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Arthur D. Gorman

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ON CAUSTICS ASSOCIATED WITH ROSSBY WAVES

ARTHUR D. GORMAN, Easton

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Summary. Rossby wave equations characterize a class of wave phenomena occurring in geophysical fluid dynamics. One technique useful in the analysis of these waves is the geometrical optics, or multi-dimensional WKB technique. Near caustics, e.g., in critical regions, this technique does not apply. A related technique that does apply near caustics is the Lagrange Manifold Formalism. Here we apply the Lagrange Manifold Formalism to study Rossby waves near caustics.

Keywords: Rossby waves, caustics, turning points, Lagrange Manifold, WKB

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1. INTRODUCTION

An important equation in the mathematical analysis of large-scale atmospheric flow processes is the Rossby wave equation

$$\frac{\partial}{\partial t}(\nabla^2\psi) + \beta(y)\frac{\partial\psi}{\partial x} = 0. \tag{1}$$

In this equation t is the time, x and y are spatial coordinates, $\psi(x, y, t)$ is the streamline and $\beta(y)$ is a vorticity parameter. Such equations cannot be solved exactly. Consequently, various techniques, each valid under situation-specific assumptions, have been developed to obtain approximate solutions and to study the phenomena the equations model. One such technique is the multi-dimensional WKB, or geometrical optics, approach developed by Keller and his students [1]. Karoly and Hoskins [2] and Yang [3], among others, have employed this approach and extended it to study Rossby-type wave equations, i.e., variations of equation 1, cf. [3] for a careful exposition which includes much of Yang's early work. Near turning or caustic points

the classical WKB technique is not valid [4], for example, near the "critical layers", layers where the phase velocity of the wave coincides with the velocity of the large-scale current [5, 6], or near sharp topographies [7]. A related approach that is valid at caustics is the Lagrange Manifold formalism developed by Maslov [8] and Arnol'd [9]. Here we apply the Lagrange Manifold approach to obtain asymptotic solutions of Rossby-type wave equations at caustic points. We also use their approach to study wave phenomena at caustics in a manner analogous to the use of the classical WKB technique in studying wave phenomena at off-caustic points.

2. Formalism

Since Yang's treatment is noteworthy for its clarity, we parallel his development. We first re-scale our independent variables to "slower" variables

$$\varepsilon\bar{r} = \varepsilon(x, y) \to (x, y), \quad \varepsilon t \to t,$$

where ε is a small parameter. Then equation 1 becomes

$$\varepsilon^3 \frac{\partial}{\partial t}(\nabla^2\psi) + \varepsilon\widehat{\beta}(y)\frac{\partial\psi}{\partial x} = 0, \tag{2}$$

$\widehat{\beta}(y) = \beta(\varepsilon y)$. Near caustics we assume equation 2 has an asymptotic solution of the form

$$\psi(x, y, t) = \int A(\bar{r}, \bar{p}, t, \varepsilon)\mathrm{e}^{\mathrm{i}\varphi/\varepsilon}\,\mathrm{d}\bar{p}. \tag{3}$$

In equation 3, $\bar{r} = (x, y)$, and $\bar{p} = (p_x, p_y)$ may be regarded as a wavevector. The amplitude

$$A(\bar{r}, \bar{p}, t, \varepsilon) \sim \sum_{k=0} A_k(\bar{r}, \bar{p}, t)(\mathrm{i}/\varepsilon)^k$$

and its derivatives are assumed bounded and

$$\varphi(\bar{r}, t, \bar{p}, \omega) = \bar{r}\cdot\bar{p} - \omega t - S(\bar{p}, \omega)$$

where $S(\bar{p}, \omega)$ is analytic may be regarded as a phase [10]. Carrying the differentiation in equation 2 across the integral in equation 3, re-grouping by powers of (i/ε) and introducing the wave vector $\bar{p}$ and frequency ω

$$\bar{p} = \nabla\varphi, \quad \omega = -\frac{\partial\varphi}{\partial t} \tag{4}$$

leads to

$$\int \Big\{\varepsilon^3\Big[\frac{\partial}{\partial t}(\nabla^2 A) - \frac{\mathrm{i}}{\varepsilon}\omega\nabla^2 A + \frac{2\mathrm{i}}{\varepsilon}\bar{p}\cdot\frac{\partial}{\partial t}(\nabla A) + \frac{2\omega}{\varepsilon^2}\bar{p}\cdot\nabla A - \frac{\bar{p}\cdot\bar{p}}{\varepsilon^2}\frac{\partial A}{\partial t} + \frac{\mathrm{i}}{\varepsilon^3}\bar{p}\cdot\bar{p}\omega A\Big] + \varepsilon\widehat{\beta}\Big(\frac{\partial A}{\partial x} + \mathrm{i}\,\frac{p_x}{\varepsilon}A\Big)\Big\}\mathrm{e}^{i\varphi/\varepsilon}\,\mathrm{d}\bar{p} \sim 0. \tag{5}$$

