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A NOTE ON THE GENERALIZED ENERGY INEQUALITY
IN THE NAVIER-STOKES EQUATIONS*

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Abstract. We prove that there exists a suitable weak solution of the Navier-Stokes equation, which satisfies the generalized energy inequality for every nonnegative test function. This improves the famous result on existence of a suitable weak solution which satisfies this inequality for smooth nonnegative test functions with compact support in the space-time.

Keywords: Navier-Stokes equations, suitable weak solution, generalized energy inequality

MSC 2000: 35Q35, 35Q30

Suppose that we solve the Navier-Stokes equations for sufficiently smooth data. It is proved in [1] that then there exists a suitable weak solution of these equations, i.e. the generalized energy inequality holds for the smooth test functions with compact support in the space-time domain. The goal of this paper is to describe briefly the construction of a suitable weak solution which satisfies the above mentioned generalized energy inequality for *every* smooth test function. A similar result is also mentioned in [6], but it is not proved there.

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with $\mathcal{C}^{2+\mu}$ boundary $\partial\Omega$ ($\mu > 0$), $T > 0$, $Q_T = \Omega \times (0, T)$. The classical formulation of the Navier-Stokes initial-boundary value problem for a viscous incompressible fluid can be written as

$$(1) \quad \frac{\partial u}{\partial t} - \nu \cdot \Delta u + (u \cdot \nabla)u + \nabla \mathcal{P} = f \quad \text{in } Q_T,$$

$$(2) \quad \nabla \cdot u = 0 \quad \text{in } Q_T,$$

$$(3) \quad u(\cdot, 0) = u_0,$$

$$(4) \quad u = 0 \quad \text{on } \partial\Omega \times (0, T),$$

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where $u = (u_1, u_2, u_3)$ and \mathcal{P} denote the velocity and the pressure, $\nu > 0$ is the viscosity coefficient and f is an external body force. Throughout the paper we suppose that $\nu = 1$.

Let $\mathcal{V} = \mathcal{C}_0^\infty(\Omega, \mathbb{R}^3) \cap \{v; \nabla \cdot v = 0\}$. As is usual in mathematical literature, H and V , respectively, denote the closures of \mathcal{V} in the norms of $[L^2(\Omega)]^3$ and $[W_0^{1,2}(\Omega)]^3$. Denote further $A = -P_H \Delta$, where P_H is the Helmholtz projection from $[L^2(\Omega)]^3$ onto H . Then $\mathcal{D}(A) = [W^{2,2}(\Omega)]^3 \cap V$ is a Banach space with the norm $\|\cdot\|_{\mathcal{D}(A)}$ which is equivalent to the norm $\|\cdot\|_{[W^{2,2}(\Omega)]^3}$.

We write $L^{\alpha,\beta}$ instead of $L^\alpha(0, T; L^\beta(\Omega))$ and $\|\cdot\|_{L^\infty, 2 \cap L^{2,6}}$ is the sum

$$\|\cdot\|_{L^\infty(0, T; L^2(\Omega))} + \|\cdot\|_{L^2(0, T; L^6(\Omega))}.$$

Throughout the paper we suppose that the following conditions are satisfied:

$$(5) \quad f \in L^2(0, T; H)$$

and for simplicity,

$$(6) \quad u_0 \in \mathcal{D}(A).$$

The suitable weak solution in [1] is defined as follows.

Definition. Let

$$(7) \quad f \in L^q(Q_T), \quad q > 5/2,$$

$$(8) \quad \nabla \cdot f = 0,$$

$$(9) \quad u_0 \in H.$$

The pair (u, \mathcal{P}) is called a suitable weak solution of (1)–(4) if

$$(10) \quad u \in L^2(0, T; V) \cap L^\infty(0, T; H),$$

$$(11) \quad u(t) \rightarrow u_0 \text{ weakly in } H \text{ for } t \rightarrow 0,$$

$$(12) \quad \mathcal{P} \in L^{\frac{5}{4}}(Q_T),$$

$$(13) \quad \text{equation (1) holds in the sense of distributions in } Q_T$$

and

$$(14) \quad \begin{aligned} & \int_{\Omega \times \{t\}} |u|^2 \varphi + 2 \int_0^t \int_\Omega |\nabla u|^2 \varphi \\ & \leq \int_{\Omega \times \{0\}} |u_0|^2 \varphi + \int_0^t \int_\Omega |u|^2 \left(\frac{\partial \varphi}{\partial t} + \Delta \varphi \right) \\ & \quad + \int_0^t \int_\Omega (|u|^2 u + 2\mathcal{P}u) \cdot \nabla \varphi + 2 \int_0^t \int_\Omega f u \varphi \end{aligned}$$

for every $\varphi \in \mathcal{C}^\infty(\overline{Q_T})$, $\varphi \geq 0$, $\varphi = 0$ in a neighbourhood of $\partial\Omega \times (0, T)$ and for every $t \in (0, T)$.

