Applications of Mathematics

Tadie

Steady vortex rings with swirl in an ideal fluid: asymptotics for some solutions in exterior domains

Applications of Mathematics, Vol. 44 (1999), No. 1, 1--13

Persistent URL: http://dml.cz/dmlcz/134402

Terms of use:

© Institute of Mathematics AS CR, 1999

Institute of Mathematics of the Academy of Sciences of the Czech Republic provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these *Terms of use*.

This paper has been digitized, optimized for electronic delivery and stamped with digital signature within the project *DML-CZ: The Czech Digital Mathematics Library* http://project.dml.cz

STEADY VORTEX RINGS WITH SWIRL IN AN IDEAL FLUID: ASYMPTOTICS FOR SOME SOLUTIONS IN EXTERIOR DOMAINS

Tadie, Copenhagen

(Received October 13, 1997)

Abstract. In this paper, the axisymmetric flow in an ideal fluid outside the infinite cylinder ($r \leqslant d$) where (r, θ, z) denotes the cylindrical co-ordinates in $\mathbb{R}^3$ is considered. The motion is with swirl (i.e. the θ-component of the velocity of the flow is non constant). The (non-dimensional) equation governing the phenomenon is (Pd) displayed below. It is known from e.g. [9] that for the problem without swirl ($f_q = 0$ in (f)) in the whole space, as the flux constant k tends to ∞,

1) $\operatorname{dist}(0z, \partial A) = O(k^{1/2})$; $\operatorname{diam} A = O(\exp(-c_0 k^{3/2}))$;

2) $(k^{1/2}\Psi)_{k\in\mathbb{N}}$ converges to a vortex cylinder U_m (see (1.2)).

We show that for the problem with swirl, as $k \nearrow \infty$, 1) holds; if $m \leqslant q+2$ then 2) holds and if $m > q+2$ it holds with U_{q+2} instead of U_m. Moreover, these results are independent of f_0, f_q and $d > 0$.

Keywords: vortex rings, potential theory, elliptic equations

MSC 2000: 76M25 31B15 35J

1. Introduction

Let (r, θ, z) denote the cylindrical coordinates in $\mathbb{R}^3$. We consider an axisymmetric (w.r.t. Oz) flow in an ideal fluid occupying the exterior domain

$$\Omega_d := \{(r, \theta, z) \mid r > d,\ \theta = 0,\ z \in \mathbb{R}\}, \quad d > 0.$$

The problem is then posed in the half plane $\Pi_d := \{x = (r, z) \mid r > d, z \in \mathbb{R}\}$.

It si known (see e.g. [1]) that if $\mathbf{q} = (u, S, v)$ denotes the velocity of the flow and $\omega = (w_1, \omega, w_2) = \operatorname{curl} \mathbf{q}$ its vorticity, then $w_1 = -S_z$; $\omega = u_z - v_r$; $w_2 = -(1/r)\{rS\}_r$.

The mass conservation ($\mathrm{div}\mathbf{q} = 0$) implies that there is a stream function Ψ such that $u = -\Psi_z/r$, $v = \Psi_r/r$ whence $\omega = \{\Psi_{rr} - \Psi_r/r + \Psi_{zz}\}/r$.

From Bernoulli's equation $|\mathbf{q}|^2/2 + p/\varrho = H(\Psi)$ where p and ϱ denote the pressure and the density and H a scalar function, the dynamical equation $\mathbf{q}\times\omega - \mathbf{q}_t = \mathrm{grad}\, H$, rS and ω/r are constant on each stream line and for $rS = C(\Psi)$ we have $\omega/r \equiv L\Psi := r^{-2}\{\Psi_{rr} - \Psi_r/r + \Psi_z\} = r^{-2}C(\Psi)C'(\Psi) - H'(\Psi)$.

So, as seen in [1], the non-dimensional equations (see [9]) governing the flow are

$$(Pd)\begin{cases} L\psi := \dfrac{1}{r^2}\Big\{\partial_r^2 - \dfrac{1}{r}\partial_r + \partial_z^2\Big\}\psi = -\lambda f(r,\Psi) \quad \text{in } \Pi_d \\ \psi|_{r=0} = 0; \quad \psi \text{ and } |\nabla\psi| \searrow 0 \quad \text{as } |x| = \sqrt{r^2+z^2} \nearrow \infty \text{ in } \Pi_d \end{cases}$$

where the stream functions are related by $\psi(x) := \Psi(x) + r^2/2 + k$, the vorticity function f is here defined for some $m, q \geqslant 0$ and $f_0, f_q \geqslant 0$ by

$$\text{(f)} \qquad f(r,t) := f_q\,\frac{\{t_+\}^q}{r^2} + f_0\,\{t_+\}^m$$

where $t_+ := \max\{t, 0\}$ and $f_q = 0$ *for the problem without swirl.* The parameter $\lambda > 0$ is a Lagrangian multiplier, determined a posteriori. The parameter $k > 0$ denotes the flux constant, measuring the flux of the fluid between the boundary $r = d$ and the boundary of the ring ∂A where

$$\text{(A)} \qquad A := \{x \in \Pi_d \mid \Psi(x) > 0\}$$

denotes the cross-section of the ring. The problem is then to find solutions $\psi \in C^1(\overline{\Pi_d})$ and the corresponding A for (Pd).

