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SINGULAR LIMIT OF A TRANSMISSION PROBLEM FOR THE PARABOLIC PHASE-FIELD MODEL

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Abstract. A transmission problem describing the thermal interchange between two regions occupied by possibly different fluids, which may present phase transitions, is studied in the framework of the Caginalp-Fix phase field model. Dirichlet (or Neumann) and Cauchy conditions are required. A regular solution is obtained by means of approximation techniques for parabolic systems. Then, an asymptotic study of the problem is carried out as the time relaxation parameter for the phase field tends to 0 in one of the domains. It is also proved that the limit formulation admits a unique solution in a suitable weak sense.

Keywords: phase-field models, maximal monotone operators, transmission problems, parabolic PDEs

MSC 2000: 35B40, 35K55, 80A22

1. Introduction

Given a couple of smooth and bounded adjoining domains Ω_1 and Ω_2 in $\mathbb{R}^N$, which we suppose filled with two different but homogeneous fluids, allowed to undergo changes of phase, we intend to study some particular cases of heat transmission problems between the two regions in the framework of the standard (linearized) phase-field model.

We recall that phase field models, first introduced by Caginalp [4], Fix [10] and others and recently rediscussed and improved from the thermodynamical point of view by several authors (we cite the work [13] by Penrose and Fife, accounting for an exhaustive explanation of the underlying physics), provide an extension of the enthalpy method for the Stefan problem with the advantage of making it possible to describe some rather fine physical phenomena which can take place during fusion-solidification processes. In the latest years a great effort has been produced by mathematicians in the study of several variants of the model, and interesting results have been obtained in the directions of existence and regularity of solutions, as well as of their dependence on the physical parameters (see the bibliography in [5]).

Here, anyway, we refer to a very simple situation, which can be derived from the standard phase-field model [4] by supposing the fluids to remain close to the thermal equilibrium (see also [13], Section 6), performing the change of variables as in [7], Section 2, and assuming that thermal and phase field interchanges are present at the common boundary of the domains. As a consequence, the equations constituting the "physical" problem can be written, for $i = 1, 2$, as follows:

$$\partial_t u_i - \Delta u_i = f_i - \lambda \Delta \chi_i \quad \text{in } \Omega_i, \tag{1}$$

$$\mu_i \partial_t \chi_i - \nu_i \Delta \chi_i + \zeta_i = \lambda u_i \quad \text{in } \Omega_i, \tag{2}$$

$$\zeta_i \in \alpha_i(\chi_i), \tag{3}$$

where χ_i is the phase field in the domain Ω_i, u_i the enthalpy density, f_i a source term, μ_i, ν_i are nonnegative diffusion parameters proper of the fluids, and we have chosen the same λ (which is a parameter representing the latent heat of the fluid) in the two domains in order to ensure regularity in space of the enthalpy in the union of the Ω_i's.

Moreover, we suppose the α_i's to be maximal monotone graphs in $\mathbb{R} \times \mathbb{R}$ (see also the derivation in [7]); they account for the constitutive relations linking the phase field to the internal energy of the bodies.

Our first task will be the study of these model equations with the addition of suitable initial and boundary value conditions, along with compatibility and transmission conditions at the interface of the two regions. We shall write the variational formulation of the problem and treat it with standard approximation techniques for evolution equations such as the Faedo-Galerkin regularization scheme and the Yosida approximation of the operators α_i. By these means, we shall get an existence result for the above statement and also some regularity properties of the solution, especially those concerning the χ_i's.

Then, we shall consider the behaviour of the solutions of a family of problems of the previous type as we let the time relaxation parameter μ_2 for the phase field tend to 0 in Ω_2. This choice, which moves towards an inhomogeneous statement ruled by two different physical situations in the regions considered, is motivated both from a physical point of view, since the parameters μ_i can be in the applications very small and we get back as a limit in Ω_2 the phase field model for fluids in the phase equilibrium [4], [7], and from the mathematical viewpoint. In fact, the main reason of interest (and of difficulty) of this kind of transmission problems consists in the necessity of a careful examination of the compatibility conditions between the phase diffusion parameters of the two fluids, which are essential in the proof of any existence result.

Indeed, also in the formulation at fixed μ_2, we will prove our existence and regularity result only under the assumption that the operators α_1 and α_2 "do not differ too much", and also the hypothesis $\nu_1 \neq \nu_2$ results in a loss of regularity. Under a so strong variation of the parameter μ_2, it will become considerably more difficult to get any a priori estimates which remain uniform in μ_2 and this will force the solution of the limit problem to be regarded in a weaker sense. The non-separability of the contributions of the two domains in the weak formulation of the problem, moreover, will cause the dependence of some estimates on μ_2 to fall also upon the Ω_1-components of the solutions, which will likewise suffer a similar loss of regularity, even if the diffusion equations in Ω_1 are not touched by the limit procedure.

Asymptotic analyses of problems similar to ours, but in the simpler case of a single domain, have been performed in the papers by Damlamian, Kenmochi and Sato [8], by Colli, Gilardi and Grasselli [7] (in the case of the phase field model with memory effects) and by Visintin [17] (for the so-called "phase relaxation" model, which does not take account of the spatial phase field diffusion). Other references can be found in the bibliographies of the papers quoted.

In Section 2, we present both the original and the limit problem in their precise mathematical formulations and list the main results of the paper; in Section 3, we carry out the regularization procedure, deducing the existence of an approximate solution to the initial problem. Some a priori estimates, involving or not the dependence on μ_2, are obtained in Section 4. In Section 5, we first pass to the limit in the approximation scheme at μ_2 fixed to get a solution of the original problem; then the asymptotic behaviour for μ_2 tending to 0 is investigated and we get a solution for the limit formulation, also specifying which regularity properties are left for it after the limit procedure. Finally, in Section 6 we prove a simple uniqueness result for the limit statement.

2. Formulation of the problems and the main results

Starting from system (1)–(3), we first describe the physical hypotheses of the transmission problem. The precise formulation subsequently presented will have a variational character, anyway; hence, some of the physical requirements that we are going to state hereafter will actually become implicit in the weak framework.

In the following, the index i will always be supposed to take the values $1, 2$, referring to one or the other of the domains; $T > 0$ will be an arbitrary final time. We denote by Γ the common boundary of the sets Ω_1, Ω_2, and define $\Gamma_i := \partial\Omega_i \setminus \overline{\Gamma}$, $\Omega := \Omega_1 \cup \Gamma \cup \Omega_2$, $Q := \Omega \times]0, T[$, $Q_i := \Omega_i \times]0, T[$, $\Sigma := \Gamma \times]0, T[$, $\Sigma_i :=$

$\Gamma_i \times]0, T[$. The whole region Ω will be supposed smooth (say C^2) and bounded and the subdomains Ω_i Lipschitz continuous. Moreover, for any function (or distribution) v defined on Ω, it will be convenient to denote by v_i its restriction to Ω_i. Conversely, given a couple of functions (w_1, w_2), defined on Ω_1 and Ω_2 respectively, we will denote by w the function on Ω whose restriction to Ω_i coincides with w_i (we shall use this notation also to denote by μ, ν the piecewise constant functions coinciding with μ_i, ν_i respectively on Ω_i). As we deal with L^p functions, we need not worry about the ambiguity on the interface Γ. This notation will be adopted in the sequel without further remarks.

Now we can state the physical transmission properties at the interface, which will be required in the deduction of the weak formulation:

$$u_1 = u_2 \quad \text{on } \Gamma, \tag{4}$$

$$\chi_1 = \chi_2 \quad \text{on } \Gamma, \tag{5}$$

$$\nu_1 \partial_{\mathbf{n}} \chi_1 = \nu_2 \partial_{\mathbf{n}} \chi_2 \quad \text{on } \Gamma, \tag{6}$$

$$\partial_{\mathbf{n}} u_1 = \partial_{\mathbf{n}} u_2 + \lambda(1 - \nu_1 \nu_2^{-1}) \partial_{\mathbf{n}} \chi_1 \quad \text{on } \Gamma, \tag{7}$$

where $\mathbf{n}$ stands for the normal unit vector on Γ pointing outwards Ω_1. Here and in the sequel, we adopt the convention of denoting with the same symbol a function defined on Ω and its traces on $\partial\Omega$ and Γ.

