Applications of Mathematics

Lucas Jódar; Enrique A. Navarro

Co-solutions of algebraic matrix equations and higher order singular regular boundary value problems

Applications of Mathematics, Vol. 39 (1994), No. 3, 189–202

Persistent URL: http://dml.cz/dmlcz/134252

CO-SOLUTIONS OF ALGEBRAIC MATRIX EQUATIONS AND HIGHER ORDER SINGULAR REGULAR BOUNDARY VALUE PROBLEMS

LUCAS JÓDAR, ENRIQUE NAVARRO, Valencia

(Received May 28, 1992)

Summary. In this paper we obtain existence conditions and a closed form of the general solution of higher order singular regular boundary value problems. The approach is based on the concept of co-solution of algebraic matrix equations of polynomial type that permits the treatment of the problem without considering an extended first order system as it has been done in the known literature.

Keywords: Algebraic matrix equation, co-solution, singular regular system, boundary value problem, Drazin inverse, closed form solution.

AMS classification: 34A08, 34B10

1. INTRODUCTION

Troughout this paper $\mathbb{C}_{m\times n}$ denotes the set of all $m \times n$ matrices with complex entries. Systems of higher order differential equations of the type

$$A_p x^{(p)}(t) + A_{p-1}x^{(p-1)}(t) + \ldots + A_0 x(t) = f(t), \tag{1.1}$$

where A_i for $0 \leqslant i \leqslant p$ are matrices in $\mathbb{C}_{n\times n}$ and $x(t)$, $f(t)$ lie in $\mathbb{C}_{n\times 1}$, appear in vibrational systems theory [4], in thermal and electrical problems [4] and in the solution of partial differential equations by means of the method of lines [14].

The aim of this paper is to find a closed form expression for the general solution of multipoint boundary value problems defined by (1.1) together with the conditions

$$\sum_{h=1}^{p} E_{ih}x^{(h-1)}(a_i) = F_i, \qquad 1 \leqslant i \leqslant q,\ 0 = a_1 < a_2 < \ldots < a_q = a, \tag{1.2}$$

where $E_{ih} \in \mathbb{C}_{n\times n}$, $F_i \in \mathbb{C}_{n\times 1}$, $f(t)$ is sufficiently differentiable and the matrix coefficients A_i, $0 \leqslant i \leqslant p$, satisfy the regularity condition

$$\text{(1.3)} \qquad \begin{array}{c}\text{there exists a complex number } \lambda_0 \text{ such that} \\ P(\lambda_0) = A_p\lambda_0^p + A_{p-1}\lambda_0^{p-1} + \ldots + A_1\lambda_0 + A_0 \text{ is invertible.}\end{array}$$

The system (1.1) has been considered by several authors [2], [5], [6], [7], [8], [10], but all of the papers are based on the consideration of an extended first order system which involves an increase of the problem dimension and a lack of flexibility in the expression of the general solution of (1.1) that is required to find a closed form of the general solution of problem (1.1)–(1.2).

The paper is organized as follows. In Section 2 we introduce the concept of rectangular co-solution of the associated algebraic matrix equation

$$\text{(1.4)} \qquad A_pZ^p + A_{p-1}Z^{p-1} + \ldots + A_0 = 0$$

which permits us to find a closed form expression of the general solution of the homogeneous system

$$\text{(1.5)} \qquad A_px^{(p)}(t) + A_{p-1}x^{(p-1)}(t) + \ldots + A_0x(t) = 0.$$

Section 3 is concerned with the construction of a particular solution of the non-homogeneous problem (1.1). Finally, in Section 4, existence conditions and a closed form expression of the general solution of the boundary value problem (1.1)–(1.3) are presented.

If S is a rectangular matrix in $\mathbb{C}_{m\times n}$, we denote by S^+ its Moore-Penrose pseudoinverse and recall that an efficient procedure for computing S^+ may be found in [12]. If T is a matrix in $\mathbb{C}_{n\times n}$, we denote by T^D its Drazin inverse. An efficient algorithm for computing the Drazin inverse of a matrix is given in [1]. An account of properties of the Drazin inverse may be found in [3]. In particular, we recall that if T is invertible then T^{-1} coincides with T^D and if T is a nilpotent matrix then $T^D = 0$.

2. On the solution of the homogeneous differential system

We begin this section with some algebraic preliminaries related to the concept of rectangular co-solution for the non-monic algebraic matrix equation (1.4), which generalizes the analogous concept defined in [9] for the monic case.