We are concerned with the variational solutions of the type found in [3], i.e. local maximizers of the functional

$$\text{(Z)} \qquad Z(u) := \int_{\Pi_d} F(r,U)\,\mathrm{d}\tau;\ \mathrm{d}\tau := r\,\mathrm{d}r\,\mathrm{d}z; \quad F(r,T) := \int_0^T f(r,s)\,\mathrm{d}s$$

on the sphere $S_1(\Pi_d) := \{u \in H_d := H(\Pi_d) \mid \|u\|^2 = 1\}$ where $U(x) := u(x) - r^2/2 - k$ and H_d denotes the completion of $C_0^\infty(\Pi_d)$ in the norm

$$\text{(1.1a)} \qquad \|u\| \equiv \|u\|_{H_d} := \left(\int_{\Pi_d} \frac{u_r^2 + u_z^2}{r^2}\,\mathrm{d}\tau\right)^{1/2}.$$

Note that for $u \in C_0^\infty(\Pi_d)$, $\|u\| = \big(-\int_{\Pi_d} uLu\,\mathrm{d}\tau\big)^{1/2}$ and H_d is a Hilbert space with the scalar product

$$\text{(1.1b)} \qquad \langle u, v\rangle_{H_d} := \int_{\Pi_d} \frac{1}{r^2}\{u_r v_r + u_z v_z\}\,\mathrm{d}\tau.$$

The problem without swirl for which Z is replaced by

$$\text{(J)} \qquad J_m(u) := f_0 \int_\Pi \frac{\{U_+\}^{m+1}}{m+1}\,\mathrm{d}\tau$$

in $\Pi := \Pi_0$ has variational solutions ψ such that ([3])

a) $\psi \in C^2(\overline{\Pi})$ if $m > 0$ and $\psi \in C^2(\overline{\Pi} \setminus \overline{\partial A}) \bigcap C^1(\overline{\Pi})$ for $m = 0$;

b) ψ is an even function of z and $\psi_z < 0$ if $z > 0$;

c) for $k > 0$, A has a finite number of simply connected components ([9]) and is simply connected if $m \geqslant 1$.

For the asymptotics of these solutions ([9]), for large values of k,

d) $|a^2 - (2/3)k| = O(k^{-1/2}\log k)a := (r_1 + r_2)/2$ where $r_1 := \inf\{r > 0 \mid (r, 0) \in A\}$ ($r_2 := \sup\{r > 0 \mid (r, 0) \in A\}$;

e) for $\varepsilon > 0$ such that $\operatorname{diam} A = 2a\varepsilon$ and $c_0 = 8\pi(2/3)^{3/2}$, $\varepsilon \leqslant C\exp\{-c_0 k^{3/2}\}$, $\lambda \leqslant Ck^{(m-2)/2}\exp\{2c_0 k^{3/2}\}$ and $|\Psi|_{C(\overline{A})} = O(k^{-1/2})$; let

$$\text{(1.}\Pi\text{)} \qquad \hat{\Pi} := \{\zeta = (\xi, \eta) \mid \xi > -1/\varepsilon, \quad \eta \in \mathbb{R}\}$$

denote the image of Π in the transformation

$$\text{(1.}\zeta\text{)} \qquad r = a(1 + \varepsilon\xi), \quad z = a\varepsilon\eta$$

and for u defined in Π let $\hat{u}(\zeta) := u(a(1+\varepsilon\xi), a\varepsilon\eta)$; when $k \nearrow \infty$ the functions $k^{1/2}\Psi$ converge in $C^1(\Pi)$ to a function U_m such that $\hat{U}_m$ is radial; namely

$$\text{(1.2)} \qquad \hat{U}_m(\sigma) = \frac{\sqrt{6}}{4\pi\varrho_m^2} Q_m(\varrho_m\,\sigma)$$

where Q_m is the unique solution of

$$\text{(1.Q)} \qquad Q'' + Q'/\sigma = -Q_+^m, \quad \sigma > 0; \quad Q(0) = 1; \quad Q'(0) = 0$$

and ϱ_m is its unique positive zero ([8]). In this context the function U will be called a *vortex cylinder*.

In the sequel, for any function φ, $\Phi(x) := \{\varphi(x) - r^2/2 - k\}$ and diverse constants C will denote generic constants.

By the maximum principle all solutions ψ are positive in their respective domains. The main results that we obtain are:

1) The variational solutions of (Pd) satisfy a)–e) where for i) in a), Π_d replaces Π, $mq > 0$ and $mq = 0$ replace respectively $m > 0$ and $m = 0$;

ii) the estimates in d)–e) are independent of d.

2) Independently of d, f_0 and f_q, the functions $k^{1/2}\hat{\Psi}_d$ converge to $\hat{U}_m$ if $m \leqslant q+2$ and to $\hat{U}_{q+2}$ if $m > q+2$.

3) For large k we deduce variational solutions of the problem in Π from those of $\{(P_d)\}_{d\in(0,1]}$, and they have the same estimates.

2. Existence of solutions

2.1. Preliminaries. For $b > 0$, let $D \equiv D_b$ denote a regular convex domain ($\partial D_b \in C^l$; $l \geqslant 2$) in Π_d such that the rectangle $(d, d+b) \times (-2b, 2b)$ is contained in $\overline{D}$. Define the spaces $L^p(D) := \{u \mid |u|_{p;D} := \left(\int_D |u|^p \,\mathrm{d}\tau\right)^{1/p} < \infty\}$ and denote by $H(D)$ the completion of $C_0^\infty(D)$ in the norm $\|u\|_D := \left(\int_D \{(u_r^2 + u_z^2)/r^2\} \,\mathrm{d}\tau\right)^{1/2}$. We have the imbeddings ([3])

$$H(D) \subset W_2^1(D) \subset L^p(D); \quad p \geqslant 1 \tag{2.1}$$

where the second imbedding is compact. In fact, if $u \in H(D)$ has its support in $R := (r_0 - \alpha, r_0) \times (-2\beta, 2\beta)$ then

$$|u|_{p,R} \leqslant C_p r_0^{(2+p)/2p} \,(2\alpha\beta)^{1/p} \|u\|_R; \quad p \geqslant 1. \tag{2.2}$$