Moreover, we need boundary value conditions on the rest of the boundaries. We shall treat both the Dirichlet

$$u_i = \chi_i = 0 \quad \text{on } \Gamma_i \tag{8}$$

and the Neumann case:

$$\partial_{\mathbf{n}_i} u_i = \partial_{\mathbf{n}_i} \chi_i = 0 \quad \text{on } \Gamma_i \tag{9}$$

(here $\mathbf{n}_i$ is the outer normal unit vector to Γ_i).

As is customary when dealing with transmission problems, the weak formulation of equations (1)–(3) will be written in compact form by resetting them in the whole domain Ω. This is the reason why we now introduce an abstract Hilbert triplet (V, H, V'), which in the case of the Dirichlet conditions reads:

$$V = H_0^1(\Omega), \quad H = L^2(\Omega), \quad V' = H^{-1}(\Omega), \tag{10}$$

while in the Neumann case the choice is naturally

$$V = H^1(\Omega), \quad H = L^2(\Omega), \quad V' = H^1(\Omega)'. \tag{11}$$

In either case, it is convenient to set $V_i := \{v_i\colon\ v \in V\}$, obtaining in this way other two Hilbert triplets

$$V_i, \quad L^2(\Omega_i), \quad V_i'. \tag{12}$$

Moreover, we assume the following hypotheses on data:

$$u_0^\mu, \chi_0^\mu \in V, \tag{13}$$
$$u_0 \in H, \tag{14}$$
$$\chi_{1,0} \in L^2(\Omega_1), \tag{15}$$
$$f, f^\mu \in L^2(Q), \tag{16}$$
$$\lambda, \mu_1, \overline{\mu}_2, \nu_i \quad \text{positive real parameters}, \tag{17}$$
$$0 < \mu_2 < \overline{\mu}_2, \tag{18}$$

where, by the last condition we mean that μ_2 will be allowed to vary in the interval $]0, \overline{\mu}_2[$.

Considering the constitutive relation (3), we require α_i to be maximal monotone, possibly multivalued, graphs in $\mathbb{R} \times \mathbb{R}$, so normalized that $0 \in \alpha_i(0)$. As a consequence, there exist convex and lower semicontinuous functions $\varphi_i\colon\ \mathbb{R} \to [0, +\infty]$ such that $\alpha_i = \partial\varphi_i$ and $\varphi_i(0) = 0$.

Operators α_i are also required to satisfy some compatibility conditions whose meaning will become clearer in the course of the approximating procedure. As in our former paper [15], we suppose that $D(\alpha_1) = D(\alpha_2) =: D_{1,2}$, where, by definition, $D(\alpha_i) = \{x \in \mathbb{R}\colon\ \alpha_i(x) \neq \emptyset\}$ is the *domain* of α_i, and that there exists a fixed constant C_1 such that

$$|y_1| \leqslant C_1(1 + |y_2|), \quad |y_2| \leqslant C_1(1 + |y_1|) \tag{19}$$

for all $x \in \operatorname{Int}(D_{1,2})$ and for all $y_1 \in \alpha_1(x), y_2 \in \alpha_2(x)$.

We state now the Cauchy conditions for the approximating problem:

$$u(0) = u_0^\mu, \tag{20}$$
$$\chi(0) = \chi_0^\mu. \tag{21}$$

We give also the minimal convergence-boundedness hypotheses on the data, required in the μ-passage to the limit; here M is a positive constant independent of μ_2:

$$u_0^\mu \to u_0 \quad \text{in } H\text{-strong,} \tag{22}$$

$$\chi_{1,0}^\mu \to \chi_{1,0} \quad \text{in } L^2(\Omega_1)\text{-strong,} \tag{23}$$

$$\mu_2^{1/2}\chi_{2,0}^\mu \to 0 \quad \text{in } L^2(\Omega_2)\text{-strong,} \tag{24}$$

$$f^\mu \to f \quad \text{in } L^2(Q)\text{-strong,} \tag{25}$$

$$\mu_2\|u_0^\mu\|_{H^1(\Omega)} \leqslant M, \tag{26}$$

$$\mu_2^{1/2}\|\chi_0^\mu\|_{H^1(\Omega)} \leqslant M, \tag{27}$$

$$\|\varphi_1(\chi_{1,0}^\mu)\|_{L^1(\Omega_1)} \leqslant M, \tag{28}$$

$$\mu_2\|\varphi_2(\chi_{2,0}^\mu)\|_{L^1(\Omega_2)} \leqslant M. \tag{29}$$

At this point, if we multiply (1)–(3) by a test function $v \in V$ and exploit relations (4)–(7) as well as (8) or (9), we easily obtain the natural weak formulation of our equations, which can be written as

P r o b l e m s (dwP$^\mu$)–(nwP$^\mu$). For any fixed $\mu_2 \in]0, \overline{\mu}_2[$, find a couple of functions

$$(u^\mu, \chi^\mu) \in L^\infty(0,T;V)^2 \cap H^1(0,T;H)^2$$

such that the following equations hold for any $v \in V$ and for almost every $t \in [0,T]$:

$$\int_\Omega \partial_t u^\mu v \, \mathrm{d}x + \int_\Omega \nabla u^\mu \cdot \nabla v \, \mathrm{d}x = \int_\Omega f^\mu v \, \mathrm{d}x + \lambda \int_\Omega \nabla \chi^\mu \cdot \nabla v \, \mathrm{d}x, \tag{30}$$

$$\int_\Omega \mu(x)\partial_t \chi^\mu v \, \mathrm{d}x + \int_\Omega \nu(x)\nabla\chi^\mu \cdot \nabla v \, \mathrm{d}x + \int_\Omega \zeta^\mu v \, \mathrm{d}x = \lambda \int_\Omega u^\mu v \, \mathrm{d}x, \tag{31}$$

$$\chi_i^\mu \in D(\alpha_i) \quad \text{and} \quad \zeta_i^\mu \in \alpha_i(\chi_i^\mu) \quad \text{a.e. in } Q_i \tag{32}$$

along with the initial conditions (20)–(21).

Here and in the sequel the first letter 'd' or 'n' of the "name" of the problem accounts for the choice of the Dirichlet or Neumann boundary conditions, respectively.

A formal substitution $\mu_2 = 0$ in equation (31) immediately suggests the form of the limit problems. However, their precise statement feels the effect of lower regularity of the solutions:

P r o b l e m s (dwP)–(nwP). Find

$$u \in H^1(0,T;V') \cap L^2(0,T;V) \quad \text{and} \tag{33}$$

$$\chi \in L^2(0,T;V) \quad \text{with} \quad \chi_1 \in H^1(0,T;V_1') \tag{34}$$

such that the following equations hold for every $v \in L^2(0,T;V)$ and $t \in [0,T]$:

$$
\begin{aligned}
&(35) \quad {}_{L^2(0,t;V')}\langle \partial_t u, v\rangle_{L^2(0,t;V)} + \int_0^t \int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x \, \mathrm{d}s \\
&\qquad = \int_0^t \int_\Omega f v \, \mathrm{d}x \, \mathrm{d}s + \lambda \int_0^t \int_\Omega \nabla\chi \cdot \nabla v \, \mathrm{d}x \, \mathrm{d}s \\
&(36) \quad {}_{L^2(0,t;V_1')}\langle \mu_1 \partial_t \chi_1, v_1\rangle_{L^2(0,t;V_1)} + \int_0^t \int_\Omega \nu(x)\nabla\chi \cdot \nabla v \, \mathrm{d}x \, \mathrm{d}s + \int_0^t \int_\Omega \zeta v \, \mathrm{d}x \\
&\qquad = \lambda \int_0^t \int_\Omega u v \, \mathrm{d}x \, \mathrm{d}s \\
&(37) \quad \chi_i \in D(\alpha_i) \quad \text{and} \quad \zeta_i \in \alpha_i(\chi_i) \quad \text{a.e. in } Q_i.
\end{aligned}
$$

