $[a,b]$ if there exists a process $F = \{F_t\colon t \in [a,b]\}$ which is an L^2-martingale and AC^2 on $[a,b]$ such that for any $\varepsilon > 0$ there exists a positive function δ on $[a,b]$ such that

$$E\Big((D)\sum\{f_\xi(W_v - W_u) - (F_v - F_u)\}^2\Big) \leqslant \varepsilon$$

whenever $D = \{([\xi, v], \xi)\}$ is a δ-fine belated partial division of $[a,b]$.

It follows from Vitali's Covering Lemma that given any positive function δ there exists a belated partial division of $[a,b]$ covering this interval up to a set of arbitrarily small positive measure, hence the uniqueness of the integral process F follows.

It was also proved in [1] that the standard properties of integrals (such as uniqueness of the integral, additivity of the integral, integrability over subintervals) hold true for the Henstock-Kurzweil-Itô integral. The proofs are similar to the classical integration theory, see [2], [3], [4]. In fact, it has been proved in Theorem 9 of [1] that the integral defined by this new approach is equivalent to the classical Itô integral.

We have a class of stochastic processes which are Henstock-Kurzweil-Itô integrable on $[a,b]$.

E x a m p l e 7. Let $\mathscr{L}_2$ denote the class of all adapted stochastic processes $\varphi = \{\varphi_t\colon t \in [a,b]\}$ on the standard filtering space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}, P)$ such that

$$\int_a^b E|\varphi(t,\omega)|^2 \,\mathrm{d}t$$

is finite. Then any adapted process from $\mathscr{L}_2$ is Henstock-Kurzweil-Itô integrable on $[a,b]$.

In fact, $\mathscr{L}_2$ is the class of all classical Itô integrable functions. We have proved in [1] that if f is classical Itô integrable, then f is also Henstock-Kurzweil-Itô integrable and the two integrals coincide.

Theorem 8. *Let an adapted process f be Henstock-Kurzweil-Itô integrable on $[a,b]$ and let $F_t = \int_a^t f_s \,\mathrm{d}W_s$. Then $D_\beta F_t = f_t$ a.e. on $[a,b)$.*

P r o o f. The idea of this proof is motivated by that of the Henstock integration theory. We need to show that the set of points B of $[a,b)$ for which $D_\beta F_t$ does not exist or is unequal to f is of Lebesgue measure zero. Let $t \in B$. By definition, there exists $\gamma(t) > 0$ such that for any positive number $\delta(t)$, there exists $[t,v] \subset [t, t+\delta(t))$ such that

$$E(|f_t(W_v - W_t) - (F_v - F_t)|^2) > \gamma(t)(v-t). \tag{1}$$

From the definition of the Henstock-Kurzweil-Itô integral (see Definition 6), given $\varepsilon > 0$, there exists a positive function β on $[a,b]$ such that whenever $D = \{((\xi_i, v_i], \xi_i)\}_{i=1}^n$ is a β-fine belated partial division of $[a,b]$, we have

$$E\Big((D)\sum |f_\xi[W_v - W_\xi] - (F_v - F_\xi)|^2\Big) \leqslant \varepsilon. \tag{2}$$

Now we consider a special D such that each $[\xi_i, v_i]$ satisfies (1) and (2). Let us denote $B_m = \{t \in [a,b]\colon \gamma(t) \geqslant \frac{1}{m}\}$, $m = 1,2,3,\ldots$, and fix B_m. Suppose each $\xi_i \in B_m$. Then by (1) and (2), we have

$$\sum_{i=1}^n (v_i - \xi_i) \leqslant m\varepsilon.$$

Let $\mathscr{G}$ be the family of collections of intervals $[\xi, v]$ induced from all β-fine belated partial divisions with $\xi \in B_m$ satisfying (1). Then $\mathscr{G}$ covers B_m in Vitali's sense. Applying the Vitali Covering Theorem, there exists a finite collection of intervals $\{[\xi_i, v_i], i = 1,2,3,\ldots,q\}$ such that

$$\mu(B_m) \leqslant \sum_{i=1}^q |v_i - \xi_i| + \varepsilon \leqslant (m+1)\varepsilon.$$

Hence $\mu(B_m) = 0$ and so $\mu(B) = 0$. Thus our proof is completed. □

Theorem 9. *Let f be an adapted process on $[a,b]$ such that*

(i) *F is an L^2-martingale with $F_a = 0$ a.e.;*

(ii) *F has the AC^2 property;*

(iii) *$D_\beta F_t = f_t$ a.e. on $[a,b)$; then f is Henstock-Kurzweil-Itô integrable on $[a,b]$ with $F_t = \int_a^t f_s \,\mathrm{d}W_s$.*

The reader is reminded that (iii) means that $D_\beta F_t(\omega) = f_t(\omega)$ for almost all $\omega \in \Omega$ for a.e. $t \in [a,b)$.