From the Sobolev inequality ([2])

$$\forall u \in H(\Pi) \quad \forall p \geqslant 2, \quad \int_\Pi \frac{|u|^p}{r^{2+p/2}} \,\mathrm{d}r \,\mathrm{d}z \leqslant (A_p \|u\|_\Pi)^p \tag{2.3}$$

where A_p depends only on p, we have the following lemma:

Lemma 2.1. *Let $u \in H(\Pi)$ be such that $A(u) := \{x \in \Pi \mid U(x) := u(x) - k - r^2/2 > 0\}$ has a non void interior. Then $\forall p \geqslant 1$ and $l > 0$ with $\mu := 8p - 6$,*

$$\forall u \in H(\Pi) \quad \int_\Pi \left(\frac{U_+^l}{r^2}\right)^p \mathrm{d}\tau \leqslant k^{\mu/2} \, A_\mu^\mu (|U|_{2pl})^{pl} \, \|u\|^{\mu/2}; \tag{2.4a}$$

$$\forall u \in H(Db) \quad \int_{Db} \left(\frac{U^l}{r^2}\right)^p \leqslant (C_{2pl})^{pl} b^{(1+pl)/2} (\operatorname{diam} D_b)^{pl} \|u\|_{Db}^{pl}. \tag{2.4b}$$

Also for $p \geqslant 2$,

$$\int_{A(u)} \frac{1}{r^{2p}} \,\mathrm{d}\tau \leqslant k^{3-2p} \big(A_{4p-6} \|u\|\big)^{4p-6}, \tag{2.4c}$$

where A_μ and C_l are from (2.3) and $|.|_l := |.|_{l;\Pi}$.

P r o o f. As $u > k$ on $A(u)$, by the Hölder inequality we have

$$\int_{A(u)} r^{-2p} U^{pl}\,\mathrm{d}\tau \leqslant (|U|_{2pl})^{pl} k^{\mu/2} \Big(\int_{A(u)} u^{\mu}/r^{2+\mu/2}\,\mathrm{d}r\,\mathrm{d}z \Big)^{1/2}$$

and (2.4a) follows. The other assertions follow from (2.2) and (2.4a). □

Maps between $H(\Pi)$ and the space V_5: Π becomes a meridional half-plane in $\mathbb{R}^N$, $N \geqslant 3$, if we define $z = x_N$; $r = \sqrt{x_1^2 + x_2^2 + \ldots + x_{N-1}^2}$.

Let V_N denote the completion of

$$C_{0,c}^{\infty}(\mathbb{R}^N) := \{\varphi \in C_0^{\infty}(\mathbb{R}^N) \mid u \text{ depends only on } (r,z)\}$$

in the norm $\|\varphi\|_{V_N} := \big(\int_{\Pi} (\varphi_r^2 + \varphi_z^2) r^{N-2}\,\mathrm{d}r\,\mathrm{d}z\big)^{1/2}$.

From [2], the map $\varphi \mapsto \overline{\varphi}$; $\overline{\varphi}(x) := r^{-(N-1)/2}\varphi(x)$ is a homeomorphism from $H(\Pi)$ to V_N with

$$\|\overline{\varphi}\|_{V_N}^2 = \|\varphi\|_{H(\Pi)}^2 + \frac{(N-1)(N-5)}{4} \int_{\Pi} |\varphi|^2 r^{-3}\,\mathrm{d}r\,\mathrm{d}z. \tag{2.5}$$

Thus for $N = 5$ the map is an isometry between $H(\Pi)$ and V_5.

2.2. Solutions in bounded $Db \subset \Pi_d$. As $d > 0$, L is uniformly elliptic in Π_d and $\forall u \in H_d$ with $Z_\alpha(u) := (1/\alpha^2) J_q(u) + J_m(u)$, we have

$$Z_{2b}(u) \leqslant Z(u) \leqslant Z_d(u). \tag{2.6}$$

Theorem 2.2. *$\forall k$, $b > 0$, the problem*

$$L\psi = -\lambda f(r,\Psi) \quad \text{in } Db; \quad \psi|_{\partial Db} = 0 \tag{Db}$$

has a solution ψ which is a maximizer of Z on $S_1(Db)$. For some $\nu \in (0,1]$, if $mq > 0$ there is

$$\overline{\psi} \in C^{2,\nu}(\overline{Db}) \quad (\in C^{1,\nu}(\overline{Db}) \bigcap C^{2,\nu}(\overline{Db} \setminus \overline{\partial A}) \quad \text{if } mq = 0) \tag{2.7}$$

such that $\psi(x) = r^2 \overline{\psi}(x)$. Moreover, $\overline{\psi}$ is an even function in z with $\overline{\psi}_z < 0$ for $z > 0$ in Db.

P r o o f. From (2.4) and (2.6), as J_m is in $H(D)$ (see [3]), Z is bounded on $S_1(Db)$ and continuous w.r.t. the weak convergences of $H(Db)$ (hence w.r.t. the strong convergences in $L^p(Db)$, $p \geqslant 1$). Thus there is $\psi \equiv \psi_b \in S_1(Db)$ such that

i) $Z(\psi) = \max\limits_{S_1(Db)} Z(u)$;

ii) Z has a Frechet derivative Z' defined by

$$\langle Z'(u), \varphi\rangle_{H(Db)} := \int_{Db} \varphi f(r, U)\,\mathrm{d}\tau \ \forall \varphi \in H(Db);$$

iii) ψ is a critical point of Z whence there is $\lambda > 0$ such that

$$\forall \varphi \in H(Db) \ \ \langle \psi, \varphi\rangle_{H(Db)} = \lambda \int_{Db} \varphi f(r, \Psi)\,\mathrm{d}\tau$$

and ψ is a weak solution of (Db) with

$$\lambda \equiv \lambda_b = \left(\int_{Db} \psi_b f(r, \Psi_b)\,\mathrm{d}\tau\right)^{-1} \leqslant \{Z(u)\}^{-1} \qquad \forall u \in H(D_b). \tag{2.8}$$