In order to unify the procedures of resolution with respect to the choice of the Dirichlet or Neumann conditions, we also introduce the abstract operator (depending on V too):

$$
D\colon\ V \to V', \quad {}_{V'}\langle Dv, w\rangle_V := \int_\Omega \nu(x)\nabla v(x) \cdot \nabla w(x) \, \mathrm{d}x. \tag{38}
$$

For the purpose of specifying the boundary regularity of the solutions, we also need to recall some definitions and properties of trace spaces of Sobolev type. First of all, we point out that, for any function in $H^1(\Omega_i)$, the trace on Γ [or $\partial\Omega_i$, Γ_i] lies in the space $H^{1/2}(\Gamma)$ [$H^{1/2}(\partial\Omega_i)$, $H^{1/2}(\Gamma_i)$] and the related trace operator is continuous; we shall denote with $H^{1/2}_{00}(\Gamma)$ [$H^{1/2}_{00}(\Gamma_i)$] the trace space on Γ [Γ_i] of the functions of $H^1(\Omega_i)$ vanishing on Γ_i [Γ] (it is endowed with the graph norm with respect to the corresponding trace operator); we also point out that $H^{1/2}_{00}(\Gamma)$ [$H^{1/2}_{00}(\Gamma_i)$] is densely and continuously embedded into $H^{1/2}(\Gamma)$ [$H^{1/2}(\Gamma_i)$, respectively]. Moreover, the continuous and dense transpose inclusion

$$
H^{-1/2}(\Gamma) = H^{1/2}(\Gamma)' \subset H^{1/2}_{00}(\Gamma)' \tag{39}
$$

holds (and also the corresponding one with Γ_i in place of Γ). Finally, we recall that the space $H(\mathrm{div}, \Omega_i)$ is defined as

$$
H(\mathrm{div}, \Omega_i) := \{\mathbf{h} \in L^2(\Omega_i)^N\colon\ \mathrm{div}\,\mathbf{h} \in L^2(\Omega_i)\} \tag{40}
$$

and that the related trace operators

$$
\mathbf{h} \mapsto \mathbf{h}\cdot\mathbf{n}_i, \quad H(\mathrm{div}, \Omega_i) \to H^{-1/2}(\partial\Omega_i), \tag{41}
$$

$$
\mathbf{h} \mapsto \mathbf{h}\cdot\mathbf{n}_i, \quad H(\mathrm{div}, \Omega_i) \to H^{1/2}_{00}(\Gamma)' \quad \text{or } H^{1/2}_{00}(\Gamma_i)' \tag{42}
$$

are continuous, $\mathbf{n}_i$ denoting in this case the outer normal unit vector to $\partial\Omega_i$.

We are now ready to state the main results of this paper:

Theorem 1. *For every* $\mu_2 \in]0, \overline{\mu}_2[$, *there exists a solution to problem* (dwP$^\mu$) *and a solution to* (nwP$^\mu$). *For either choice of the boundary conditions, the corresponding solution will be denoted by the same symbol* (u^μ, χ^μ). *Moreover,* $D\chi^\mu \in L^2(0,T;H)$, *and therefore* $\chi^\mu \in C^0([0,T];V)$ *and*

$$\mu_i \partial_t \chi_i^\mu - \nu_i \Delta \chi_i^\mu + \zeta_i^\mu = \lambda u_i^\mu \tag{43}$$

holds a.e. in Q_i. *Furthermore, for a.e.* $t \in [0,T]$, *we can recover* (6) *and, in the Neumann case, also* (9). *Finally, if* $\nu_1 = \nu_2$, *we have that* $\chi \in L^2(0,T;H^2(\Omega))$.

Here is the corresponding result for the limit systems:

Theorem 2. *Under the convergence and boundedness hypotheses* (22)–(29), *problem* (dwP) *[*(nwP)*] admits a solution* (u,χ) *which is the limit of solutions of problems* (dwP$^\mu$) *[respectively,* (nwP$^\mu$)*] in the following sense:*

$$u^\mu \to u \quad \text{in } L^\infty(0,T;H)\text{-weak}^*, \tag{44}$$

$$\chi_1^\mu \to \chi_1 \quad \text{in } L^\infty(0,T;L^2(\Omega_1))\text{-weak}^*, \tag{45}$$

$$u^\mu \to u \quad \text{in } L^2(0,T;V)\text{-strong}, \tag{46}$$

$$\chi^\mu \to \chi \quad \text{in } L^2(0,T;V)\text{-strong}, \tag{47}$$

$$\zeta_i^\mu \to \zeta_i \quad \text{in } L^2(Q_i)\text{-weak}. \tag{48}$$

We also have the following additional convergences for the Ω_2*-part:*

$$\mu_2^{1/2} \chi_2^\mu \to 0 \quad \text{in } L^\infty(0,T;V_2)\text{-weak}^*, \tag{49}$$

$$\mu_2 \chi_2^\mu \to 0 \quad \text{in } H^1(0,T;L^2(\Omega_2))\text{-weak}. \tag{50}$$

Moreover,

$$-\nu_2 \Delta \chi_2 + \zeta_2 = \lambda \chi_2 \tag{51}$$

a.e. in Q_2 *and in the Neumann case* (9) *holds, only for* $i = 2$, *in the sense of* $H^{-1/2}(\Gamma_2)$. *Finally, we have*

$$u(0) = u_0, \tag{52}$$

$$\chi_1(0) = \chi_{1,0} \tag{53}$$

Remark. In the limit formulations, the initial value condition for χ_2 disappears since the limit problems are "stationary" in Ω_2 as concerns the phase field.

R e m a r k. We specify that, thanks to (42), the relation (6) appearing in the statement of Theorem 1 is a priori valid in the space $H^{1/2}_{00}(\Gamma)'$. Nevertherless, condition (9) (which is obviously a relation in $H^{-1/2}(\Gamma_i)$) and [9], Prop. 3.3, allow to say that it actually holds in a slightly more regular space $H^{-1/2}(\Gamma)$.

To conclude, we state our uniqueness result for the limit problem:

Theorem 3. *The solution to problem* (dwP) *[or* (nwP)*] with initial values* (52)–(53) *is unique.*

R e m a r k. In the course of proof of Theorem 2, we will show that the convergences listed in the statement hold up to the extraction of suitable subsequences; Theorem 3 actually tells us that they are valid for the whole original sequences.

3. Approximation of the problems

We want to use the Faedo-Galerkin regularization scheme in order to obtain a family of approximate solutions to problem (dwP$^\mu$) [or (nwP$^\mu$)]. We point out that our regularization procedure is developed separately for every *fixed* $\mu_2 \in]0, \overline{\mu}_2[$; so, we do not require any uniformity with respect to μ_2.

First of all, let us consider the standard elliptic eigenvalue problems (depending on the choice of V):

$$\begin{cases} v_n \in V \\ Dv_n = \lambda_n v_n \quad \text{in } V', \end{cases} \tag{54}$$

where the eigenvalues λ_n are ordered in an increasing sequence and each counted according to its multiplicity, so that in both cases their associated eigenvectors v_n, with $\|v_n\|_H = 1$, form an orthogonal (with respect to the weighted natural inner product) basis of V; if we define $V_n := \operatorname{span}\{v_1, \ldots, v_n\}$ and $V_\infty := \bigcup\limits_{n=1}^{\infty} V_n$, we get that V_∞ is dense in V.