The coefficient of the $(\mathrm{i}/\varepsilon)^0$ term is Maslov's Hamiltonian,

$$H = (\bar{p}\cdot\bar{p})\omega + p_x\widehat{\beta}(y). \tag{6}$$

The integral is evaluated at any point $\bar{r} = (x, y)$ using the stationary phase technique ($\nabla_p\varphi = 0$), which turns Maslov's Hamiltonian into an eikonal equation

$$(\bar{p}\cdot\bar{p})\omega + p_x\widehat{\beta}(y) = 0 \tag{7}$$

and determines the Lagrange Manifold,

$$\bar{r} = \nabla_p S(\bar{p}, \omega), \tag{8}$$

a coordinate transformation between configuration space and momentum space once $S(\bar{p}, \omega)$ is known. To determine $S(\bar{p}, \omega)$ we first apply Hamilton's equations

$$\frac{\mathrm{d}\bar{r}}{\mathrm{d}\gamma} = \nabla_p H \qquad \frac{\mathrm{d}\bar{p}}{\mathrm{d}\gamma} = -\nabla_r H$$
$$\frac{\mathrm{d}t}{\mathrm{d}\gamma} = -\frac{\partial H}{\partial\omega} \qquad \frac{\mathrm{d}\omega}{\mathrm{d}\gamma} = \frac{\partial H}{\partial t}$$

where γ is a ray-path parameter, to find the trajectories (maps)

$$\bar{r} = \bar{r}(\gamma, \bar{\sigma}) \qquad \bar{p} = \bar{p}(\gamma, \bar{\sigma})$$
$$t = t(\gamma, \bar{\sigma}) \qquad \omega = \omega(\gamma, \bar{\sigma})$$

where $\bar{\sigma}$ is a parametrized initial condition. Next, inverting the frequency, time and wave-vector transformations obtains

$$\gamma = \gamma(\bar{p}, \omega) \quad \bar{\sigma} = \bar{\sigma}(\bar{p}, \omega).$$

Then, substituting into the coordinate space maps determines the Lagrange manifold explicitly

$$\bar{r} = \bar{r}(\gamma(\bar{p}, \omega), \bar{\sigma}(\bar{p}, \omega)) = \nabla_p S(\bar{p}, \omega) \tag{9}$$

where the frequency appears as a parameter. Finally, an integration along the trajectories obtains

$$S(\bar{p},\omega) = \int_{\bar{p}_0}^{\bar{p}} \bar{r} \cdot \mathrm{d}\bar{p} \tag{10}$$

and hence the phase

$$\varphi(\bar{r},t,\bar{p},\omega) = \bar{r}\cdot\bar{p} - \omega t - S(\bar{p},\omega). \tag{11}$$

In this approach the entire caustic curve, or locus of turning points, may be determined by setting the Hessian determinant of the phase to zero

$$\det\Big(\frac{\partial^2\varphi}{\partial p_x \partial p_y}\Big) = 0. \tag{12}$$

Each real $\bar{p}$ satisfying this equation specifies a caustic point in wavevector space. The corresponding point in configuration space may be obtained by substituting this wavevector into the Lagrange Manifold, equation 9. The locus of these configuration space points is the caustic curve.

The determine a transport equation for the amplitudes we Taylor expand the Hamiltonian near the Lagrange Manifold

$$(\bar{p}\cdot\bar{p})\omega + p_x\widehat{\beta}(y) = (\bar{p}\cdot\bar{p})\omega + p_x\widehat{\beta}\Big(\frac{\partial S}{\partial p_y}\Big) + (\bar{r} - \nabla_p S)\cdot\overline{D} = (\bar{r} - \nabla_p S)\cdot\overline{D} \tag{13}$$

where

$$\overline{D} = \int_0^1 \nabla_r H(\xi(\bar{r} - \nabla_p S) + \nabla_p S, \bar{p}, \omega, t)\,\mathrm{d}\xi \tag{14}$$

that is, the remainder of the Taylor series. Next substituting into equation 5 and performing a partial integration obtains

$$\int \mathrm{d}\bar{p}\mathrm{e}^{\mathrm{i}\varphi/\varepsilon}\Big\{\varepsilon\Big[-\nabla_p\cdot\overline{D}A - \overline{D}\cdot\nabla_p A + 2\omega\bar{p}\cdot\nabla A - (\bar{p}\cdot\bar{p})\frac{\partial A}{\partial t} + \widehat{\beta}\frac{\partial A}{\partial x}\Big]$$
$$\mathrm{i}\varepsilon^2\Big(-\omega\nabla^2 A + 2\bar{p}\cdot\frac{\partial}{\partial t}(\nabla A)\Big) + \varepsilon^3\frac{\partial}{\partial t}(\nabla^2 A)\Big\} \sim 0.$$