Definition 2.1. We say that (X,T) with $X \in \mathbb{C}_{n\times r}$, $X \neq 0$, $T \in \mathbb{C}_{r\times r}$, is an (n,r) co-solution of equation (1.4) if

$$A_pXT^p + A_{p-1}XT^{p-1} + \ldots + A_1XT + A_0X = 0. \tag{2.1}$$

R e m a r k 1. Note that if (X,T) is an (n,r) co-solution of equation (1.4), then $x(t) = X\exp(tT)v$ with $v \in \mathbb{C}_{r\times 1}$ defines a solution of the homogeneous system (1.5) because

$$\begin{aligned} &A_px^{(p)}(t) + A_{p-1}x^{(p-1)}(t) + \ldots + A_0x(t) \\ &\quad = (A_pXT^p + A_{p-1}XT^{p-1} + \ldots + A_0X)\exp(tT)v = 0. \end{aligned}$$

Let us suppose that the regularity condition (1.3) is satisfied and let us consider in (1.5) the transformation defined by

$$x(t) = \exp(t\lambda_0)y(t). \tag{2.2}$$

Then an easy computation yields that $y(t)$ satisfies

$$\begin{aligned} &B_py^{(p)}(t) + B_{p-1}y^{(p-1)}(t) + \ldots + B_1y^{(1)}(t) + y(t) = 0, \\ &B_j = \frac{1}{j!}\left[P(\lambda_0)\right]^{-1}P^{(j)}(\lambda_0), \qquad 1 \leqslant j \leqslant p. \end{aligned} \tag{2.3}$$

The following lemma relates co-solutions of monic and non-monic algebraic matrix equations of polynomial type.

Lemma 1. *If (Z,S) is an (n,r) co-solution of the equation*

$$W^p + B_1W^{p-1} + \ldots + B_{p-1}W + B_p = 0 \tag{2.4}$$

and S is not a nilpotent matrix, then for $q \geqslant 1$ the pair $(ZS^{q+1}S^D, S^D)$ is an (n,r) co-solution of the matrix equation

$$B_pY^p + B_{p-1}Y^{p-1} + \ldots + B_1Y + I = 0. \tag{2.5}$$

P r o o f. Since (Z,S) is an (n,r) co-solution of (2.4), it follows that

$$ZS^p + B_1ZS^{p-1} + \ldots + B_{p-1}ZS + B_pZ = 0.$$

Postmultiplying this equation by $S^{q-1}(S^D)^{p-1}$ one gets

$$\tag{2.6} \begin{aligned} &B_p Z S^{q-1}(S^D)^{p-1} + B_{p-1} Z S^{q}(S^D)^{p-1} + \dots \\ &+ B_1 Z S^{p+q-2}(S^D)^{p-1} + Z S^{p+q-1}(S^D)^{p-1} = 0. \end{aligned}$$

Taking into account that from the properties of the Drazin inverse [3], p. 8 we have

$$\tag{2.7} SS^D = S^D S, \quad S^D S S^D = S^D,$$

we see that (2.6), (2.7) implies

$$\begin{aligned} &B_p Z S^{q+1} S^D (S^D)^p + B_{p-1} Z S^{q+1} S^D (S^D)^{p-1} + \dots \\ &+ B_1 Z S^{q+1} S^D S^D + Z S^{q+1} S^D = 0. \end{aligned}$$

Thus the result is established. □

For the sake of clarity of presentation we include a definition given in [9].

Definition 2.2 ([9]). Let (X_i, T_i) be an (n, m_i) co-solution of equation (2.4) for $1 \leqslant i \leqslant k$. We say that $\{(X_i, T_i),\ 1 \leqslant i \leqslant k\}$ is a k-complete set of co-solutions of equation (2.4), if the generalized block Vandermonde matrix

$$\tag{2.8} V = \begin{bmatrix} X_1 & X_2 & \dots & X_k \\ X_1 T_1 & X_2 T_2 & \dots & X_k T_k \\ \vdots & & & \\ X_1 T_1^{p-1} & X_2 T_2^{p-1} & \dots & X_k T_k^{p-1} \end{bmatrix}$$

is invertible in $\mathbb{C}_{np \times np}$.

The proof of the next result may be found in [9].