Recall now problems (dwP$^\mu$) and (nwP$^\mu$). First, replace the operators α_i in relation (32) with their Yosida approximates (see for instance [2] or [3]) α_i^ε. We remark that, for any $x \in \mathbb{R}$,

$$\alpha_i^\varepsilon(x) := \frac{x - J_i^\varepsilon(x)}{\varepsilon}, \tag{55}$$

where $J_i^\varepsilon := (\mathrm{Id} + \varepsilon\alpha_i)^{-1}$ is the *resolvent* of the operator α_i. We also recall that α_i^ε are Lipschitz continuous functions and inherit the property $\alpha_i^\varepsilon(0) = 0$. Denote by

φ_i^ε the primitive of α_i^ε such that $\varphi_i^\varepsilon(0) = 0$. We remark (see [3] again) that

$$\varphi_i^\varepsilon(x) \leqslant \varphi_i(x) \quad \text{for all } x \in \mathbb{R}. \tag{56}$$

Replace also (u^μ, χ^μ) with $(u^{\mu,\varepsilon,n}, \chi^{\mu,\varepsilon,n})$, where we require

$$u^{\mu,\varepsilon,n} = \sum_{j=1}^{n} a_{jn}^{\mu,\varepsilon}(t) v_j, \quad \chi^{\mu,\varepsilon,n} = \sum_{j=1}^{n} b_{jn}^{\mu,\varepsilon}(t) v_j \tag{57}$$

and $a_{jn}^{\mu,\varepsilon}, b_{jn}^{\mu,\varepsilon}$ are regarded as real-valued functions on $[0,T]$. Finally, take v in V_n instead of in the whole V. These modifications change system (30)–(32) into its Faedo-Galerkin regularization, which is worth while to write at once in the following very compact vectorial form, where we have chosen $v = v_h$ for $h = 1, \ldots, n$ and do not superscribe the indices μ, ε, n in order to simplify the notation:

$$\begin{cases} M\mathbf{a}' + N\mathbf{a} = \mathbf{f} + \lambda N\mathbf{b}, \\ \widetilde{M}(\mu)\mathbf{b}' + \widetilde{N}\mathbf{b} + \mathbf{T}^\varepsilon(\mathbf{b}) = \lambda M\mathbf{a}. \end{cases} \tag{58}$$

Here $(\mathbf{a}, \mathbf{b})$ are the vectors $(a_{hn}^{\mu,\varepsilon}(t), b_{hn}^{\mu,\varepsilon}(t))_{h=1,\ldots,n}$, and we have set

$$m_{hj} := (v_h, v_j)_H = \delta_{hj}, \quad n_{hj} := (\nabla v_h, \nabla v_j)_H, \tag{59}$$

$$\widetilde{m}_{hj}(\mu) := \int_\Omega \mu(x) v_h(x) v_j(x)\, \mathrm{d}x, \quad \widetilde{n}_{hj} := \int_\Omega \nu(x) \nabla v_h(x) \cdot \nabla v_j(x)\, \mathrm{d}x = \lambda_j \delta_{hj} \tag{60}$$

and finally

$$f_h := \int_\Omega f v_h\, \mathrm{d}x, \quad T_h^\varepsilon(\mathbf{b}) := \sum_{i=1}^{2} \int_{\Omega_i} \alpha_i^\varepsilon \Big(\sum_{j=1}^{n} b_{jn}^{\mu,\varepsilon}(t) v_j(x) \Big) v_h(x)\, \mathrm{d}x, \tag{61}$$

where $\mathbf{T}^\varepsilon$ is Lipschitz since such are the α_i^ε. We also approximate the initial data (u_0^μ, χ_0^μ) with a sequence

$$(u_0^{\mu,n}, \chi_0^{\mu,n}) \in V_n^2, \quad \text{with} \quad (u_0^{\mu,n}, \chi_0^{\mu,n}) \to (u_0^\mu, \chi_0^\mu) \quad \text{in } V^2, \tag{62}$$

that is

$$(u_0^{\mu,n}, \chi_0^{\mu,n}) = \Big(\sum_{j=1}^{n} a_{jn,0}^{\mu} v_j, \sum_{j=1}^{n} b_{jn,0}^{\mu} v_j \Big). \tag{63}$$

Moreover, thanks to the quadratic growth of the φ_i^ε and to (62), we can also suppose that

$$\int_{\Omega_i} \varphi_i^\varepsilon(\chi_{i,0}^{\mu,n})\, \mathrm{d}x \leqslant 1 + \int_{\Omega_i} \varphi_i^\varepsilon(\chi_{i,0}^{\mu})\, \mathrm{d}x \tag{64}$$

at least for n sufficiently large (depending on μ_2 and ε).

Notice now that, for every choice of $\mu_2 > 0$ and for every $n \in \mathbf{N}$, the matrix $\widetilde{M}(\mu)$ is associated with the scalar product in V_n

$$g(v,w) := \int_\Omega \mu(x)v(x)w(x)\,\mathrm{d}x; \tag{65}$$

consequently, it is positive definite and, in particular, invertible.

So, for every ε, μ_2, n, Cauchy's theorem for ordinary differential equations yields a solution of system (58) with initial values (63); that is, we get a family of approximate solutions

$$(u^{\mu,\varepsilon,n}, \chi^{\mu,\varepsilon,n}) \in C^1([0,T]; V_n)^2$$

of the Faedo-Galerkin regularization of system (30)–(32).

4. A priori estimates

With suitable choices of the test function v in (30)–(31)-regularized, we now deduce several a priori estimates for the solutions of the approximate problems as obtained in the previous section, with the purpose of removing the (ε, n)-approximations. The procedure will be carried out at one time for either choice of the boundary conditions, since at this level there will be no remarkable differences. The convergences stated in Theorem 2 will also follow, with the important exception of (46)–(47) which can only be deduced in the weak sense. Throughout this section, we will omit the superscripts μ, ε, n over the solutions of the approximate problem, but not over the initial conditions $(u_0^{\mu,n}, \chi_0^{\mu,n})$ and over f^μ, which need a more careful treatment.

In the following, we shall repeatedly exploit the elementary inequality

$$ab \leqslant \sigma a^2 + \frac{1}{4\sigma} b^2 \quad \text{for every } a, b \in \mathbb{R}, \sigma > 0. \tag{66}$$

We also recall the Poincaré inequality in the form which will be used later:

Lemma 1. *There exists a purely geometric constant C_Ω such that, for any $v \in H^1(\Omega)$, we have*

$$\|v_2\|^2_{L^2(\Omega_2)} \leqslant C_\Omega [\|\nabla v_2\|^2_{L^2(\Omega_2)} + \|v_1\|^2_{H^1(\Omega_1)}], \tag{67}$$

where, if we also suppose $v \in H^1_0(\Omega)$, the second norm on the right hand side can be omitted; however, we will write it also in that case in order to unify the computations.

Theorem 4. *There exists a constant $K > 0$, independent of μ_2, ε, such that, for every $\mu_2 \in\,]0, \overline{\mu}_2[$, the following estimates hold for any n sufficiently (in a way possibly depending on μ_2) large:*

$$\|u\|_{L^\infty(0,T;H)} \leqslant K, \tag{68}$$

$$\|\chi\|_{L^2(0,T;V)} \leqslant K, \tag{69}$$

$$\|u\|_{L^2(0,T;V)} \leqslant K, \tag{70}$$

$$\mu_i^{1/2}\|\chi_i\|_{L^\infty(0,T;L^2(\Omega_i))} \leqslant K. \tag{71}$$

P r o o f. Let us choose respectively $v = u$ and $v = m\chi$ in the heat and phase field Galerkin-approximate equations (30)–(31), where $m > 0$ will be fixed later. Integrate then in time between 0 and an arbitrary $t \in [0, T]$ and carry out the computations; summing together the results, exploiting (66) and recalling the monotonicity of α_i, it is easy to obtain

$$\begin{aligned} &\frac{1}{2}\|u(t)\|_H^2 + \|\nabla u\|^2_{L^2(0,t;H)} + \sum_{i=1}^2 \Big[\frac{m\mu_i}{2}\|\chi_i(t)\|^2_{L^2(\Omega_i)} + m\nu_i\|\nabla\chi_i\|^2_{L^2(0,t;L^2(\Omega_i))}\Big] \\ &\leqslant \frac{1}{2}\|u_0^{\mu,n}\|_H^2 + \sum_{i=1}^2 \frac{m\mu_i}{2}\|\chi_{i,0}^{\mu,n}\|^2_{L^2(\Omega_i)} + \frac{1}{2}\|f^\mu\|^2_{L^2(0,t;H)} + \frac{1}{2}\|u\|^2_{L^2(0,t;H)} \\ &\quad + \frac{\lambda^2}{2}\|\nabla\chi\|^2_{L^2(0,t;H)} + \frac{1}{2}\|\nabla u\|^2_{L^2(0,t;H)} + \frac{m\sigma}{2}\|\chi\|^2_{L^2(0,t;H)} + \frac{\lambda^2 m}{2\sigma}\|u\|^2_{L^2(0,t;H)}. \end{aligned} \tag{72}$$