Then introducing into this integral the non-Hamiltonian flow

$$\begin{aligned}
\dot{\bar{r}} &= (2\omega p_x + \widehat{\beta}(y), 2\omega p_y)\\
\dot{\bar{p}} &= -\overline{D}\\
\dot{t} &= -\bar{p}\cdot\bar{p}
\end{aligned}$$

leads to a transport equation for the amplitudes

$$\frac{\mathrm{d}A_k}{\mathrm{d}t} - \nabla_p \cdot \overline{D}A_k + \left(2\bar{p} \cdot \frac{\partial}{\partial t}(\nabla A_{k-1}) - \omega\nabla^2 A_{k-1}\right) + \frac{\partial}{\partial t}(\nabla^2 A_{k-2}) = 0. \tag{15}$$

$\psi(\bar{r}, t)$ may now be determined at any caustic point by asymptotic evaluation of the integrals

$$I = \int A_k(\bar{r}, \bar{p}, t)\mathrm{e}^{\mathrm{i}\varphi/\varepsilon}\,\mathrm{d}\bar{p}, \tag{16}$$

cf. [11].

We illustrate using an example considered by Yang, namely

$$\frac{\partial\widehat{\beta}}{\partial y} = -\delta$$

where δ is a constant, or equivalently

$$\widehat{\beta}(y) = -\delta y.$$

For this case we obtain the Hamiltonian

$$H = (p_x^2 + p_y^2)\omega - \delta y p_x$$

and for initial position $\bar{r}_0 = (0, 0)$ and direction cosines $\bar{p}_0 = (p\cos\theta, p\sin\theta)$, we determine the flow

$$\begin{aligned} x(\gamma, \theta) &= \frac{1}{3}\omega\delta^3 p\gamma^3\cos\theta - \omega\delta p\gamma^2\sin\theta + 2p\gamma\omega\cos\theta \\ y(\gamma, \theta) &= -\omega\delta p\gamma^2\cos\theta + 2p\omega\gamma\sin\theta. \end{aligned}$$

Then from equations 8, 11 and 15, respectively, we determine the Lagrange Manifold

$$\begin{aligned} y &= \frac{\omega}{\delta p_x}(p_y^2 + p_x^2 - p_0^2) = \frac{\partial S}{\partial p_y} \\ x &= \frac{-\omega p_y^3}{3\delta p_x^2} + \frac{p_0^2\omega p_y}{\delta p_x^2} + \frac{\omega p_y}{\delta} - \frac{\omega}{\delta}\left(\frac{4}{3} + \frac{2p_0^2}{3p_x^2}\right)(p_0^2 - p_x^2)^{1/2} = \frac{\partial S}{\partial p_x} \end{aligned}$$

phase

$$\varphi = xp_x + yp_y - \omega t - \frac{\omega}{\delta}\left(\frac{p_y^3}{3p_x} + p_x p_y - \frac{p_0^2 p_y}{p_x} + \left(\frac{2p_x}{3} - \frac{2p_0^2}{3p_x}\right)(p_0^2 - p_x^2)^{1/2}\right)$$

and transport equation

$$\frac{\mathrm{d}A_k}{\mathrm{d}t} - \nabla_p \cdot \overline{D} A_k + \left(2\bar{p} \cdot \frac{\partial}{\partial t}(\nabla A_{k-1}) - \omega \nabla^2 A_{k-1}\right) + \frac{\partial}{\partial t}(\nabla^2 A_{k-2}) = 0$$

where

$$\overline{D} = (0, \delta p_x).$$

The Lagrange Manifold technique also leads to some interesting correspondences between off-caustic and on-caustic phenomenology. If the eikonal equation is solved for ω, we obtain the same dispersion equation on the caustic as Yang obtains off the caustic

$$\omega = -\frac{\widehat{\beta}(y) p_x}{p_x^2 + p_y^2}. \tag{17}$$

Differentiating ω with respect to p_x and p_y we obtain the directional phase velocities