Theorem 1 ([9]). *Let C be the companion matrix defined by*

$$\tag{2.9} C = \begin{bmatrix} 0 & I & 0 & \dots & 0 \\ 0 & 0 & I & \dots & 0 \\ \vdots & & & & \\ -B_p & -B_{p-1} & -B_{p-2} & \dots & -B_1 \end{bmatrix}$$

and let $M = (M_{ij})$ be an invertible matrix in $\mathbb{C}_{np \times np}$ such that $M_{ij} \in \mathbb{C}_{n \times n_j}$, $1 \leqslant i \leqslant p$, $1 \leqslant j \leqslant k$, $n_1 + \ldots + n_k = np$ and

$$\tag{2.10} M \operatorname{diag}(S_1, S_2, \dots, S_k) = CM,$$

where $J = \operatorname{diag}(S_1, \dots, S_k)$ is the Jordan canonical form of C with $S_j \in \mathbb{C}_{n_j \times n_j}$, $1 \leqslant j \leqslant k$. Then $\{(M_{1j}, S_j);\ 1 \leqslant j \leqslant k\}$ is a k-complete set of co-solutions of (2.4). The general solution of the system

$$W^{(p)}(t) + B_1 W^{(p-1)}(t) + \ldots + B_{p-1} W^{(1)}(t) + B_p W(t) = 0 \tag{2.11}$$

is given by

$$W(t) = \sum_{j=1}^{k} M_{1j} \exp(tS_j) q_j, \qquad q_j \in \mathbb{C}_{n_j \times 1}. \tag{2.12}$$

Remark 2. It is interesting to recall that the Jordan canonical form of a matrix can be efficiently computed by using MACSYMA, [11], and the matrix exponential $\exp(tS_j)$ of a Jordan block S_j has a well known expression in terms of the eigenvalue associated to S_j, [13], p. 66.

Remark 1 and Lemma 1 imply that for arbitrary vectors $v_j \in \mathbb{C}_{n_j \times 1}$, the expression

$$\sum_{j=1}^{k} M_{1j} S_j^{q+1} S_j^D \exp(tS_j^D) v_j \tag{2.13}$$

defines solutions of the non-monic system (2.3). On the other hand, let us order the Jordan blocks of the matrix J defined in Theorem 1 in the following way:

$$\begin{array}{c} S_1, \dots, S_h \text{ are invertible blocks of } J, \\ S_{h+1}, \dots, S_k \text{ are nilpotent blocks of } J, \end{array} \tag{2.14}$$

and taking into account that $S_j^D = S_j^{-1}$ for $1 \leqslant j \leqslant h$ and $S_j^D = 0$ for $h+1 \leqslant j \leqslant k$, the expression (2.13) takes the form

$$\sum_{j=1}^{h} M_{1j} S_j^q \exp(tS_j^{-1}) v_j, \qquad v_j \in \mathbb{C}_{n_j \times 1}.$$

This result proves that in fact

$$y(t) = \sum_{j=1}^{h} M_{1j} S_j^{p-1} \exp(tS_j^{-1}) v_j, \qquad v_j \in \mathbb{C}_{n_j \times 1} \tag{2.15}$$

describes the general solution of (2.3).

From Lemma 1, for any positive integer $q \geqslant 1$, the pair $(M_{1j} S_j^q, S_j^{-1})$ is an (n, n_j) co-solution of (2.5) for $1 \leqslant j \leqslant h$ and thus for arbitrary vectors $v_j \in \mathbb{C}_{n_j \times 1}$, the right-hand side of (2.15) defines a solution of (2.3). Now we prove that any solution $z(t)$ of (2.3) may be represented by (2.15) with appropriate vectors v_j in $\mathbb{C}_{n_j \times 1}$ for $1 \leqslant j \leqslant h$. Let $z(t)$ be a solution of (2.3) and let $z^{(i)}(0) = c_i$, $0 \leqslant i \leqslant p-1$. Considering the change defined by

$$u(t) = \begin{bmatrix} y(t) \\ y'(t) \\ \vdots \\ y^{(p-1)}(t) \end{bmatrix},$$

$$Bu'(t) = u(t), \quad u(0) = \begin{bmatrix} c_0 \\ \vdots \\ c_{p-1} \end{bmatrix}, \tag{2.16}$$

$$B = \begin{bmatrix} -B_1 & -B_2 & \dots & -B_{p-1} & -B_p \\ I & 0 & \dots & 0 & 0 \\ \vdots & & & & \\ 0 & 0 & \dots & I & 0 \end{bmatrix}$$

and recalling Theorem 3.1.3 of [2], p. 37, we conclude that the problem (2.16) is solvable and the solution is unique if and only if

$$\begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_{p-1} \end{bmatrix} \in \mathrm{Image}(BB^D). \tag{2.17}$$