By taking large p in (2.4), the elliptic theory implies that $\psi_b \in C^{1,\nu}(\overline{Db})$ for any $\nu \in (0, 1]$. Let $\varphi := \overline{\psi_b}$, the image of ψ_b in the isometry in (2.5); then

$$\triangle_5 \varphi := \varphi_{rr} + \frac{3\varphi_r}{r} + \varphi_{zz} = -\lambda f(r, \Psi_b) \quad in \quad Db_5; \quad \varphi|_{\partial Db_5} = 0 \tag{D_5}$$

and φ satisfies (2.7). The proof is completed by the fact that the equation in (D_5) is even in z (see [4]). □

2.3. Solutions in Π_d for a fixed $k > 0$. For a $b > 0$ and $b_i := ib$, $i \in \mathbb{N}$, let $Di := Db_i$ and let (ψ_i, λ_i) be the corresponding solutions of (Di) where ψ_i is extended by 0 outside Di. From (2.8),

$$\forall i > 1, \qquad \lambda_i \leqslant \{Z(\psi_1)\}^{-1}. \tag{2.9}$$

Lemma 2.3. *There is a bounded domain* $\Omega_k \subset \Pi_d$ *such that*

$$A_i := A(\psi_i) \subset \Omega_k \qquad \forall i \in \mathbb{N}. \tag{2.10}$$

Consequently, Z is uniformly bounded and continuous on $S_1(\Pi_d)$.

P r o o f. Let D be any of the Di, (ψ, λ) the corresponding solution and $A := A(\psi)$. Let $r_2 := \sup\{r > 0 \mid (r, 0) \in \overline{A}\}$. The Green function G of L in D satisfies for $S^2 := (r - r_2)^2 + z^2$ with $x = (r, z)$, $x_2 = (r_2, 0)$ and $\sigma := 4\sqrt{rr_2}/s$ the inequality $(16/\sqrt{rr_2})\, G(x_2, x) \leqslant \sinh^{-1}(1/\sigma) = \log\{1/\sigma + \sqrt{1 + 1/\sigma^2}\}$ (see [6])

whence for a small $\alpha > 0$ and $A\alpha := \{x \in A;\ s < \alpha r_2\}$ we have $r_2^2/2 + k = \psi(x_2) \leqslant (r_2/2\pi)\lambda \int_{A\alpha} \log 1/s\, f(r,\Psi)\,\mathrm{d}\tau + (r_2\lambda/2\pi k)\log(24r_2/\alpha)\int_A \psi f(r,\Psi)\,\mathrm{d}\tau$ whence

$$k + r_2^2/2 \leqslant C\, r_2 \frac{r_2}{2\pi k}\{\log(24r_2/\alpha)\} + \lambda\alpha r_2^{m'} \tag{2.11}$$

for some $m' := m'(q,m)$. From (2.9), λ is bounded and so is r_2; in fact if we suppose that r_2 is very big, then for $\alpha > 0$ such that $1/r_2 < \lambda\alpha r_2^{m'} < 1$, (2.11) implies that $r_2 \leqslant \{\log r_2 + \log\lambda\}$. From [3], it is known that if $(r,z) \in A$ then $|z| < k^{-1}$. The existence of Ω_k is obtained.

Let $\varphi \in S_1(\Pi_d)\bigcap C_0^\infty(\Pi_d)$; there is $l \in \mathbb{N}$ such that $\varphi \in S_1(Dl)$ whence from (2.1), (2.4) and (2.9),

$$Z(\varphi) \leqslant Z(\psi_l) \leqslant C(\Omega_k). \tag{2.11$'$}$$

The uniform continuity then follows as in the case of J_m in [3]. □

Theorem 2.4. *(Pd) has a solution ψ which is a maximizer of Z on $S_1(\Pi_d)$ such that*

1) *there is $\varphi \in C^{2,\nu}(\overline{\Pi_d})$ if $mq > 0$ and $\varphi \in C^{1,\nu}(\overline{\Pi_d})\bigcap C^{2,\nu}(\overline{\Pi_d}\setminus\overline{\partial A(\varphi)})$ if $mq = 0$ such that $\psi(x) = r^2\varphi(x)$ in Π_d; ψ is an even function of z and $\psi_z < 0$ if $z > 0$;*

2) *the cross-section A is simply connected if $m,q \geqslant 1$ and has a finite number of components otherwise.*

Proof. As $\forall i \quad H(Di) \subset H(D_{i+1}) \subset H(\Pi_d)$,

$$\lim_{i\nearrow\infty} Z(\psi_i) = \max_{S_1(\Pi_d)} Z(u) := \sigma_d; \tag{2.12}$$

so the uniform continuity and boundedness of Z on S_1 implies that there is a subsequence (ψ_i') which converges weakly to a $\psi \in H(\Pi_d)$ with $\|\psi\|_{H(\Pi_d)} \leqslant 1$. If we suppose that $\psi \notin S_1$ then $u := \psi/\|\psi\| \in S_1$ with $Z(u) > \sigma_d$ which is absurd. As (ψ_l') converges strongly in $L^p(\Omega_k)\quad \forall p \geqslant 1$, (λ_l') converges to a $\lambda > 0$ and ψ is a weak solution of (Pd) with $\lambda := \left(\int_{\Pi_d} \psi f(r,\Psi)\,\mathrm{d}\tau\right)^{-1}$. The proof is completed by similar arguments as for the last theorem.

2) As the domain is away from $r = 0$, if $m,q \geqslant 1$ then a slight extension of the results in [3] and [5] shows that A is simply connected.