Proof. Let $D_\beta F_t = f_t$ for all $t \in [a,b)$ except possibly for a set B which has Lebesgue measure zero. Let $\xi \in [a,b) \setminus B$. Given $\varepsilon > 0$, there exists a positive function δ on $[a,b]$ such that whenever $(\xi, v]$ is δ-fine, we have

$$E(|f_\xi(W_v - W_\xi) - (F_v - F_\xi)|^2) \leqslant \varepsilon|v - \xi|.$$

Let $D = \{((\xi_i, v_i], \xi_i), i = 1,2,3,\ldots,n\}$ be a δ-fine belated partial division of $[a,b]$ with all $\xi_i \in [a,b] \setminus B$. Then

$$\begin{aligned} E\biggl(\biggl|\sum_{i=1}^{n} f_{\xi_i}(W_{v_i} - W_{\xi_i}) - (F_{v_i} - F_{\xi_i})\biggr|^2\biggr) \\ &= E\biggl(\sum_{i=1}^{n} |f_{\xi_i}(W_{v_i} - W_{\xi_i}) - (F_{v_i} - F_{\xi_i})|^2\biggr) \text{ by (i)} \\ &\leqslant \varepsilon \sum_{i=1}^{n} |v_i - \xi_i| \leqslant \varepsilon(b-a). \end{aligned}$$

Thus if $B = \varphi$, it is clear from the above that f is Itô integrable with $F_t = \int_a^t f_t \,\mathrm{d}W_t$. In general, B is nonempty with $\mu(B) = 0$.

Now let

$$B_m = \{t \in [a,b)\colon\ m - 1 < E[f_t^2] \leqslant m\},$$

where $\mu(B_m) = 0$ and $B = \bigcup\limits_{m=1}^{\infty} B_m$.

Since F has the AC^2 property, given any positive integer m, there exists $\eta_m > 0$ with $\eta_m \leqslant (\varepsilon/2^m)^2 \cdot m^{-2}$ such that whenever $\{(u_i, v_i]\}$ is a finite collection of disjoint left-open subintervals of $[a,b]$ with $\sum |v_i - u_i| \leqslant \eta_m$, we have

$$E\biggl(\biggl|\sum [F_{v_i} - F_{u_i}]\biggr|^2\biggr) \leqslant \Bigl(\frac{\varepsilon}{2^m}\Bigr)^2.$$

Take an open set $G_m \supset B_m$ such that $\mu(G_m) \leqslant \eta_m$.

Fix a positive integer m. Let $D = \{((\xi_i, v_i], \xi_i)\}$ be a β-fine belated partial division of $[a,b)$ such that $\xi_i \in B_m$ for all i. Then we have

$$\begin{aligned} E\Big(\Big|\sum_i f_{\xi_i}[W_{v_i} - W_{\xi_i}] - (F_{v_i} - F_{\xi_i})\Big|^2\Big) &\leqslant 2E\Big(\Big|\sum_i f_{\xi_i}(W_{v_i} - W_{u_i})\Big|^2\Big) + 2E\Big(\Big|\sum_i (F_{v_i} - F_{\xi_i})\Big|^2\Big) \\ &\leqslant 2\sum_i E[f_{\xi_i}^2](v_i - u_i) + 2\Big(\frac{\varepsilon}{2^m}\Big)^2 \leqslant 4\Big(\frac{\varepsilon}{2^m}\Big)^2. \end{aligned}$$

So, considering any β-fine belated partial division over $[a,b]$, denoted by $D_1 = \{((\xi_i, v_i], \xi_i)\}$, we have

$$\begin{aligned} E\Big(\Big|\sum_i f_{\xi_i}(W_{v_i} - W_{\xi_i}) - (F_{v_i} - F_{\xi_i})\Big|^2\Big) &\leqslant 2E\Big(\Big|\sum_{\xi\in[a,b]\setminus B} f_{\xi_i}(W_{v_i} - W_{\xi_i}) - (F_{v_i} - F_{\xi_i})\Big|^2\Big) \\ &\quad + 2E\Big(\Big|\sum_{m=1}^{\infty}\sum_{\xi_i\in B_m} f_{\xi_i}(W_{v_i} - W_{\xi_i}) - (F_{v_i} - F_{\xi_i})\Big|^2\Big) \\ &\leqslant 2\varepsilon(b-a) + 2\varepsilon, \end{aligned}$$

thereby showing that f is Itô integrable with $F_t = \int_a^t f_t \,\mathrm{d}W_t$. □

Combining Theorems 8 and 9, we have the following characterization of all Henstock-Kurzweil-Itô integrable stochastic processes:

Theorem 10. *Let f be an adapted process on $[a,b]$. Then f is Henstock-Kurzweil-Itô integrable on $[a,b]$ if and only if there exists an L^2-martingle F on $[a,b]$ with $F_a = 0$ a.s. and AC^2 on $[a,b]$ such that $D_\beta F_t = f_t$ almost everywhere on $[a,b)$.*

Example 11. From Example 3, $X_t = \frac{1}{2}W_t^2 - \frac{1}{2}t$ is an L^2-martingale on $[a,b]$. Hence the process

$$F_t = \frac{1}{2}W_t^2 - \frac{1}{2}W_a^2 - \frac{1}{2}(t-a),$$

where $F_a = 0$, is an L^2-martingale on $[a,b]$. It can be also easily verified that F is AC^2 on $[a,b]$. Furthermore, it was shown that

$$D_\beta X_t = W_t$$

on $[a,b]$, hence

$$D_\beta F_t = W_t$$

on $[a,b]$. By Theorem 10, we have

$$\int_a^b W_t \,\mathrm{d}W_t = F_b = \frac{1}{2}\left(W_b^2 - W_a^2\right) - \frac{1}{2}(b-a).$$

Example 12. Let $f \in \mathscr{L}_2$, the class of all classical Itô integrable adapted processes on the standard filtering space. Then there exists an L^2-martingale F on $[a,b]$ which is also AC^2 on $[a,b]$, such that $D_\beta F_t = f_t$ a.e. on $[a,b)$.

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