Since

$$B = \begin{bmatrix} 0 & 0 & \dots & 0 & I \\ 0 & 0 & \dots & I & 0 \\ \vdots & & & & \\ I & 0 & \dots & 0 & 0 \end{bmatrix} C \begin{bmatrix} 0 & 0 & \dots & 0 & I \\ 0 & 0 & \dots & I & 0 \\ \vdots & & & & \\ I & 0 & \dots & 0 & 0 \end{bmatrix},$$

from (2.10) one gets

$$B = (M'_{ij}) \operatorname{diag}(S_1, \dots, S_k)(M'_{ij})^{-1}, \qquad M'_{ij} = M_{p+1-i,j},\ 1 \leqslant i \leqslant p,\ 1 \leqslant j \leqslant k. \tag{2.18}$$

From [2], p. 16, it follows that

$$B^D = (M'_{ij}) \operatorname{diag}(S_1^{-1}, S_2^{-1}, \dots, S_h^{-1}, 0, \dots, 0)(M'_{ij})^{-1},$$
$$BB^D = (M'_{ij}) \operatorname{diag}(I, I, \dots, I, 0, \dots, 0)(M'_{ij})^{-1}$$

and condition (2.17) means

$$
(2.19) \qquad \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_{p-1} \end{bmatrix} \in \text{Image} \begin{bmatrix} M'_{11} & M'_{12} & \dots & M'_{1h} \\ M'_{21} & M'_{22} & \dots & M'_{2h} \\ \vdots & & & \\ M'_{p1} & M'_{p2} & \dots & M'_{ph} \end{bmatrix}.
$$

On the other hand, the derivatives of $y(t)$ defined by (2.15) take the form

$$
y^{(i)}(t) = \sum_{j=1}^{h} M_{1j} S_j^{p-i-1} \exp(t S_j^{-1}) v_j.
$$

If we evaluate this expression at $t = 0$ and impose $y^{(i)}(0) = c_i$ for $0 \leqslant i \leqslant p-1$, it follows that vectors v_j, for $1 \leqslant j \leqslant h$, must verify

$$
(2.20) \qquad \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_{p-1} \end{bmatrix} = \begin{bmatrix} M_{11} S_1^{p-1} & M_{12} S_2^{p-1} & \dots & M_{1h} S_h^{p-1} \\ M_{11} S_1^{p-2} & M_{12} S_2^{p-2} & \dots & M_{1h} S_h^{p-2} \\ \vdots & & & \\ M_{11} & M_{12} & \dots & M_{1h} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_h \end{bmatrix}
$$

and by virtue of

$$
(2.21) \qquad M_{ij} = M_{1j} S_j^{i-1}, \qquad 1 \leqslant i \leqslant p
$$

it follows that the conditions (2.19) and (2.20) are equivalent and the general solution of (2.3) is given by

$$
y(t) = \sum_{j=1}^{h} M_{1j} S_j^{p-1} \exp(t S_j^{-1}) v_j = \sum_{j=1}^{h} M_{pj} \exp(t S_j^{-1}) v_j,
$$

where v_j is an arbitrary vector in $\mathbb{C}_{n_j \times 1}$ for $1 \leqslant j \leqslant h$. This together with (2.2) yields the following result:

Theorem 2. *Let us use the notation of Theorem* 1 *and let us assume that the regularity condition* (1.3) *is satisfied for some complex number* λ_0. *Then the general solution of* (1.5) *is given by*

$$
(2.22) \qquad x(t) = \sum_{j=1}^{h} M_{pj} \exp\left((\lambda_0 I + S_j^{-1}) t\right) v_j,
$$

where v_j *is an arbitrary vector in* $\mathbb{C}_{n_j \times 1}$ *for* $1 \leqslant j \leqslant h$.

3. THE NON-HOMOGENEOUS DIFFERENTIAL SYSTEM

Let us consider the system (1.1) under the regularity condition (1.3) satisfied by some complex number λ_0. Let us observe the notation of Theorems 1 and 2. If k_j denotes the index of the Jordan block S_j for $h+1 \leqslant j \leqslant k$, where in accordance with (2.14), $S_{h+1}, \ldots, S_k$ are the nilpotent Jordan blocks of the matrix J introduced in Theorem 1, we define the number ϱ by

$$\varrho = \max\left\{k_j = \operatorname{Ind}(S_j);\ h+1 \leqslant j \leqslant k\right\}. \tag{3.1}$$

We suppose that $f(t)$ is a $\varrho+p-1$ times continuously differentiable function. Taking into account the transformation (2.2), the system (1.1) takes the form

$$L\big[y(t)\big] = \hat{f}(t), \tag{3.2}$$

where

$$L\big[y(t)\big] = B_p y^{(p)}(t) + B_{p-1} y^{(p-1)}(t) + \ldots + B_1 y^{(1)}(t) + y(t), \tag{3.3}$$