Assume only that $m,q \geqslant 0$.

If A has an infinite number of disjoint connected components then for some $\theta > 0$, $A^\theta := A\bigcap\{z = -\theta\}$ and $A^0 := A\bigcap\{z = 0\}$ have an infinite number of components (t_i, t_{i+1}) and (r_i, r_{i+1}) with $t_i < r_i < r_{i+1} < t_{i+1}\ \forall i \in \mathbb{N}$. As Ω_k is bounded, the sequences (t_i) and (r_i) converge to the same limit. We then have a contradiction as $\forall i$, $\Psi(r_i,0) = 0$ and $\Psi(t_i,0) = -\theta$ ([9]). □

2.4. Estimates for ψ in Π_d for large $k > 0$. Let $(\psi, \lambda) \equiv (\psi_k, \lambda_k)$ denote the solution in Π_d corresponding to $k > 0$.

Lemma 2.5. *For any $d > 0$ with $c_0 := 8\pi(2/3)^{3/2}$, as $k \nearrow \infty$, we have*

$$\lambda \leqslant Ck^{(m-2)/2} \exp\{2c_0 k^{3/2}\}. \tag{2.13}$$

P r o o f. Let $v(x) := \psi_0(r-d, z)$ where ψ_0 denotes the solution of the problem without swirl with $m = 0$ (see [9]). We have $v \in S_1(\Pi_d)$ and for large k, $J_m(v) \geqslant Ck^{(2-m)/2)} \exp(-2c_0 k^{3/2})$ (see [9]).

The estimate then derives from the fact that $\lambda \leqslant \{Z(v)\}^{-1} \leqslant \{J_m(v)\}^{-1}$. $\square$

Define $r_1 := \inf\{r > 0;\ (r, 0) \in A\}$ ($r_2 := \sup\{r > 0;\ (r, 0) \in A\}$).

Theorem 2.6. *For any $d > 0$, as $k \nearrow \infty$,*

$$\left| r_i^2 - \frac{2}{3}k \right| = O(k^{-1/2} \log k); \tag{2.14}$$

$$\operatorname{diam} A \leqslant Ck^{1/2} \exp\{-c_0 k^{3/2}\}; \tag{2.15}$$

$$|\psi|_{C(\overline{A})} = O(k); \tag{2.16}$$

$$\lambda k |f(.\,, \Psi)|_{C(A)} |A|_\tau = \lambda k |f(.\,, \Psi)|_{C(A)} \int_A \mathrm{d}\tau = O(1); \tag{2.17}$$

$$|\Psi|_{C(A)} \leqslant Ck^{-1/2}. \tag{2.18}$$

P r o o f. As the Green function P of L in Π_d has the same estimates as that in D, (2.11) and (2.13) imply that for large k, $r_i^2/2 + k \leqslant Cr_i\{k^{1/2} + \log r_2\}$ after taking a suitable value for α (note that (2.11) shows also that λ_k cannot be bounded as $k \nearrow \infty$). The last estimate implies that $r_i = O(k^{1/2})$ and (2.16) is similarly obtained as the estimate holds for any $x = (r, z) \in A$. The capacity theory ([7]) shows that for large k, as $(r_2 - r_1)/r_1$ is bounded and A is moving away from $z = 0$ as $k \nearrow \infty$, if $\operatorname{diam} A = 2\varepsilon r_0$ where $r_0 := (r_1 + r_2)/2$, then we have the estimate

$$\varepsilon \leqslant C \exp\{-c_0 k^{3/2}\}.$$

In fact, the capacity of a closed subset E of Π relative to the operator L is defined as the quantity

$$\begin{aligned} \operatorname{Cap}_L(E) &:= \inf\left\{ -\int_{\Pi \setminus E} uLu \,\mathrm{d}\tau \mid u \in C_0^\infty(\Pi),\ u|_E \geqslant 1 \right\} \\ &= \inf\{\|u\|^2 \mid u \in H(\Pi);\ u|_E \geqslant 1\}. \quad ([7]) \end{aligned}$$

For $E := [a(1-\varepsilon),\, a(1+\varepsilon)] \times \{z=0\}$, if $\varepsilon > 0$ is small enough, then

$$\mathrm{Cap}_L(E) = 2\pi\big(\log(16/e^2\varepsilon)\big)^{-1}\{1 + O(\varepsilon \log 1/\varepsilon)\}.$$

(Theorem 3 of [7].)

Thus we have $2\pi\big(r_1 \log(16e^{-2}/\varepsilon)\big)^{-1} = \mathrm{Cap}_L\, A \leqslant (r_1^2/2 + k)^{-2}$ whence $\varepsilon \leqslant 16e^{-2}\exp\{-2\pi k^{3/2}s^{-1/2}(s^2/2+1)^2\}$ with $r_1^2 \simeq sk$ for large k. $y(s) := 2\pi s^{-1/2}(s^2/2+1)^2$ has its minimum $c_0 = 8\pi(2/3)^{3/2}$ at $s_0 = 2/3$. As $y''(s_0) > 0$, if $r_1^2 \simeq (s_0+\tau^2)k$ for large k, (2.13) and (2.17) imply that $k^{m'}\exp(-\tau^2 k^{3/2}) = O(1)$ and this leads to (2.14). (2.17) follows from the fact that $\lambda k \int_A f(r,\Psi)\,\mathrm{d}\tau = O(1)$ for large k.