The following result is proved in [1].

Theorem 1. *Let (7) and (8) hold and $u_0 \in H \cap W_0^{\frac{2}{5}, \frac{5}{4}}(\Omega)$. Then there exists a suitable weak solution of (1)–(4).*

The following theorem is the main result of this paper.

Theorem 2. *Let (5)–(6) hold. Then there exists a suitable weak solution (u, \mathcal{P}) of (1)–(4). Furthermore,*

(15) *the function $u: [0, T] \rightarrow H$ is weakly continuous,*

(16) *$\mathcal{P} \in L^{r, s^*}$,*

where

$$(17) \quad \frac{2}{r} + \frac{3}{s^*} = 3, \quad 1 < r < 2, \quad \frac{3}{2} < s^* < 3$$

and

$$(18) \quad \int_{\Omega \times \{t_2\}} |u|^2 \varphi + 2 \int_{t_1}^{t_2} \int_{\Omega} |\nabla u|^2 \varphi \\ \leq \int_{\Omega \times \{t_1\}} |u|^2 \varphi + \int_{t_1}^{t_2} \int_{\Omega} |u|^2 \left(\frac{\partial \varphi}{\partial t} + \Delta \varphi \right) \\ + \int_{t_1}^{t_2} \int_{\Omega} (|u|^2 u + 2\mathcal{P}u) \cdot \nabla \varphi + 2 \int_{t_1}^{t_2} \int_{\Omega} f u \varphi$$

for every $\varphi \in \mathcal{C}^\infty(\overline{Q_T})$, $\varphi \geq 0$, for almost every $t_1 \in [0, T]$ and every $t_2 \in [0, T]$, $t_1 < t_2$. Moreover, (18) holds for $t_1 = 0$.

First, we present a few lemmas. The first lemma is proved in [5].

Lemma 1. *If $g \in L^{\infty, 2} \cap L^{2, 6}$ and $\alpha \in [2, \infty]$, $\beta \in [2, 6]$, $\frac{2}{\alpha} + \frac{3}{\beta} \geq \frac{3}{2}$, then*

$$\|g\|_{L^{\alpha, \beta}} \leq c \|g\|_{L^{2, 2}}^{\frac{2}{\alpha} + \frac{3}{\beta} - \frac{3}{2}} \|g\|_{L^{\infty, 2} \cap L^{2, 6}}^{\frac{5}{2} - (\frac{2}{\alpha} + \frac{3}{\beta})}$$

where $c = c(\Omega)$.

The following lemma is an immediate consequence of Lemma 1.

Lemma 2. If $g \in L^{\infty,2} \cap L^{2,6}$, $p \in [2, \infty]$, $q \in [2, 6]$ and

$$(19) \quad \frac{2}{p} + \frac{3}{q} = \frac{3}{2},$$

then

$$(20) \quad \|g\|_{L^{p,q}} \leq c \|g\|_{L^{\infty,2} \cap L^{2,6}},$$

where $c = c(\Omega)$.

Lemma 3. Let $\psi \in L^2(0, T; \mathcal{D}(A))$, $\psi' \in L^2(0, T; H)$, (5) hold and $w_0 \in V$. Then there exists a unique solution (w, \mathcal{Q}) of the problem

$$(21) \quad w' - \Delta w + \psi \nabla w + \nabla \mathcal{Q} = f,$$

$$(22) \quad w(0) = w_0,$$

where $w \in L^2(0, T; \mathcal{D}(A)) \cap L^\infty(0, T; V)$, $w' \in L^2(0, T; H)$, $\nabla \mathcal{Q} \in L^2(0, T; L^2(\Omega)^3)$, $\mathcal{Q} \in L^{2,6}$, $\int_\Omega \mathcal{Q} = 0$ for almost every t and

$$(23) \quad \|w\|_{L^2(0,T;V)} + \|w\|_{L^\infty(0,T;H)} \leq c_1 \cdot (\|f\|_{L^2(0,T;H)} + \|w_0\|_H).$$