B_j is defined by (2.3) for $1 \leqslant j \leqslant p$ and

$$\hat{f}(t) = \big[P(\lambda_0)\big]^{-1} \exp(-t\lambda_0) f(t). \tag{3.4}$$

Let M be the matrix introduced in Theorem 1, and let (M'_{ij}) be the block matrix defined in (2.18) with $M'_{ij} = M_{p+1-i,j}$, $1 \leqslant i \leqslant p$, $1 \leqslant j \leqslant k$. Let us denote

$$W = (W_{ij}) = (M'_{ij})^{-1}, \qquad W_{ij} \in \mathbb{C}_{n_j \times n},\ 1 \leqslant i \leqslant k,\ 1 \leqslant j \leqslant p; \tag{3.5}$$

then we are interested in obtaining a particular solution of (3.2) of the form

$$\begin{aligned} g(t) = {} & \sum_{j=h+1}^{k} \sum_{i=0}^{k_j-1} M_{pj} (S_j)^i W_{j1} \hat{f}^{(i)}(t) \\ & - \sum_{j=1}^{h} M_{pj} \exp(tS_j^{-1}) \int_0^t \exp(-uS_j^{-1}) S_j^{-1} W_{j1} \hat{f}(u)\, du\,. \end{aligned} \tag{3.6}$$

Taking the derivatives of $g(t)$ for $1 \leqslant r \leqslant p$, we find that

$$\begin{aligned} g^{(r)}(t) = {} & -\sum_{j=1}^{h} M_{pj} (S_j^{-1})^r \exp(tS_j^{-1}) \int_0^t \exp(-uS_j^{-1}) S_j^{-1} W_{j1} \hat{f}(u)\, du \\ & -\sum_{j=1}^{h} \sum_{q=1}^{r} M_{pj} (S_j^{-1})^q W_{j1} \hat{f}^{(r-q)}(t) \\ & + \sum_{j=h+1}^{k} \sum_{i=0}^{k_j-1} M_{pj} (S_j)^i W_{j1} \hat{f}^{(i+r)}(t). \end{aligned} \tag{3.7}$$

From (3.7) and taking into account that from (2.18) one gets

$$B(M'_{ij}) = (M'_{ij})\,\mathrm{diag}(S_1, S_2, \ldots, S_k),$$

we conclude that

$$\begin{aligned} L[g(t)] = &-\sum_{j=1}^{h}\Bigg\{\Bigg[\sum_{i=1}^{p} B_i M_{pj}(S_j^{-1})^i + M_{pj}\Bigg] \\ &\times \exp(tS_j^{-1})\int_0^t \exp(-uS_j^{-1})S_j^{-1}W_{j1}\hat{f}(u)\,\mathrm{d}u\Bigg\} \\ &+\sum_{j=1}^{k} M_{pj}W_{j1}\hat{f}(t) + \sum_{z=1}^{p-1}\sum_{v=1}^{p-z} B_{z+v}\Bigg\{\sum_{j=1}^{k} M_{p-v,j}W_{j1}\Bigg\}\hat{f}^{(z)}(t) \\ &+\sum_{w=0}^{k_j-1}\sum_{j=h+1}^{k}\Bigg\{\sum_{i=1}^{p} B_i M_{pj}S_j^{p-i} + M_{pj}S_j^{p}\Bigg\}S_j^{w}W_{j1}\hat{f}^{(p+w)}(t) \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} &\sum_{i=1}^{p} B_i M_{pj}S_j^{p-i} + M_{pj}S_j^{p} = 0, \qquad h+1 \leqslant j \leqslant k, \\ &\sum_{i=1}^{p} B_i M_{pj}(S_j^{-1})^i + M_{pj} = 0, \qquad 1 \leqslant j \leqslant h, \\ &\sum_{j=1}^{k} M_{pj}W_{j1} = I, \\ &\sum_{j=1}^{k} M_{p-v,j}W_{j1} = 0, \qquad 1 \leqslant v \leqslant p-1. \end{aligned}$$

Hence (3.8) implies

$$L[g(t)] = \hat{f}(t).$$

From linearity one concludes that the general solution of (3.2)–(3.4) is given by

$$y(t) = \sum_{j=1}^{h} M_{pj}\exp(S_j^{-1}t)v_j + g(t), \qquad v_j \in \mathbb{C}_{n_j\times 1}. \tag{3.9}$$

Taking into account the transformation defined by (2.2), one gets that the general solution of the regular system (1.1), (1.3) is given by

$$x(t) = \sum_{j=1}^{h} M_{pj}\exp\big((\lambda_0 I + S_j^{-1})t\big)v_j + \exp(t\lambda_0)g(t), \qquad v_j \in \mathbb{C}_{n_j\times 1}. \tag{3.10}$$