As $\psi \in S^1(\Pi)$ and for large k we have $|A(\psi)|_\tau := \int_{A(\psi)} \mathrm{d}\tau \simeq |A(\psi_0)|_\tau$ where ψ_0 denotes the solution for the problem without swirl with $m = 0$, for large k we obtain $(|A(\psi)|_\tau)^{-1}J_0(\psi) \leqslant (|A(\psi_0)|_\tau)^{-1}J_0(\psi_0) \leqslant |\Psi_0|_{C(A(\psi_0))} = Ck^{-1/2}$ and (2.18) follows. $\square$

Theorem 2.7. *Suppose that for large k, $|\Psi|_{C(A)} = O(k^{-\alpha})$ for some $\alpha > 0$ and define*

$$g(\Psi) := \begin{cases} f_0\Psi_+^m & \text{if} \quad m < q+2; \\ f_q'\Psi_+^{q+1/\alpha} := (2f_q/3)\Psi_+^{q+1/\alpha} & \text{if } m > q+2; \\ f_{q0}\Psi_+^m := ((2f_q/3)+f_0)\Psi_+^m & \text{if } m = q+2. \end{cases} \tag{2.19}$$

Then as $k \nearrow \infty$, ψ becomes the solution for the problem without swirl

$$L\psi = -\lambda g(\Psi) \quad \text{in } \Pi_d. \tag{2.20}$$

P r o o f. From (2.14), for large k, in A we have $f(r,\Psi) \simeq (2f_q/3)\Psi^{q+1/\alpha} + f_0\Psi^m \simeq k^{-(1+\alpha q)}\{2f_q/3 + f_0 k^{\alpha(q-m)+1}\}$, hence $f(r,\Psi) \simeq g(\Psi)$ in A for large k. As $\|\psi\|^2_{H(\Pi_d)} = \lambda\int_A \psi f(r,\Psi)\,\mathrm{d}\tau$, we then have $\lim\limits_{k\nearrow\infty}\big\{\langle\psi,\psi\rangle_{H(\Pi_d)} - \lambda\int_A \psi g(\Psi)\,\mathrm{d}\tau\big\} = 0$ and (2.20) follows. $\square$

3. Estimates in the stretched plane $\hat{\Pi}_d$

Define $\forall k > 0$ $a \equiv a(k) > 0$ such that

$$\nabla\Psi(a,0) = 0 \quad \text{and} \quad \Psi(a,0) = \max_{x\in A} \Psi(x). \tag{3.1}$$

Let $\hat{\Pi}_d$ be the image of Π_d in the transformation (1.ζ) where ε satisfies $\operatorname{diam} A = 2a\varepsilon$. $\hat{D}$ will denote the image of any $D \subset \Pi_d$. Define $u(\zeta) := u(a(1+\varepsilon\xi), a\varepsilon\eta)$ for u defined in Π_d and $\hat{f}(U) := f(a(1+\varepsilon\xi), U(\zeta))$. For large k,

$$\hat{A} \subset B^1 := \{\zeta \mid |\zeta|^2 = \xi^2 + \eta^2 < 1\} \text{ and } \operatorname{diam} \hat{A} = 2. \tag{3.2a}$$

For $x, x_0 \in \Pi$ with $x = (r,z)$ and $x_0 = (r_0, z_0)$, the Green function of L in Π is

$$P(x,x_0) = \frac{rr_0}{2\pi} \int_0^\pi \frac{\cos\theta \, \mathrm{d}\theta}{\{r^2 + r_0^2 - 2rr_0\cos\theta + (z-z_0)^2\}^{1/2}}$$

([3], [6]). So, provided that $\varepsilon|\zeta_0|, \varepsilon|\zeta| \in (0,1)$, the Green function P of L in $\hat{\Pi}$ satisfies for $P(\zeta,\zeta_0) := P((a(1+\varepsilon\xi), a\varepsilon\eta), (a(1+\varepsilon\xi_0), a\varepsilon\eta_0))$ ([6], [9])

$$P(\zeta,\zeta_0) = \frac{a}{2\pi}\Big\{\log\frac{8e^{-2}}{\varepsilon|\zeta-\zeta_0|} + R_1(\zeta,\zeta_0)\log\frac{8}{\varepsilon|\zeta-\zeta_0|} + R_2(\zeta,\zeta_0)\Big\} \tag{3.2b}$$

where for $|\alpha| \in \mathbb{N}$, $|D^\alpha R_i| = O(\varepsilon)$. Under those conditions we have the following estimates for large k:

$$P(\zeta_0,\zeta) = \frac{a}{2\pi}\log\frac{8e^{-2}}{\varepsilon|\zeta-\zeta_0|} + O\Big(\varepsilon\log\frac{1}{\varepsilon}\Big) \tag{3.3}$$

$$= Ck^2 + \frac{a}{2\pi}\log\frac{1}{|\zeta-\zeta_0|} + O\Big(\varepsilon\log\frac{1}{\varepsilon}\Big). \tag{3.4a}$$

If $\zeta' \in \hat{A}$, we get for large k and $|\nabla_{\zeta'}P(\zeta,\zeta')| := \sqrt{P_{\xi'}^2 + P_{\eta'}^2}$

$$|\nabla_{\zeta'}P(\zeta,\zeta')| \leqslant \text{const } a\Big\{\varepsilon\log\frac{1}{\varepsilon} + \frac{1}{|\zeta-\zeta'|} + \varepsilon\log\frac{e}{|\zeta-\zeta'|}\Big\}. \tag{3.4b}$$