Moreover, if $w_0 \in \mathcal{D}(A)$, then

$$(24) \quad \|\mathcal{Q}\|_{L^{r,s^*}} + \|\nabla \mathcal{Q}\|_{L^{r,s}} \leq c_2 \cdot (\|f\|_{L^2(0,T;H)} + \|w_0\|_{\mathcal{D}(A)}) (\|\psi\|_{L^{\infty,2} \cap L^{2,6}} + 1),$$

where r, s^* satisfy (17), $c_2 = c_2(\Omega)$,

$$(25) \quad \frac{2}{r} + \frac{3}{s} = 4, \quad 1 < s < \frac{3}{2}$$

and c_1, c_2 do not depend on ψ .

Proof. Using ([7], Proposition 2) we get that there exists a unique w , $w \in L^2(0, T; \mathcal{D}(A)) \cap L^\infty(0, T; V)$, $w' \in L^2(0, T; H)$, which is a weak solution of (21), (22). Consequently, $\nabla \mathcal{Q} \in L^2(0, T; L^2(\Omega))$, $\mathcal{Q} \in L^2(0, T; L^6(\Omega))$ and w, \mathcal{Q} solve (21), (22). Multiplying (21) by w , we get

$$\|w(t)\|_H^2 + 2 \int_s^t \|\nabla w\|_{L^2(\Omega)}^2 \leq \|w(s)\|_H^2 + 2 \int_s^t \|f\|_H \|w\|_H$$

for every $0 \leq s < t \leq T$. Set $s = 0$. Then

$$\|w(t)\|_H^2 + 2 \int_0^t \|\nabla w\|_{L^2(\Omega)}^2 \leq \|w(0)\|_H^2 + 2 \int_0^t \|f\|_H \|w\|_H$$

for every $t \in [0, T]$. The last inequality implies

$$\|w(t)\|_H^2 + \int_0^T \|\nabla w\|_{L^2(\Omega)}^2 \leq c_1 \cdot (\|w_0\|_H^2 + \int_0^T \|f\|_H^2)$$

and (23) follows immediately.

Using the inequality

$$\|\psi \nabla w\|_{L^{r,s}} \leq c \|\nabla w\|_{L^{2,2}} \|\psi\|_{L^{p,q}} \leq c \|\nabla w\|_{L^{2,2}} \|\psi\|_{L^\infty \cap L^{2,6}},$$

where $p = \frac{2r}{2-r}$, $q = \frac{2s}{2-s}$ and r, s satisfy (17) and (25), it is possible to see that also $f - \psi \nabla w \in L^{r,s}$ and

$$\|f - \psi \nabla w\|_{L^{r,s}} \leq c(\|f\|_{L^2(0,T;H)} + c \|\nabla w\|_{L^{2,2}} \|\varphi\|_{L^\infty \cap L^{2,6}}).$$

By virtue of (23), inequality (24) for $\|\nabla \mathcal{Q}\|_{L^{r,s}}$ now follows from the famous $L^r - L^s$ estimates for the Stokes equations (see [2]). Using now the fact that $\int_\Omega \mathcal{Q} = 0$, we get (24) for $\|\mathcal{Q}\|_{L^{r,s^*}}$, where r, s^* satisfy (17). \square

Lemma 4. *The unique solution (w, \mathcal{Q}) obtained in Lemma 3 satisfies the generalized energy equality for every $\varphi \in \mathcal{C}^\infty(\overline{Q_T})$, $\varphi \geq 0$ and every $t_1, t_2 \in [0, T]$, $t_1 < t_2$:*

$$\begin{aligned} (26) \quad & \int_{\Omega \times \{t_2\}} |w|^2 \varphi + 2 \int_{t_1}^{t_2} \int_\Omega |\nabla w|^2 \varphi \\ &= \int_{\Omega \times \{t_1\}} |w|^2 \varphi + \int_{t_1}^{t_2} \int_\Omega |w|^2 \left(\frac{\partial \varphi}{\partial t} + \Delta \varphi \right) \\ & \quad + \int_{t_1}^{t_2} \int_\Omega (|w|^2 \cdot \psi + 2\mathcal{Q}w) \cdot \nabla \varphi + 2 \int_{t_1}^{t_2} \int_\Omega f w \varphi. \end{aligned}$$

Proof. Multiplying (21) by $2w\varphi$ and integrating by parts, we get (26). \square

Definition. Let $n \in \mathbb{N}$, $\delta_n = T/n$ and $\varphi \in \mathcal{C}(0, T; H)$. Define

$$(27) \quad \Psi_n(\varphi)(t) = \begin{cases} \varphi(0) & \text{for } t \in (0, \delta_n), \\ \varphi(t - \delta_n) & \text{for } t \in [\delta_n, T]. \end{cases}$$

Clearly, $\Psi_n(\varphi) \in \mathcal{C}(0, T; H)$.