Note that (2.21) yields

$$M_{pj} = M_{1j} S_j^{p-1}, \qquad 1 \leqslant j \leqslant h. \tag{3.11}$$

This and (3.6), (3.10), enable us to write the general solution of (1.1), (1.3) in the form

$$\begin{aligned} x(t) = &\sum_{j=1}^{h} M_{1j} S_j^{p-1} \exp\left((\lambda_0 I + S_j^{-1})t\right) v_j \\ &- \sum_{j=1}^{h} M_{pj} \exp\left((\lambda_0 I + S_j^{-1})t\right) \int_0^t \exp(-u S_j^{-1}) S_j^{-1} W_{j1} \hat{f}(u)\, du \\ &+ \exp(t\lambda_0) \sum_{j=h+1}^{k} M_{1j} S_j^{p-1} \sum_{i=0}^{k_j - 1} S_j^i W_{j1} \hat{f}^{(i)}(t). \end{aligned} \tag{3.12}$$

Note that the term appearing in the last expression of (3.12) takes the form

$$\exp(t\lambda_0) \sum_{j=h+1}^{k} \sum_{i=0}^{k_j-1} M_{1j} S_j^{p+i-1} W_{j1} \hat{f}^{(i)}(t). \tag{3.13}$$

By (3.13), if S_j is a Jordan block with index $k_j < p$ then since $S_j^{p+i-1} = 0$, the terms of (3.13) corresponding to the block S_j do not appear in (3.13). On the other hand, if $h+1 \leqslant j \leqslant k$ and $k_j < p$, since $S_j^v = 0$ for $v \geqslant k_j$, if we denote by $h+1 \leqslant j_1 < j_2 < \ldots < j_q \leqslant k$ such that

$$k_{j_1} = \operatorname{Ind}(S_{j_1}) \geqslant p,\ k_{j_2} = \operatorname{Ind}(S_{j_2}) \geqslant p,\ \ldots,\ k_{j_q} = \operatorname{Ind}(S_{j_q}) \geqslant p \tag{3.14}$$

and the other Jordan blocks S_j with $h+1 \leqslant j \leqslant k$ have indices $k_j = \operatorname{Ind}(S_j) < p$ for

$$h+1 \leqslant j \leqslant k,\ j \neq j_r,\ 1 \leqslant r \leqslant q,\ k_j = \operatorname{Ind}(S_j) < p, \tag{3.15}$$

then (3.13) may be written in the form

$$P(t) = \exp(t\lambda_0) \sum_{r=1}^{q} \sum_{i=0}^{k_{j_r} - p} M_{pj_r} S_{j_r}^i W_{j_r 1} \hat{f}^{(i)}(t). \tag{3.16}$$

Apart from this simplification of (3.13), it is important to remark that we do not need to assume that $f(t)$ is a $\varrho + p - 1$ times continuously differentiable function because by (3.16) the derivatives $\hat{f}^{(i)}(t)$ are unnecessary for $i > \varrho$ when $\varrho \geqslant p$, and if

$\varrho < p$, we only need to compute $p-1$ derivatives of $f(t)$. Summarizing we establish the following result:

Theorem 3. *Let us observe the notation of Theorem 1, let $S_1, \ldots, S_h$ be the invertible Jordan blocks of J and let $S_{h+1}, \ldots, S_k$ be the nilpotent Jordan blocks of J. Let k_j be the index of the matrix S_j for $h+1 \leqslant j \leqslant k$, and let j_r for $1 \leqslant r \leqslant q$ satisfy $h+1 \leqslant j_r \leqslant k$ and $k_{j_r} \geqslant p$, and $\mathrm{Ind}(S_j) = k_j < p$ for $h+1 \leqslant j \leqslant k$ and $j \neq j_r$, $1 \leqslant r \leqslant q$. Let ϱ be defined by (3.1) and let $f(t)$ be a w times continuously differentiable function where*

$$w = \max\{p-1, \varrho\}. \tag{3.17}$$

Then the general solution of the regular problem (1.1)–(1.3) is given by

$$\begin{aligned} x(t) = {} & \exp(t\lambda_0)\Bigg\{\sum_{j=1}^{h} M_{pj}\exp(S_j^{-1}t) \\ & \times \left[v_j - \int_0^t \exp(-S_j^{-1}u)S_j^{-1}W_{j1}\hat{f}(u)\,\mathrm{d}u\right]\Bigg\} + P(t), \end{aligned} \tag{3.18}$$

where $P(t)$ is defined by (3.16) and v_j is an arbitrary vector in $\mathbb{C}_{n_j \times 1}$ for $1 \leqslant j \leqslant h$. If J has no nilpotent blocks S_j with index $k_j \geqslant p$, then $P(t) = 0$.