Lemma 3.1. *For large k,*

$$\Psi(\zeta_0) = \frac{\lambda a^4\varepsilon^2}{2\pi}\int_{\hat{A}} \hat{f}(\Psi)\log\frac{1}{|\zeta-\zeta_0|}\,\mathrm{d}\xi\,\mathrm{d}\eta + O\Big(\varepsilon\log\frac{1}{\varepsilon}\Big) \tag{3.5a}$$

$$= O(k^{-1/2}) \quad \text{in } \hat{A}, \tag{3.5b}$$

$$\text{and } \big||\psi|_{C(A)} - 4k/3\big| = O(k^{-1/2}). \tag{3.6}$$

P r o o f. From (3.4), for large k, with $\delta := \varepsilon \log 1/\varepsilon$ and $\zeta_0 \in \hat{A}$, we have $\psi(\zeta_0) = Ck + (\lambda a^4 \varepsilon^2/2\pi) \int_{\hat{A}} \log(1/|\zeta - \zeta_0|)\, \hat{f}(\Psi)(1+\varepsilon\xi)\, \mathrm{d}\xi\, \mathrm{d}\eta + O(\delta)$. From (2.17), for large k, we have $\lambda a^5 \varepsilon^2 |\hat{f}(\Psi)|_{\hat{A}} = O(1)$ hence the integral term above is $O(k^{-1/2})$. The fact that $|\psi|_{C(A)} = O(k)$ then leads to (3.5a). The formula (3.6) follows from (2.14) and (3.5b). □

For any $k > 0$, define $u_k := k^{1/2}\Psi$. In $\hat{\Pi}_d$

$$\forall k > 0 \quad \nabla u_k(0) = 0 \quad \text{and} \quad |u_k|_{C(\hat{A})} = O(1) \quad \text{for large } k. \tag{3.7}$$

As $\forall k > 0$ $\lambda \int_A \psi f(r, \Psi)\, \mathrm{d}\tau = 1$, by (2.19), (3.5b) and (3.6) each of the quantities

$$\begin{cases} \dfrac{4}{3}\lambda a^2 \varepsilon^2 k^{(2-m)/2} f_0 u_k(0)^m |\hat{A}| & \text{if } m < q+2, \\ \dfrac{4}{3}\lambda a^3 \varepsilon^2 k^{-q/2} f_q' u_k(0)^{q+2} |\hat{A}| & \text{if } m > q+2, \\ \text{and } \dfrac{4}{3} f_{q0} \lambda a^3 \varepsilon^2 k^{(2-m)/2} u_k(0)^m |\hat{A}| & \text{if } m = q+2 \end{cases} \tag{3.8}$$

converges to 1 as $k \nearrow \infty$.

Theorem 3.2. *Let $\mu \in (0,1]$; then $(u_k)_{k\in\mathbb{N}}$ converges to u, such that*
1) $u \in C^{2,\mu}(\hat{\Pi}_d)$ if $mq > 0$ and $u \in C^{1,\mu}(\hat{\Pi}_d) \bigcap C^{2,\mu}(\hat{\Pi} \setminus \partial\hat{A}(u))$ if $mq = 0$;
2) u is radial and independent of d, f_0 and f_q. In fact, for $\sigma := |\zeta|$ we have

$$u(\sigma) = \begin{cases} \dfrac{\sqrt{6}}{4\pi \varrho_m^2} Q_m(\varrho_m \sigma) & \text{if } m \leqslant q+2 \\ \dfrac{\sqrt{6}}{4\pi \varrho_{q+2}^2} Q_{q+2}(\varrho_{q+2}\sigma) & \text{if } m > q+2 \end{cases} \tag{3.9}$$

where Q_l and ϱ_l are defined in (1.Q).

P r o o f. Let B be a (bounded) ball centered at the origin in $\hat{\Pi}_d$ and let k be large. From the equation $L\Psi = -\lambda f(r, \Psi)$, with $A_k := A(\psi)$ we obtain $(\partial_\xi^2 - \varepsilon\partial_\xi/(1+\varepsilon\xi) + \partial_\eta^2)u_k = -\lambda a^4 \varepsilon^2 k^{1/2}(1+\varepsilon\xi)^2 \hat{f}(\Psi)$ in $\hat{A}_k$.

From (2.17) and (3.8), the second member of this equation is bounded uniformly on $\hat{A}_k$ and the elliptic theory implies that $\|u_k\|_{W_p^2(B)}$ is uniformly bounded as easy calculations show that $|u_k(\zeta)| \leqslant C|\zeta|$ for $\zeta \notin \hat{A}_k$.

In fact, from (3.4b), as $\varepsilon^2 a^5 \lambda |\hat{f}(\Psi)|_{C(\hat{A})} = O(1)$, we obtain

$$\begin{aligned} |\nabla\psi(\zeta)| &\leqslant \text{const}\, \varepsilon^2 a^4 \lambda |\hat{f}(\Psi)|_{C(\hat{A})} \\ &\qquad \times \left\{ \varepsilon \log(1/\varepsilon) + \int_{\hat{A}} (1/|\zeta - \zeta'|)\, \mathrm{d}\xi'\, \mathrm{d}\eta' + \varepsilon \int_{\hat{A}} \log(e/|\zeta - \zeta'|)\, \mathrm{d}\xi'\, \mathrm{d}\eta' \right\} \\ &\leqslant (\text{const}/a)\{\varepsilon \log(1/\varepsilon) + O(1+\varepsilon)\} = O(k^{-1/2}). \end{aligned}$$

For $\zeta'' \in \partial\hat{A}$ satisfying $\operatorname{dist}(\zeta, \partial\hat{A}) = |\zeta - \zeta''|$, as $u_k(\zeta'') = 0$, we conclude $|u_k(\zeta)| = |u_k(\zeta) - u_k(\zeta'')| \leqslant |\nabla u_k||\zeta - \zeta''| \leqslant |\nabla u_k||\zeta| \leqslant \text{const}|\zeta|$ as $|\nabla u_k| \leqslant \text{const} k^{1/2}|\nabla\psi| + O(k^{3/2}\varepsilon)$. The existence of $u \in C^{1,\nu}(B)$ as the limit of a subsequence of (u_k) follows from the Sobolev imbedding theorems. The regularity of u follows from the elliptic theory.