Moreover, owing to (67), we derive

$$\|\chi\|^2_{L^2(0,T;H)} \leqslant (1 + C_\Omega)\|\chi_1\|^2_{L^2(0,t;L^2(\Omega_1))} + C_\Omega\|\nabla\chi\|^2_{L^2(0,t;H)}. \tag{73}$$

Thanks to the previous computation, the choice of

$$m = \frac{3\lambda^2}{2\min\limits_{i=1,2}\{\nu_i\}}, \qquad \sigma = \frac{2\min\limits_{i=1,2}\{\nu_i\}}{3C_\Omega}$$

in (72) allows us to kill also the $\nabla\chi$-term.

Finally, observe that the terms related to the initial and source (f^μ) data are bounded thanks to (22)–(25) and to (62); the double approximation (before in μ_2 and then in n), nevertheless, prevents us from finding estimates holding uniformly in n and μ_2 at the same time. Anyway, the Gronwall lemma easily yields (68), and also (70)–(71) follow at once from (72). Finally, (69) is a consequence of (71), (72) and (73). □

Theorem 5. *For some $K > 0$ independent of μ_2, ε, n, we get the further estimates, holding at least for n sufficiently large (depending on μ_2, ε):*

$$\mu_2^{1/2} \|\chi_1\|_{H^1(0,T;L^2(\Omega_1))} \leqslant K, \tag{74}$$

$$\mu_2 \|\chi_2\|_{H^1(0,T;L^2(\Omega_2))} \leqslant K, \tag{75}$$

$$\mu_2^{1/2} \|\chi\|_{L^\infty(0,T;V)} \leqslant K, \tag{76}$$

$$\mu_2 \|\varphi_i^\varepsilon(\chi_i)\|_{L^\infty(0,T;L^1(\Omega_i))} \leqslant K. \tag{77}$$

P r o o f. Use $\partial_t \chi$ as a test function in the Galerkin regularization of equation (31). Integrate as before over $[0, t]$, obtaining by easy calculations

$$\begin{aligned}
&\sum_{i=1}^{2} \Big[\mu_i \|\partial_t \chi_i\|^2_{L^2(0,t;L^2(\Omega_i))} + \frac{\nu_i}{2} \|\nabla \chi_i(t)\|^2_{L^2(\Omega_i)}\Big] + I_1 \\
&\quad = \sum_{i=1}^{2} \frac{\nu_i}{2} \|\nabla \chi_{i,0}^{\mu,n}\|^2_{L^2(\Omega_i)} + \lambda \int_0^t \int_\Omega u \partial_t \chi \,\mathrm{d}x \,\mathrm{d}s \\
&\quad \leqslant \sum_{i=1}^{2} \Big[\frac{\nu_i}{2} \|\nabla \chi_{i,0}^{\mu,n}\|^2_{L^2(\Omega_i)} + \frac{\lambda^2}{2\mu_i} \|u_i\|^2_{L^2(0,t;L^2(\Omega_i))} + \frac{\mu_i}{2} \|\partial_t \chi_i\|^2_{L^2(0,t,L^2(\Omega_i))}\Big]
\end{aligned} \tag{78}$$

where we have set

$$I_1 := \sum_{i=1}^{2} \int_0^t \int_{\Omega_i} \alpha_i^\varepsilon(\chi_i) \partial_t \chi_i \,\mathrm{d}x \,\mathrm{d}s \tag{79}$$

and, thanks to the Lipschitz continuity of the Yosida approximates, we have

$$I_1 = \sum_{i=1}^{2} \Big[\int_{\Omega_i} \varphi_i^\varepsilon(\chi_i(t)) \,\mathrm{d}x - \int_{\Omega_i} \varphi_i^\varepsilon(\chi_{i,0}^{\mu,n}) \,\mathrm{d}x\Big]. \tag{80}$$

Substituting this expression into (78), its second term, moved to the right hand side, reads

$$\begin{aligned}
\sum_{i=1}^{2} \int_{\Omega_i} \varphi_i^\varepsilon(\chi_{i,0}^{\mu,n}) \,\mathrm{d}x &\leqslant 1 + \sum_{i=1}^{2} \int_{\Omega_i} \varphi_i^\varepsilon(\chi_{i,0}^{\mu}) \,\mathrm{d}x \quad \text{thanks to (64)} \\
&\leqslant 1 + \sum_{i=1}^{2} \int_{\Omega_i} \varphi_i(\chi_{i,0}^{\mu}) \,\mathrm{d}x \quad \text{thanks to (56).}
\end{aligned}$$

Notice now that it is necessary to multiply (78) by μ_2, in order to evaluate the norm of u_2 independently of μ_2; unfortunately, because of the non-separability of the equations in Ω_1 and Ω_2, the factor μ_2 will fall also on the integrals on Ω_1 on the left hand side, preventing us from finding for them estimates independent of μ_2. This feature is characteristic of the problem and we will find it again.

Anyway, the boundedness hypotheses (27)–(29), along with (62), allow us to complete immediately the proof of the theorem. □

We prove now the regularity in time for the enthalpy u:

Theorem 6. *For some fixed $K > 0$, the following relations hold for sufficiently large n:*

$$\mu_2 \|u\|_{H^1(0,T;H)} \leqslant K, \tag{81}$$

$$\mu_2 \|u\|_{L^\infty(0,T;V)} \leqslant K. \tag{82}$$

Proof. Take $v = \partial_t(u - \lambda\chi)$ in equation (30) and integrate over $[0,t]$. Proceeding as in the former proofs it is easy to derive

$$\|\partial_t u\|^2_{L^2(0,t;H)} + \frac{1}{2}\|\nabla(u-\lambda\chi)(t)\|^2_H = \frac{1}{2}\|\nabla(u_0^{\mu,n} - \lambda\chi_0^{\mu,n})\|^2_H$$
$$+ \int_0^t \int_\Omega f^\mu \partial_t u \, \mathrm{d}x \, \mathrm{d}s - \lambda \int_0^t \int_\Omega f^\mu \partial_t \chi \, \mathrm{d}x \, \mathrm{d}s + \lambda \int_0^t \int_\Omega \partial_t u \, \partial_t \chi \, \mathrm{d}x \, \mathrm{d}s.$$

Splitting the integral terms with the aid of (66), we obtain

$$\begin{aligned} &\frac{1}{2}\|\partial_t u\|^2_{L^2(0,t;H)} + \frac{1}{2}\|\nabla(u-\lambda\chi)(t)\|^2_H \\ &\quad \leqslant \frac{1}{2}\|\nabla(u_0^{\mu,n} - \lambda\chi_0^{\mu,n})\|^2_H + \Big(1 + \frac{\lambda^2}{2}\Big)\|f^\mu\|^2_{L^2(0,t;H)} \\ &\qquad + \Big(\frac{1}{2} + \lambda^2\Big)\|\partial_t \chi\|^2_{L^2(0,t;H)}. \end{aligned} \tag{83}$$

Hence, multiplying the previous relation by μ_2^2 and taking into account the regularity hypotheses (25)–(27), the approximation ones (62) and the estimates (74)–(75), we easily get (81), while (68) and (76), also imply (82) as desired. □

The above estimates are sufficient to remove the Galerkin approximation; so we can now start the limit procedures.