Example 1. Let us consider the system

$$\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} x^{(2)}(t) + \begin{bmatrix} -1 & 2 \\ -3 & 3 \end{bmatrix} x'(t) + \begin{bmatrix} 0 & 0 \\ 3 & -2 \end{bmatrix} x(t) = \begin{bmatrix} 2\exp(2t) \\ \exp(2t) \end{bmatrix}. \tag{3.19}$$

Easy computation shows that taking $\lambda_0 = 1$, the matrix $P(1)$ appearing in (1.3) satisfies $P(1) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and the corresponding matrices B_1 and B_2 defined in (2.3) take the form

$$B_1 = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}.$$

In accordance with the notation of Theorem 1 we have

$$C = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & -1 & 0 \end{bmatrix}, \quad M = \begin{bmatrix} 0 & 1 & 2 & 1 \\ -1 & 0 & 3 & 1 \\ 0 & 1 & -2 & 0 \\ -1 & -1 & -3 & 0 \end{bmatrix},$$

$$W = \begin{bmatrix} -1 & \frac{1}{4} & \frac{5}{4} & -\frac{5}{4} \\ 1 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & -\frac{1}{4} & -\frac{1}{4} & \frac{1}{4} \\ -1 & 1 & 2 & -1 \end{bmatrix},$$

$$S_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad S_2 = (-1), \quad S_3 = (0),$$

$$M_{21} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}, \quad M_{22} = -\begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad M_{23} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$$W_{11} = \begin{bmatrix} -1 & \frac{1}{4} \\ 1 & -\frac{1}{2} \end{bmatrix}, \quad W_{21} = [0, -\tfrac{1}{4}], \quad W_{31} = [-1, 1].$$

Note that $J = \operatorname{diag}(S_1, S_2, S_3)$ has no nilpotent Jordan blocks with index $k_j \geqslant 2$ and thus the function $P(t)$ from Theorem 3 is zero. By Theorem 3, the general solution of system (3.19) is given by

$$x(t) = \begin{bmatrix} 0 & 1 \\ -1 & t-1 \end{bmatrix} \exp(2t) v_1 - \begin{bmatrix} 2 \\ 3 \end{bmatrix} v_2 - \frac{1}{2} \begin{bmatrix} -1 \\ t - \frac{3}{2} \end{bmatrix} \exp(2t),$$

where v_1 is an arbitrary vector in $\mathbb{C}_{2\times 1}$ and v_2 is an arbitrary complex number.

4. Boundary value problems

Let us observe the notation of the previous sections and for the sake of convenience let us write the general solution of (1.1), (1.3) in the form

$$x(t) = \sum_{j=1}^{h} M_{pj} \exp\big((\lambda_0 I + S_j^{-1})t\big) v_j + Q(t), \tag{4.1}$$

where v_j is an arbitrary vector in $\mathbb{C}_{n_j \times 1}$ and $Q(t)$ is given by

$$Q(t) = P(t) - \exp(t\lambda_0) \sum_{j=1}^{h} M_{pj} \int_0^t \exp\big((t-u)S_j^{-1}\big) S_j^{-1} W_{j1} \hat{f}(u)\, du, \tag{4.2}$$

while $P(t)$ is given by (3.16) if J has nilpotent Jordan blocks S_j with index $k_j \geqslant p$, and $P(t) = 0$ in the other case.

If we assume that the function $x(t)$ defined by (4.1)–(4.2) satisfies the boundary value conditions of (1.2), it follows that the vectors v_j must verify

$$\sum_{s=1}^{p}\sum_{j=1}^{h} E_{is}M_{pj}(\lambda_0 I+S_j^{-1})^{s-1}\exp\left(a_i(\lambda_0 I+S_j^{-1})\right)v_j = F_i-\sum_{s=1}^{p} E_{is}Q^{(s-1)}(a_i) \tag{4.3}$$

for $1 \leqslant i \leqslant q$.

Let us denote by G_i and S_{ij} the matrices

(4.4)

$$S_{ij} = \sum_{s=1}^{p} E_{is}M_{pj}(\lambda_0 I + S_j^{-1})^{s-1}\exp\left(a_i(\lambda_0 I + S_j^{-1})\right), \qquad 1 \leqslant i \leqslant q,\ 1 \leqslant j \leqslant h,$$

$$G_i = F_i - \sum_{s=1}^{p} E_{is}Q^{(s-1)}(a_i), \qquad 1 \leqslant i \leqslant q.$$

Then (4.3)–(4.4) imply that the vectors v_j must solve the algebraic system

$$(S_{ij})\begin{bmatrix} v_1 \\ \vdots \\ v_h \end{bmatrix} = \begin{bmatrix} G_1 \\ \vdots \\ G_q \end{bmatrix}, \tag{4.5}$$

where (S_{ij}) is the block partitioned matrix with entries S_{ij} defined in (4.4) and vectors G_i for $1 \leqslant i \leqslant q$ are defined by (4.4).