Let (w_n, \mathcal{Q}_n) be the solution of the problem

$$(28) \quad w'_n - \Delta w_n + \Psi_n(w_n) \nabla w_n + \nabla \mathcal{Q}_n = f,$$

$$(29) \quad w_n(0) = u_0.$$

It is possible to see that $w_n, \Psi_n(w_n) \in L^2(0, \delta_n; \mathcal{D}(A))$ and by applying Lemma 3 inductively on each time interval $(k\delta_n, (k+1)\delta_n)$, $k = 1, \dots, n-1$, we get that $w_n, \Psi_n(w_n) \in L^2(0, T; \mathcal{D}(A))$. Using (23) and (27), we get that for sufficiently big K

$$(30) \quad \|w_n\|_{L^2(0, T; V)}, \|w_n\|_{L^\infty(0, T; H)}, \|\Psi_n(w_n)\|_{L^2(0, T; V)} \leq K,$$

and using (24) and (30), we obtain

$$(31) \quad \|\mathcal{Q}/\mathcal{R}\|_{L^{r, s^*}}, \|\nabla \mathcal{Q}_n\|_{L^{r, s}} \leq K.$$

Further, using Lemma 3, we get from (27) and (28) that

$$(32) \quad \|w'_n\|_{L^{\frac{4}{3}}(0, T; V^*)}, \|\Psi_n(w_n)'\|_{L^{\frac{4}{3}}(0, T; V^*)} \leq K.$$

Note that K does not depend on n and r, s, s^* satisfy (17) and (25). Using (30), (32) and ([8], Theorem 2.1 in Chapter III), we come to the conclusion that

$$(33) \quad w_n \text{ stay in a compact subset of } L^2(0, T; H).$$

Therefore, there exist $u, u^* \in L^2(0, T; V) \cap L^\infty(0, T; H)$ and $\{w_{n_k}\} \subset \{w_n\}$, $\{\Psi_{n_k}(w_{n_k})\} \subset \{\Psi_n(w_n)\}$ (for simplicity we will use $\{w_{n_k}\} = \{w_n\}$, $\{\Psi_{n_k}(w_{n_k})\} = \{\Psi_n(w_n)\}$) such that

$$(34) \quad w_n \rightarrow u \quad \text{weakly in } L^2(0, T; V),$$

$$(35) \quad w_n \rightarrow u \quad \text{strongly in } L^2(0, T; H),$$

$$(36) \quad \Psi_n(w_n) \rightarrow u^* \quad \text{weakly in } L^2(0, T; V),$$

$$(37) \quad \Psi_n(w_n) \rightarrow u^* \quad \text{strongly in } L^2(0, T; H).$$

Lemma 5. *The following equality holds:*

$$(38) \quad u = u^*.$$

Proof. We know (see (35), (37)) that $\{w_n\}, \{\Psi_n(w_n)\}$ are relatively compact sets in $L^2(0, T; H)$. Using ([4], Theorem 2.13.1, condition (ii)), we get that w_n^k are

2-mean equicontinuous (w_n^k is the k -th component of w_n , $k = 1, 2, 3$). It means that for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every $h \in \mathbb{R}$, $|h| < \delta$,

$$\int_{Q_T} |w_n^k(x, t+h) - w_n^k(x, t)|^2 < \varepsilon^2.$$

(If necessary, w_n are defined by zero outside Q_T .) It follows from the last inequality and from (27) that $u = u^*$.

It follows immediately from Lemma 1, (34) and (35) that

$$(39) \quad w_n \rightarrow w \text{ strongly in } L^{\alpha, \beta}, \text{ where } \frac{2}{\alpha} + \frac{3}{\beta} > \frac{3}{2}, \quad \alpha \in (2, \infty), \quad \beta \in (2, 6).$$

By virtue of (31), there exists $\mathcal{P} \in L^{r, s^*}$, $\nabla \mathcal{P} \in L^{r, s}$ such that

$$(40) \quad \mathcal{Q}_n \rightarrow \mathcal{P} \text{ weakly in } L^{r, s^*}$$

and

$$(41) \quad \nabla \mathcal{Q}_n \rightarrow \nabla \mathcal{P} \text{ weakly in } L^{r, s}.$$