5. Proofs of the existence results

P r o o f o f T h e o r e m 1. It can be performed at one time for both choices of boundary conditions. We begin by removing the Galerkin approximation. Exploiting estimates (68)–(71), (74)–(76) and (81)–(82), we easily get, for all fixed $\mu_2 \in]0, \overline{\mu}_2[$, the following weak or weak* convergences:

$$u^{\mu,\varepsilon,n} \to u^{\mu,\varepsilon} \quad \text{in } L^\infty(0,T;V)\text{-weak}^* \text{ and in } H^1(0,T;H)\text{-weak}, \tag{84}$$

$$\chi^{\mu,\varepsilon,n} \to \chi^{\mu,\varepsilon} \quad \text{in } L^\infty(0,T;V)\text{-weak}^* \text{ and in } H^1(0,T;H)\text{-weak}, \tag{85}$$

holding at least for appropriate subsequences. Thanks to the generalized Aubin compactness theorem (see [16], section 8, Corollary 4), we get also

$$(u^{\mu,\varepsilon,n}, \chi^{\mu,\varepsilon,n}) \to (u^{\mu,\varepsilon}, \chi^{\mu,\varepsilon}) \quad \text{in } C^0([0,T]; H^{1/2}(\Omega))^2\text{-strong} \tag{86}$$

and, recalling (62), the initial value conditions (20)–(21). Moreover, thanks to the Lipschitz continuity of α_i^ε, we obtain

$$\alpha_i^\varepsilon(\chi_i^{\mu,\varepsilon,n}) \to \alpha_i^\varepsilon(\chi_i^{\mu,\varepsilon}) \quad \text{in } C^0([0,T]; L^2(\Omega_i))\text{-strong}. \tag{87}$$

We can now pass to the limit in n in the Galerkin regularization of equations (30)–(31). We omit the checkings which are very similar to the corresponding ones in [15]. We just point out that, thanks to the density of V_∞ in V, we are allowed to take the test functions v for the limit equations in the whole space V.

To conclude the proof of Theorem 1, we have to come back from the Yosida approximations α_i^ε to the initial operators α_i. Clearly, estimates (68)–(71), (74)–(76) and (81)–(82), being uniform also in ε, allow us to get, for any fixed $\mu_2 \in]0, \overline{\mu}_2[$ and for suitable subsequences, the convergences

$$(u^{\mu,\varepsilon}, \chi^{\mu,\varepsilon}) \to (u^\mu, \chi^\mu) \tag{88}$$

in $L^\infty(0,T;V)^2$-weak*, $H^1(0,T;H)^2$-weak and in $C^0([0,T]; H^{1/2}(\Omega))^2$-strong. The ε-limit procedure is now the same as before, except for the α_i^ε-parts which require a new estimate. First of all, we recall a result of [15], adapting the compatibility assumption (19) to the Yosida approximate graphs α_i^ε.

Lemma 2. *We have*

$$|\alpha_1^\varepsilon(x)| \leqslant C_1(1 + |\alpha_2^\varepsilon(x)|), \quad |\alpha_2^\varepsilon(x)| \leqslant C_1(1 + |\alpha_1^\varepsilon(x)|), \tag{89}$$

for all $x \in \mathbb{R}$ and $\varepsilon > 0$.

It is now easy to prove the following final apriori estimate:

Lemma 3. *There exists $K>0$ independent of μ_2,ε, such that*

$$\|\alpha_i^\varepsilon(\chi_i^{\mu,\varepsilon})\|_{L^2(Q_i)}\leqslant K; \tag{90}$$

hence, for subsequences,

$$\alpha_i^\varepsilon(\chi_i^{\mu,\varepsilon})\to\zeta_i^\mu\quad \text{in } L^2(Q_i)\text{-weak}, \tag{91}$$

where, moreover, the ζ_i^μ's satisfy relation (32).

Proof. Since α_i^ε are Lipschitz continuous and $\chi^{\mu,\varepsilon}\in V$ for a.e. $t\in[0,T]$ (thanks to (85)), we can choose $v=\sum\limits_{h=1}^{2}\alpha_h^\varepsilon(\chi^{\mu,\varepsilon})$ in the Yosida approximation of (31). Now, the proof is analogous to that of a similar result of [15], to which we refer again for more details.

Integrating over $[0,t]$ and removing the nonnegative terms from the left hand side; observing also that, due to monotonicity (recall that $\alpha_i^\varepsilon(0)=0$), we have $\alpha_1^\varepsilon(x)\alpha_2^\varepsilon(x)\geqslant 0$ for all $x\in\mathbb{R}$, it is easy to get, after some computations,

$$\begin{aligned}\sum_{i=1}^2\sum_{h=1}^2\mu_i\int_{\Omega_i}\varphi_h^\varepsilon(\chi_i^{\mu,\varepsilon}(t))\,\mathrm{d}x+\sum_{i=1}^2\int_0^t\int_{\Omega_i}(\alpha_i^\varepsilon(\chi_i^{\mu,\varepsilon}))^2\,\mathrm{d}x\,\mathrm{d}s\\ \leqslant\sum_{i=1}^2\sum_{h=1}^2\mu_i\int_{\Omega_i}\varphi_h^\varepsilon(\chi_{i,0}^\mu)\,\mathrm{d}x+\frac{C_2}{\sigma}+\sigma\sum_{i=1}^2\sum_{h=1}^2\|\alpha_h^\varepsilon(\chi_i^{\mu,\varepsilon})\|^2_{L^2(0,t;L^2(\Omega_i))},\end{aligned} \tag{92}$$

where $\sigma>0$ is arbitrary and C_2 depends only on the norm of $u^{\mu,\varepsilon}$ in $L^2(Q)$, which has already been estimated.

We could have troubles with the terms on the right hand side when $i\neq h$. We observe anyway that condition (89) entails, for C_3, C_4 depending only on C_1,

$$\|\alpha_h^\varepsilon(\chi_i^{\mu,\varepsilon})\|^2_{L^2(0,t;L^2(\Omega_i))}\leqslant C_3\|\alpha_i^\varepsilon(\chi_i^{\mu,\varepsilon})\|^2_{L^2(0,t;L^2(\Omega_i))}+C_4 \tag{93}$$

and

$$\begin{aligned}\int_{\Omega_i^{\mu,+}}\varphi_h^\varepsilon(\chi_{i,0}^\mu(x))\,\mathrm{d}x&=\int_{\Omega_i^{\mu,+}}\int_0^{\chi_{i,0}^\mu(x)}\alpha_h^\varepsilon(s)\,\mathrm{d}s\,\mathrm{d}x\\ &\leqslant\int_{\Omega_i^{\mu,+}}\int_0^{\chi_{i,0}^\mu(x)}C_1(\alpha_i^\varepsilon(s)+1)\,\mathrm{d}s\,\mathrm{d}x\\ &\leqslant C_1\|\chi_{i,0}^\mu\|_{L^1(\Omega_i^{\mu,+})}+C_1\int_{\Omega_i^{\mu,+}}\varphi_i^\varepsilon(\chi_{i,0}^\mu(x))\,\mathrm{d}x,\end{aligned} \tag{94}$$

where we have set $\Omega_i^{\mu,+} := \{x \in \Omega_i\colon\ \chi_{i,0}^{\mu}(x) \geqslant 0\}$, and we notice that analogous relations still hold with $\Omega_i \setminus \Omega_i^{\mu,+}$ in place of $\Omega_i^{\mu,+}$. With the aid of (23)–(24), (28)–(29) and (56), choosing σ small enough, we immediately get an L^2-estimate for $\alpha_i^{\varepsilon}(\chi_i^{\mu,\varepsilon})$ and, consequently, (91). Recalling the strong convergence for $\chi^{\mu,\varepsilon}$ and employing [2], Prop. 1.1, page 42, also (32) follows, which proves the lemma. □

Now, the proof of Theorem 1 is almost complete; from equation (31) and from estimates (68), (74)–(75) and (90), it immediately follows that

$$D\chi^{\mu,\varepsilon} = -\mu\partial_t\chi^{\mu,\varepsilon} - \zeta^{\mu,\varepsilon} + \lambda u^{\mu,\varepsilon}$$

is bounded in $L^2(Q_2)$ with respect to ε (but not with respect to μ_2, at least as the contribution of Ω_1 is concerned); thus, possibly extracting a subsequence, we deduce

$$D\chi^{\mu,\varepsilon} \to D\chi^{\mu} \quad \text{in } L^2(Q)\text{-weak} \tag{95}$$

in ε. It follows that (43) holds in $L^2(Q_i)$ and hence q.o. in Q_i as well as the weak transmission condition (6). The regularity $\chi^{\mu} \in C^0([0,T];V)$ is now a consequence of the well-known interpolation results.