$$\begin{aligned} C_{gx} &= \frac{\partial \omega}{\partial p_x} = \frac{\widehat{\beta}(y)(p_x^2 - p_y^2)}{(p_x^2 + p_y^2)^2} \\ C_{gy} &= \frac{\partial \omega}{\partial p_y} = \frac{2\widehat{\beta}(y) p_x p_y}{(p_x^2 + p_y^2)^2}, \end{aligned} \tag{18}$$

which also follow from Hamilton's equations, if we replace the raypath parameter γ with time t, [12]. Then Hamilton's equations become

$$\frac{dx}{\mathrm{d}t} = \frac{\partial H}{\partial p_x} \Big/ \left(-\frac{\partial H}{\partial \omega}\right) = C_{gx} \tag{19a}$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\partial H}{\partial p_y} \Big/ \left(-\frac{\partial H}{\partial \omega}\right) = C_{gy} \tag{19b}$$

$$\frac{\mathrm{d}p_x}{\mathrm{d}t} = -\frac{\partial H}{\partial x} \Big/ \left(-\frac{\partial H}{\partial \omega}\right) = 0 \tag{19c}$$

$$\frac{\mathrm{d}p_y}{\mathrm{d}t} = -\frac{\partial H}{\partial y} \Big/ \left(-\frac{\partial H}{\partial \omega}\right) = \frac{\partial \widehat{\beta}}{\partial y} p_x (p_x^2 + p_y^2)^{-1} \tag{19d}$$

$$\frac{\mathrm{d}\omega}{\mathrm{d}t} = \frac{\partial H}{\partial \gamma} \Big/ \left(-\frac{\partial H}{\partial \omega}\right) = 0 \tag{19e}$$

$$\frac{\mathrm{d}t}{\mathrm{d}t} = -\frac{\partial H}{\partial \omega} \Big/ \left(-\frac{\partial H}{\partial \omega}\right) = 1 \tag{19f}$$

Further, from the definitions of wavevector and frequency, equation 4, we develop a "conservation of crests" equation

$$\frac{\partial \bar{p}}{\partial t} + \nabla \omega = 0. \tag{20}$$

Equation 17 through equation 20 are identical to the off-caustic equations developed by Yang. Consequently, the properties he has determined in his consideration of the off-caustic problem apply here as well. That is, on the caustic:

(i) The characteristic direction coincides with the group velocity, i.e., the wave packet direction, equations 19a, b.

(ii) If the medium is homogeneous, the local wavenumbers along the caustic curve are conserved. Any change in the local wavenumber in a given direction must be the result of an inhomogeneity of the medium in that direction, equation 19c, d.

(iii) If the medium is autonomous, the frequency along the caustic curve is conserved. Any change in the frequency must be the result of a temporal inhomogeneity in the medium, equation 19e.

3. Addendum

The Lagrange Manifold formalism applies to linear differential equations containing a small parameter or to such equations which may be re-scaled to contain a small parameter. Geophysical fluid dynamics includes many such equations related to equation 1 in which the time appears as an explicit parameter of the medium. We note that the above formalism still applies in these cases. For example, if the medium is moving with an ambient flow characterized by

$$\nu(x,y,t) = (U(x,y,t), V(x,y,t))$$

the corresponding, already re-scaled, Rossby-type equation is

$$\varepsilon^3\Big(\frac{\partial}{\partial t} + U(x,y,t)\frac{\partial}{\partial x} + V(x,y,t)\frac{\partial}{\partial y}\Big)\nabla^2\psi + \varepsilon\widehat{\beta}(y)\frac{\partial\psi}{\partial x} = 0. \tag{21}$$

Assuming a solution of the form in equation 3 and proceeding as above results in the nonautonomous Hamiltonian

$$H = (\bar{p}\cdot\bar{p})(\omega - p_x U(x,y,t) - p_y V(x,y,t)) - p_x\widehat{\beta}(y). \tag{22}$$

Invoking the stationary phase condition obtains an eikonal equation which may be solved to obtain the same dispersion equation that Yang obtains in the off-caustic case,

$$\omega = \frac{p_x\widehat{\beta}(y)}{(\bar{p}\cdot\bar{p})} + \frac{p_x U(x,y,t)}{(\bar{p}\cdot\bar{p})} + \frac{p_y V(x,y,t)}{\bar{p}\cdot\bar{p}}. \tag{23}$$

From equations 22 and 23, Hamilton's Equations determine relationships similar to those in equations 18–20. The solution algorithm of the procedure above also applies in this case. For brevity, we do not duplicate it.

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Author's address: *Arthur D. Gorman*, Department of Mathematics, Lafayette College, Easton, Pennsylvania (18042), USA.