Now by Theorem 2.3.2 of [15], p. 24, the algebraic system (4.5) is compatible if and only if

$$[I - (S_{ij})(S_{ij})^{+}]\begin{bmatrix} G_1 \\ \vdots \\ G_q \end{bmatrix} = 0, \tag{4.6}$$

and under the condition (4.6) the general solution of (4.5) is given by

$$\begin{bmatrix} v_1 \\ \vdots \\ v_h \end{bmatrix} = (S_{ij})^{+}\begin{bmatrix} G_1 \\ \vdots \\ G_q \end{bmatrix} + \left[I - (S_{ij})^{+}(S_{ij})\right]^{D}, \tag{4.7}$$

where D is an arbitrary vector in $\mathbb{C}_{d\times 1}$ with

$$d = \sum_{j=1}^{h} n_j. \tag{4.8}$$

By the previous comments the following result has been established:

Theorem 4. *Let us observe the notation of Theorem 3 and let us consider the boundary value problem* (1.1)–(1.3) *where* $f(t)$ *is a* w *times continuously differentiable function on the interval* $[0, a]$. *If* (S_{ij}) *and* G_i *are defined by* (4.4) *for* $1 \leqslant i \leqslant q$, $1 \leqslant j \leqslant h$, *then the boundary value problem* (1.1)–(1.3) *is solvable if and only if the condition* (4.6) *is satisfied. Under this condition the general solution of the problem is given by* (4.1)–(4.2), *where vectors* $v_1, \ldots, v_h$ *are determined by* (4.7) *and* D *is an arbitrary vector in* $\mathbb{C}_{d\times 1}$ *with* d *defined by* (4.8).

Acknowledgement. This work has been partially supported by the D.G.I.C.Y.T. grant PS90-0140.

References

[1] *K. M. Anstreicher and U. G. Rotblum*: Using Gauss-Jordan elimination to compute the index, generalized nullspaces and the Drazin inverse. Linear Algebra Appl. *85* (1987), 221–239.

[2] *S. L. Campbell*: Singular Systems of Differential Equations. Pitman Pubs. Co., 1980.

[3] *S. L. Campbell and C. D. Meyer, jr.*: Generalized Inverses of Linear Transformations. Pitman Pubs. Co., 1979.

[4] *R. C. Dorf*: Modern Control Systems. Addison-Wesley Pubs. Co., 1974.

[5] *R. J. Duffin*: A minimax theory of overdamped networks. J. Rat. Mech. and Anal. *4* (1955), 221–223.

[6] *R. J. Duffin*: Chrystal's theorem on differential equations systems. J. Math. Anal. and Appl. *8* (1963), 325–331.

[7] *I. Gohberg, M. A. Kaashoek, L. Lerer and L. Rodman*: Common multiples and common divisors of matrix polynomials, I. Spectral method. Ind. J. Math. *30* (1981), 321–356.

[8] *J. W. Hooker and C. E. Langenhop*: On regular systems of linear differential equations with constant coefficients. Rocky Mountain J. Maths. *12* (1982), 591–614.

[9] *L. Jódar and E. Navarro*: Rectangular co-solutions of polynomial matrix equations and applications. Applied Maths. Letters *4(2)* (1991), 13–16.

[10] *C. E. Langenhop*: The Laurent expansion of a nearly singular matrix. Linear Algebra Appls. *4* (1971), 329–340.

[11] MACSYMA. MACSYMA Symbolic Inc., 1989.

[12] *C. B. Moler*: MATLAB User's guide, Tech. Rep. CS81. Department of Computer Science, Univ. of Mexico, Alburquerque, New Mexico 87131, 1980.

[13] *J. M. Ortega*: Numerical Analysis, A second course. Academic Press, 1972.

[14] *K. Rektorys*: The Method of Discretization in Time and Partial Differential Equations. D. Reidel Pubs. Co., 1982.

[15] *C. R. Rao and S. K. Mitra*: Generalized Inverse of Matrices and its Applications. John Wiley, 1971.

Author's address: Lucas Jódar, Enrique Navarro, Departamento de Matemática Aplicada, Universidad Politécnica de Valencia, Apdo. 22.012 Valencia, Spain.