Notice also that, on the contrary, (7) cannot be recovered since we have not enough regularity for u. In the Neumann case, we also get back (9) in the weak $H^{-1/2}(\partial\Omega)$ sense. Finally, the $H^2(\Omega)$-regularity of χ in the case $\nu_1 = \nu_2$ is a consequence of the well-known results for elliptic equations. □

P r o o f o f T h e o r e m 2. Estimates (68)–(71), (75)–(76) and (90) immediately provide (44)–(45), (48), (49)–(50), as well as the additional weak convergence (in μ_2):

$$(u^{\mu}, \chi^{\mu}) \to (u, \chi) \quad \text{in } L^2(0,T;V)^2\text{-weak.} \tag{96}$$

We remark that the dependence on μ_2 of our a priori estimates gives rise to a lack of regularity in time which forbids to get, at this level, any strong convergence. Consequently, at present we can say nothing about the validity of relation (37).

Taking now $v \in \mathcal{D}(Q)$ (in the Dirichlet case; in the Neumann case, choose instead $v \in \mathcal{D}(0,T;C^{\infty}(\overline{\Omega}))$, recalling that Ω is smooth) in (30) and integrating in time over $[0,T]$, we obtain

$$\int_0^T \int_{\Omega} \partial_t u^{\mu} v \,\mathrm{d}x\,\mathrm{d}s \leqslant C_5 \|v\|_{L^2(0,T;V)} \quad \text{for all } \mu_2, \tag{97}$$

where C_5 is a positive constant depending on the bounds (25), (69)–(70); consequently, we get the convergence

$$\partial_t u^{\mu} \to \partial_t u \quad \text{in } L^2(0,T;V')\text{-weak,} \tag{98}$$

whence $u \in H^1(0,T;V')$ and (35) holds for any $v \in L^2(0,T;V)$ and for any $t \in [0,T]$. Moreover (96), (98) and a well-known continuous embedding theorem entail

$$u^\mu \to u \quad \text{in } C^0([0,T];H)\text{-weak,} \tag{99}$$

while the Aubin compactness theorem ([12], theorem 5.1, page 58) yields

$$u^\mu \to u \quad \text{in } L^2(0,T;H)\text{-strong.} \tag{100}$$

Furthermore, owing to (99), we can recover from (22) and (20) the Cauchy condition (52).

Let us now subtract (35) from the $[0,T]$-integral of (30); recalling (96) and choosing $v = u^\mu - u$, we obtain, with the aid of (66),

$$\begin{aligned} &\frac{1}{2}\|u^\mu(T) - u(T)\|_H^2 + \|\nabla(u^\mu - u)\|_{L^2(0,T;H)}^2 \\ &\quad \leqslant \frac{1}{2}\|u_0^\mu - u_0\|_H^2 + \frac{\lambda^2}{2}\|\nabla(\chi^\mu - \chi)\|_{L^2(0,T;H)}^2 \\ &\qquad + \frac{1}{2}\|\nabla(u^\mu - u)\|_{L^2(0,T;H)}^2 + \frac{1}{2}\|f^\mu - f\|_{L^2(0,T;H)}^2 + \frac{1}{2}\|u^\mu - u\|_{L^2(0,T;H)}^2 \end{aligned} \tag{101}$$

(observe that (99) justifies the integration by parts), whence, owing to (22), (25) and (100), we easily get

$$\|u^\mu(T) - u(T)\|_H^2 + \|\nabla(u^\mu - u)\|_{L^2(0,T;H)}^2 \leqslant R_1^\mu + \lambda^2\|\nabla(\chi^\mu - \chi)\|_{L^2(0,T;H)}^2. \tag{102}$$

Here, R_1^μ is a numerical sequence tending to 0 with μ_2.

We now see that a similar procedure works also for the phase field equation; first, observe that, thanks to the C^1 regularity of Ω, there exists a "reflection-like" operator $R\colon V_1 \to V$, that is a linear and continuous operator such that

$$(Rv_1)_{|\Omega_1} = v_1 \quad \text{for all } v_1 \in V_1. \tag{103}$$

So, if we choose $w_1 \in \mathcal{D}(0,T;V_1)$ and set $w := Rw_1$, substitute w in place of v in (31), and integrate in time, we get

$$\int_0^T \int_\Omega \mu_1 \partial_t \chi_1^\mu w_1 \,\mathrm{d}x\,\mathrm{d}s \leqslant C_6\|w\|_{L^2(0,T;V)} \leqslant C_7\|w_1\|_{L^2(0,T;V_1)}, \tag{104}$$

where C_6 depends only on the norms of ζ, $\nabla\chi$ and u in $L^2(Q)$ and on the norm of $\mu_2\partial_t\chi_2^\mu$ in $L^2(Q_2)$, which are all bounded, and C_7 is C_6 times the norm of the operator R. We conclude, like before, that

$$\partial_t\chi_1^\mu \to \partial_t\chi_1 \quad \text{in } L^2(0,T;V_1')\text{-weak} \tag{105}$$

and, recalling (96), thanks again to the Aubin theorem,

$$\chi_1^\mu \to \chi_1 \quad \text{in } L^2(0,T;L^2(\Omega_1))\text{-strong,} \tag{106}$$

$$\chi_1^\mu \to \chi_1 \quad \text{in } C^0([0,T];L^2(\Omega_1))\text{-weak.} \tag{107}$$

This is enough to pass to the limit in (31) and get back (36) in the specified sense.

Now, equation (31), after an integration over $[0,T]$, and the addition and simultaneous subtraction of some terms, can be rewritten as

$$\begin{aligned} &{}_{L^2(0,T;V_1')}\langle \mu_1 \partial_t(\chi_1^\mu - \chi_1), v_1\rangle_{L^2(0,T;V_1)} + \mu_2 \int_0^T \int_{\Omega_2} \partial_t \chi_2^\mu v_2 \,\mathrm{d}x\,\mathrm{d}s \\ &\quad + \sum_{i=1}^2 \nu_i \int_0^T \int_{\Omega_i} \nabla(\chi_i^\mu - \chi_i)\cdot \nabla v_i \,\mathrm{d}x\,\mathrm{d}s + \sum_{i=1}^2 \int_0^T \int_{\Omega_i} \zeta_i^\mu v_i \,\mathrm{d}x\,\mathrm{d}s \\ &= -{}_{L^2(0,T;V_1')}\langle \mu_1 \partial_t \chi_1, v_1\rangle_{L^2(0,T;V_1)} - \sum_{i=1}^2 \nu_i \int_0^T \int_{\Omega_i} \nabla\chi_i \cdot \nabla v_i \,\mathrm{d}x\,\mathrm{d}s \\ &\quad + \lambda \int_0^T \int_\Omega (u^\mu - u) v \,\mathrm{d}x\,\mathrm{d}s + \lambda \int_0^T \int_\Omega uv \,\mathrm{d}x\,\mathrm{d}s, \end{aligned} \tag{108}$$

where, by virtue of (105), we are again allowed to take the test function v in $L^2(0,T;V)$. In particular, for $v = \chi^\mu - \chi$, thanks also to (107), we get