Let $m < q + 2$. From (2.19) and (3.5), for large k and $\delta := \varepsilon\log(1/\varepsilon)$, we have $u_k(\zeta) \simeq (1/2\pi)(\sqrt{2/3})\lambda a^3\varepsilon^2 k^{(2-m)/2} f_0 \int_{\hat{A}} u_k^m \log(1/(|\zeta' - \zeta|)\,\mathrm{d}\xi'\,\mathrm{d}\eta' + O(\delta)$. So, as from (3.8) $\lim\limits_{k\nearrow\infty}(\sqrt{2/3})\lambda a^3\varepsilon^2 k^{(2-m)/2} f_0 = \sqrt{6}/(4|\hat{A}(u)|u(0)^m) := \nu_m$, u is a fixed point of N where

$$\forall\zeta \in \hat{\Pi}_d \qquad N\varphi(\zeta) = \frac{\nu_m}{2\pi}\int_{\hat{A}(u)} \varphi_+^m \log\frac{1}{|\zeta' - \zeta|}\,\mathrm{d}\xi'\,\mathrm{d}\eta'. \tag{3.10}$$

Then from [8], u is radial and $\nu_m = \sqrt{6}/(4\pi u(0)^m)$. For $\sigma := |\zeta|$, $u'' + u'/\sigma = -\nu_m u_+^m$, $\sigma > 0$; $u'(0) = 0$, $u(0) := u_0$ for some $u_0 > 0$.

With $t := \{\nu_m u_0^{m-1}\}^{1/2}\sigma$ and $W(t) =: u(\sigma)/u_0$, we have $W'' + W'/t = -W_+^m$; $W(0) = 1$, $W'(0) = 0$ whence $u(\sigma) = u_0 Q_m(\{\nu_m u_0^{m-1}\}^{1/2}\sigma)$. $u(1) = 0 \Longrightarrow \sqrt{6}/(4\pi u_0) = \varrho_m^2$ and

$$u(0) = \frac{\sqrt{6}}{4\pi\varrho_m^2}. \tag{3.11}$$

Because $u(0)$ is independent of the choice of the subsequence, (u_k) converges to u. The cases when $m \geqslant q + 2$ follow by the same arguments. □

4. Existence of variational solutions in Π for large k and their estimates

Lemma 4.1. *Let $\mu \in (0, 1]$; there is $K > 0$ such that for $k > K$*

$$L\psi = -\lambda f(r, \Psi) \quad \text{in } \Pi \tag{4.1}$$

with $\psi \in H(\Pi)$ has a solution $\psi \in C^{2,\mu}(\Pi)$ if $mq > 0$ ($\in C^{1,\mu}(\Pi)\bigcap C^{2,\mu}(\Pi \setminus \partial A(\psi))$ if $mq = 0$) which is a maximizer of Z on $S_1(\Pi)$.

Proof. Theorem 2.6 implies that there is $K > 0$ such that $\forall d \in (0, 1]$ and $k > K$ we have $r_1 := \inf\{r > 0 \mid (r, 0) \in A(\psi_d)\} > 1$.

Consider a decreasing sequence $(d_i)_{i\in\mathbb{N}}$ in $(0, 1]$ such that $d_i \searrow 0$ and a fixed $k > K$. The proof is similar to that of Theorem 2.4 as $\forall i \in \mathbb{N}$, $S_1(\Pi_{d_i}) \subset S_1(\Pi_{d_{i+1}}) \subset S_1(\Pi)$. □

Theorem 4.2. *Theorems* 2.6, 2.7 *and* 3.2 *hold for the variational solutions* $\{\psi_k\}_{k>K}$ *of the problems in* Π.

P r o o f. This follows from the facts that the Green function of L in Π has the same upper bounds as that in Π_d and the estimates (3.3)–(3.4) hold for it as well. It has to be noted that $\hat{\Pi}_d \simeq \hat{\Pi}$ for large k. $\square$

To my uncles G. Tatuam, P. Wabo and JB. Topa Chatue, in memoriam.

References

[1] *M. S. Berger*: Mathematical Structures of Nonlinear Sciences, an Introduction. Kluwer Acad. Publ. NTMS 1, 1990.

[2] *L. E. Fraenkel*: On Steady Vortex Rings with Swirl and a Sobolev Inequality. Progress in PDE: Calculus of Variations, Applications (C. Bandle et al., eds.). Longman Sc. & Tech., 1992, pp. 13–26.

[3] *L. E. Fraenkel & M. S. Berger*: A global theory of steady vortex rings in an ideal fluid. Acta Math. *132* (1974), 13–51.

[4] *B. Gidas B, WM. Ni & L. Nirenberg*: Symmetry and related properties via the maximum principle. Comm. Math. Phys. *68* (1979), 209–243.

[5] *J. Norbury*: A family of steady vortex rings. J. Fluid Mech. *57* (1973), 417–431.

[6] *Tadie*: On the bifurcation of steady vortex rings from a Green function. Math. Proc. Camb. Philos. Soc. *116* (1994), 555–568.

[7] *Tadie*: Problèmes elliptiques à frontière libre axisymetrique: estimation du diamètre de la section au moyen de la capacité. Potential Anal. *5* (1996), 61–72.

[8] *Tadie*: Radial functions as fixed points of some logarithmic operators. Potential Anal. *9* (1998), 83–89.

[9] *Tadie*: Steady vortex rings in an ideal fluid: asymptotics for variational solutions. Integral methods in sciences and engineering Vol 1 (Oulu 1996). Pitman Res. Notes Math. 374, Longman, Harlow, 1997, pp. 179–184.

Author's address: *Tadie*, Matematisk Institut, Universitetsparken 5, 2100 Copenhagen, Denmark, e-mail `tad@math.ku.dk`.