Applying Lemma 4 to (28), (29), we obtain the generalized energy equality for (w_n, \mathcal{Q}_n) :

$$(42) \quad \begin{aligned} & \int_{\Omega \times \{t_2\}} |w_n|^2 \varphi + 2 \int_{t_1}^{t_2} \int_{\Omega} |\nabla w_n|^2 \varphi \\ &= \int_{\Omega \times \{t_1\}} |w_n|^2 \varphi + \int_{t_1}^{t_2} \int_{\Omega} |w_n|^2 \left(\frac{\partial \varphi}{\partial t} + \Delta \varphi \right) \\ & \quad + \int_{t_1}^{t_2} \int_{\Omega} (|w_n|^2 \Psi(w_n) + 2 \mathcal{Q}_n w_n) \cdot \nabla \varphi + 2 \int_{t_1}^{t_2} \int_{\Omega} f w_n \varphi. \end{aligned}$$

P r o o f of Theorem 2. To prove that $u = \lim w_n$ satisfies (10), (13) and (15) we proceed similarly as in ([8], Theorem 3.1 in Chapter III). Further, $\mathcal{P} \in L^{r, s^*}$ is the associated pressure of the weak solution u . Using (35) we get that

$$(43) \quad \|w_n(t)\|_H \rightarrow \|u(t)\|_H$$

for almost every $t \in [0, T]$. The relation (34) implies that

$$\int_{t_1}^{t_2} \int_{\Omega} |\nabla w_n|^2 \geq \int_{t_1}^{t_2} \int_{\Omega} |\nabla u|^2$$

for every $t_1, t_2 \in [0, T], t_1 < t_2$. It is possible to prove (note that $\varphi \geq 0$ on Q_T) that also

$$(44) \quad \liminf \int_{t_1}^{t_2} \int_{\Omega} |\nabla w_n|^2 \varphi \geq \int_{t_1}^{t_2} \int_{\Omega} |\nabla u|^2 \varphi$$

for every $t_1, t_2, 0 \leq t_1 \leq t_2 \leq T$. Applying (34)–(40), (43) and (44) to (42), we get also

$$(45) \quad \begin{aligned} \int_{\Omega \times \{t_2\}} |u|^2 \varphi + 2 \int_{t_1}^{t_2} \int_{\Omega} |\nabla u|^2 \varphi \\ \leq \int_{\Omega \times \{t_1\}} |u|^2 \varphi + \int_{t_1}^{t_2} \int_{\Omega} |u|^2 \left(\frac{\partial \varphi}{\partial t} + \Delta \varphi \right) \\ + \int_{t_1}^{t_2} \int_{\Omega} (|u|^2 u + 2 \mathcal{P}u) \cdot \nabla \varphi + 2 \int_{t_1}^{t_2} \int_{\Omega} f u \varphi \end{aligned}$$

for almost every $t_1, t_2 \in [0, T], t_1 < t_2$. Using the fact that the function

$$t \in [0, T] \rightarrow \int_{\Omega \times \{t\}} |u|^2 \varphi$$

is semi-lower continuous we get (45) for almost every $t_1 \in [0, T]$ and for every $t_2 \in [0, T], t_1 < t_2$. Moreover, as (43) holds for $t = 0$, (45) is also satisfied for $t_1 = 0$. Theorem 2 is proved. \square

Remark 1. Estimate (24) is important for the proof of Theorem 2. We prove (24) using $L^r - L^s$ estimates published in [2]. To fulfil all assumptions necessary to apply the results of [2] and for the sake of simplicity, we suppose a smooth initial condition (6). Note that it is possible to suppose a more general initial condition (see [2], relation (2.5)).

Remark 2. The general energy inequality proved in [1] was used for the study of the set of internal singular points of a suitable weak solution. Analogously, using the general energy inequality proved in this paper may be useful in the study of boundary singular points.

Remark 3. Notice that we get the classical energy inequality

$$(46) \quad \int_{\Omega \times \{t_2\}} |u|^2 + 2 \int_{t_1}^{t_2} \int_{\Omega} |\nabla u|^2 \leq \int_{\Omega \times \{t_1\}} |u|^2 + 2 \int_{t_1}^{t_2} \int_{\Omega} f u$$

if we suppose $\varphi \equiv 1$ in (45).

R e m a r k 4. A draft version of this paper was published in [3]. The present main theorem is stronger than that of [3].

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