$$\begin{aligned} &\frac{\mu_1}{2}\|\chi_1^\mu(T) - \chi_1(T)\|^2_{L^2(\Omega_1)} + \frac{\mu_2}{2}\|\chi_2^\mu(T)\|^2_{L^2(\Omega_2)} \\ &\quad + \sum_{i=1}^2 \nu_i \|\nabla(\chi_i^\mu - \chi_i)\|^2_{L^2(0,T;L^2(\Omega_i))} \\ &= -\, {}_{L^2(0,T;V_1')}\langle \mu_1 \partial_t \chi_1, \chi_1^\mu - \chi_1\rangle_{L^2(0,T;V_1)} - \sum_{i=1}^2 \int_0^T \int_{\Omega_i} \zeta_i^\mu (\chi_i^\mu - \chi_i) \,\mathrm{d}x\,\mathrm{d}s \\ &\quad - \sum_{i=1}^2 \nu_i \int_0^T \int_{\Omega_i} \nabla\chi_i \cdot \nabla(\chi_i^\mu - \chi_i) \,\mathrm{d}x\,\mathrm{d}s \\ &\quad + \lambda \int_0^T \int_\Omega (u^\mu - u)(\chi^\mu - \chi) \,\mathrm{d}x\,\mathrm{d}s + \lambda \int_0^T \int_\Omega u(\chi^\mu - \chi) \,\mathrm{d}x\,\mathrm{d}s \\ &\quad + \frac{\mu_1}{2}\|\chi_{1,0}^\mu - \chi_{1,0}\|^2_{L^2(\Omega_1)} + \frac{\mu_2}{2}\|\chi_{2,0}^\mu\|^2_{L^2(\Omega_2)} + \mu_2 \int_0^T \int_{\Omega_2} \partial_t \chi_2^\mu \chi_2 \,\mathrm{d}x\,\mathrm{d}s. \end{aligned} \tag{109}$$

Let us now set, for $\psi \in L^2(Q_2)$,

$$\Phi_2^T(\psi) := \begin{cases} \displaystyle\int_0^T \int_{\Omega_2} \varphi_2(\psi) \,\mathrm{d}x\,\mathrm{d}s & \text{if } \varphi_2(\psi) \in L^1(Q_2) \\ +\infty & \text{otherwise,} \end{cases} \tag{110}$$

so that, thanks to (32), we have

$$\zeta_2^\mu \in \partial \Phi_2^T(\chi_2^\mu) \quad \text{in } L^2(0,T;L^2(\Omega_2)) \tag{111}$$

and, by the definition of subdifferential,

$$I_2 := \int_0^T \int_{\Omega_2} \zeta_2^\mu (\chi_2 - \chi_2^\mu)\, \mathrm{d}x\, \mathrm{d}s \leqslant \Phi_2^T(\chi_2) - \Phi_2^T(\chi_2^\mu). \tag{112}$$

Adding now m times (102), with $m > 0$ to be chosen later, to relation (109), recalling the convergences (50), (96), (100) and (106), the definition (110) and the hypotheses (23)–(24), we easily get

$$\begin{aligned}
& m\|u^\mu(T) - u(T)\|_H^2 + m\|\nabla(u^\mu - u)\|_{L^2(0,T;H)}^2 \\
& \quad + \frac{\mu_1}{2}\|\chi_1^\mu(T) - \chi_1(T)\|_{L^2(\Omega_1)}^2 + \frac{\mu_2}{2}\|\chi_2^\mu(T)\|_{L^2(\Omega_2)}^2 \\
& \quad + \sum_{i=1}^2 \nu_i \|\nabla(\chi_i^\mu - \chi_i)\|_{L^2(0,T;L^2(\Omega_i))}^2 \\
& \leqslant m\lambda^2 \|\nabla(\chi^\mu - \chi)\|_{L^2(0,T;H)}^2 + I_2 + R_2^\mu,
\end{aligned} \tag{113}$$

where R_2^μ is a new sequence collecting mR_1^μ and all the infinitesimal terms on the right hand side of (109). Recalling now (112), due to the convexity and lower semicontinuity of Φ_2^T and (96) we conclude that $\limsup_{\mu_2 \to 0} I_2 \leqslant 0$. Now, if we take $m = \min_{i=1,2}\{\nu_i\}/2\lambda^2$ in (113), also the V-norm of $\chi^\mu - \chi$ is controlled, so that we immediately get (46) and

$$\|\nabla\chi^\mu - \nabla\chi\|_{L^2(0,T;H)} \to 0,$$

whence we deduce (47) by recalling (106) and the Poincaré inequality (67); furthermore, relation (37) is, as before, a consequence of (47), (48) and [2], Prop. 1.1, page 42.

Recalling (98) and (105), we get also the regularities (33)–(34); furthermore, proceeding as for the derivation of (95), we get that

$$D\chi_2^\mu \to D\chi_2 \quad \text{in } L^2(Q_2)\text{-weak},$$

so that (51) holds along with the Neumann boundary condition (9) (only on Γ_2; for the definitions of the trace spaces, look back at Section 2) in the specified sense.

This concludes the proof of Theorem 2. □

P r o o f o f T h e o r e m 3. Also this proof can be performed simultaneously for both choices of the boundary conditions. Let us suppose to have a couple of solutions $(\hat{u}, \hat{\chi}, \hat{\zeta})$, $(\check{u}, \check{\chi}, \check{\zeta})$ to problem (wP), satisfying also the initial conditions (52)–(53). Set $u := \hat{u} - \check{u}$, $\chi := \hat{\chi} - \check{\chi}$, $\zeta := \hat{\zeta} - \check{\zeta}$.

Substitute first $(\hat{u}, \hat{\chi})$ and then $(\check{u}, \check{\chi})$ in equation (35), take the difference and test with $v = 2u$, easily obtaining for every $t \in [0, T]$ the inequality

$$\|u(t)\|_H^2 + \|\nabla u\|_{L^2(0,t;H)}^2 \leqslant \lambda^2 \|\nabla \chi\|_{L^2(0,t;H)}^2. \tag{114}$$

Applying a similar procedure to equation (36) and taking into account the monotonicity of the α_i, we get, for $\sigma > 0$ and for every $t \in [0, T]$,

$$\frac{\mu_1}{2} \|\chi_1(t)\|_{L^2(\Omega_1)}^2 + \sum_{i=1}^{2} \nu_i \|\nabla \chi_i\|_{L^2(0,t;L^2(\Omega_i))}^2 \leqslant \frac{\lambda^2}{4\sigma} \|u\|_{L^2(0,t;H)}^2 + \sigma \|\chi\|_{L^2(0,t;H)}^2. \tag{115}$$

Now, take the sum of m times (114) with (115), with $m > 0$, as usual, to be chosen; exploiting (67), we derive

$$\begin{aligned} &\frac{\mu_1}{2} \|\chi_1(t)\|_{L^2(\Omega_1)}^2 + \sum_{i=1}^{2} \nu_i \|\nabla \chi_i\|_{L^2(0,t;L^2(\Omega_i))}^2 + m\|u(t)\|_H^2 + m\|\nabla u\|_{L^2(0,t;H)}^2 \\ &\quad \leqslant \frac{\lambda^2}{4\sigma} \|u\|_{L^2(0,t;H)}^2 + \sigma C_\Omega \|\nabla \chi\|_{L^2(0,t;H)}^2 + m\lambda^2 \|\nabla \chi\|_{L^2(0,t;H)}^2 \\ &\qquad + \sigma(1 + C_\Omega) \|\chi_1\|_{L^2(0,t;L^2(\Omega_1))}^2. \end{aligned}$$

At this point, if we take, for instance,

$$m = \frac{\min\limits_{i=1,2} \{\nu_i\}}{3\lambda^2}, \quad \sigma = \frac{\min\limits_{i=1,2} \{\nu_i\}}{3C_\Omega}, \tag{116}$$

an application of the Gronwall lemma allows us to complete the proof. □

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