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NONSINGULARITY, POSITIVE DEFINITENESS, AND POSITIVE INVERTIBILITY UNDER FIXED-POINT DATA ROUNDING

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Abstract. For a real square matrix A and an integer $d \geqslant 0$, let $A_{(d)}$ denote the matrix formed from A by rounding off all its coefficients to d decimal places. The main problem handled in this paper is the following: assuming that $A_{(d)}$ has some property, under what additional condition(s) can we be sure that the original matrix A possesses the same property? Three properties are investigated: nonsingularity, positive definiteness, and positive invertibility. In all three cases it is shown that there exists a real number $\alpha(d)$, computed solely from $A_{(d)}$ (not from A), such that the following alternative holds:

- if $d > \alpha(d)$, then nonsingularity (positive definiteness, positive invertibility) of $A_{(d)}$ implies the same property for A;
- if $d < \alpha(d)$ and $A_{(d)}$ is nonsingular (positive definite, positive invertible), then there exists a matrix A' with $A'_{(d)} = A_{(d)}$ which does *not* have the respective property.

For nonsingularity and positive definiteness the formula for $\alpha(d)$ is the same and involves computation of the NP-hard norm $\|\cdot\|_{\infty,1}$; for positive invertibility $\alpha(d)$ is given by an easily computable formula.

Keywords: nonsingularity, positive definiteness, positive invertibility, fixed-point rounding

MSC 2000: 15A12, 15A48, 65G40, 65G50

1. Fixed-point rounding

For a real number a and a nonnegative integer d define

$$a_{(d)} = \begin{cases} \lfloor 10^d a + 0.5 \rfloor 10^{-d} & \text{if } a \geqslant 0, \\ -(-a)_{(d)} & \text{if } a < 0, \end{cases} \tag{1}$$

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where

$$\lfloor b \rfloor = \max\{c\colon\ c \leqslant b,\ c \text{ integer}\}.$$

It is obvious that $a_{(d)}$ is the result of rounding a to d decimal places. The following two properties are almost straightforward, but we include them for the sake of completeness because of their repeated use in the sequel. Throughout the paper we denote

$$\delta = 0.5 \cdot 10^{-d}. \tag{2}$$

Proposition 1. *If $a \in \mathbb{R}$ and d is a nonnegative integer, then*

$$a_{(d)} - \delta \leqslant a \leqslant a_{(d)} + \delta. \tag{3}$$

P r o o f. Let $a \geqslant 0$. Then (1) implies that

$$10^d a_{(d)} = \lfloor 10^d a + 0.5 \rfloor,$$

thus $10^d a_{(d)}$ is the integer part of $10^d a + 0.5$, hence

$$10^d a_{(d)} \leqslant 10^d a + 0.5 < 10^d a_{(d)} + 1,$$

which gives

$$a_{(d)} - 0.5 \cdot 10^{-d} \leqslant a < a_{(d)} + 0.5 \cdot 10^{-d},$$

and this in view of (2) means that

$$a_{(d)} - \delta \leqslant a < a_{(d)} + \delta. \tag{4}$$

If $a < 0$, then the inequality (4) holds for $-a$, hence

$$(-a)_{(d)} - \delta \leqslant -a < (-a)_{(d)} + \delta$$

and in virtue of (1) we obtain

$$a_{(d)} - \delta < a \leqslant a_{(d)} + \delta. \tag{5}$$

Hence in both cases (4), (5) we have (3). $\square$

Proposition 2. *If $a \in \mathbb{R}$ and d is a nonnegative integer, then each b with*

$$a_{(d)} - \delta < b < a_{(d)} + \delta \tag{6}$$

satisfies

$$b_{(d)} = a_{(d)}.$$

Proof. From (6) it follows that

$$10^d a_{(d)} < 10^d b + 0.5 < 10^d a_{(d)} + 1,$$

and since $10^d a_{(d)}$ is integer due to (1), this implies that

$$10^d a_{(d)} = \lfloor 10^d b + 0.5 \rfloor$$

and

$$a_{(d)} = \lfloor 10^d b + 0.5 \rfloor 10^{-d}.$$

Hence, if $b \geqslant 0$, then $a_{(d)} = b_{(d)}$ due to (1). If $b < 0$, then the result just proved gives $(-a)_{(d)} = (-b)_{(d)}$, hence again $a_{(d)} = b_{(d)}$ by (1). □

Now, let $A = (a_{ij})$ be a square matrix (we shall consider only square matrices in the sequel). We define

$$A_{(d)} = ((a_{ij})_{(d)}),$$

hence the matrix $A_{(d)}$ arises from A by rounding off all its coefficients to d decimal places. The main question handled in this paper is the following: assume a real matrix A is not exactly known and we have only its rounded value $A_{(d)}$ at our disposal; if $A_{(d)}$ has some property, under what additional condition(s) can we be sure that the original matrix A possesses this property as well? We shall give answers for the cases of three common properties, namely, nonsingularity, positive definiteness, and positive invertibility. In the case of nonsingularity we shall show in Theorem 5 that there exists a real number α computed from $A_{(d)}^{-1}$ such that if $d > \alpha$, then nonsingularity of $A_{(d)}$ implies nonsingularity of A, and if $d < \alpha$, then there exists a singular matrix A' satisfying $A'_{(d)} = A_{(d)}$; hence, in the former case we are done, whereas in the latter we learn that the original matrix cannot be distinguished, by means of rounding to d decimal places, from a singular matrix. In Theorem 9 we shall show that literally the same result (with the same α) holds for positive definiteness. Both theorems handle the cases $d > \alpha$ and $d < \alpha$ only, but the remaining case $d = \alpha$ occurs with probability 0 because d is integer whereas α is a real number.

As will be shown, in both theorems α is given by

$$\alpha = \log_{10}(0.5 \cdot \|A_{(d)}^{-1}\|_{\infty,1}). \tag{7}$$

This draws our attention to the very infrequent norm $\|\cdot\|_{\infty,1}$ which, in fact, had not been studied until its unexpected applications arose in interval analysis ([3], [9]). In Section 2 we therefore briefly state the basic properties of this norm, the above main results for nonsingularity and positive definiteness being the matter of Sections 3 and 4. In the last Section 5 we handle the case of positive invertibility. The result (Theorem 12) is again formulated in the form of alternatives ($d > \beta$ or $d < \beta$), but this time β does not involve computation of the norm $\|\cdot\|_{\infty,1}$ and both inequalities are easily verifiable.

Finally, we should respond to a potential question "why fixed-point rounding?". The problem of nonsingularity under floating-point data rounding was addressed in our paper [7]. As the reader may check there, also in this case there was a basic distinction between two cases ($d > \gamma$ and $d < \gamma$), but the formula for γ was too cumbersome even to formulate, the more to evaluate it (only nonsingularity was handled in [7], but the result carries over to positive definiteness and positive invertibility as well in the same way as it is done here). Therefore the fixed-point case handled in this paper is not only more elegant, but also perhaps more apt to use. Nevertheless, as we shall see, not only in the floating-point case, but also in the fixed-point one, an inherent exponentiality is hidden behind the formulae derived, which shows that in both cases extracting properties of the original real matrix from the properties of the rounded matrix is far more computationally difficult a task than it might perhaps be expected.

For clarity, decimal rounding is handled throughout this paper. The results for the case of binary rounding (which is of basic importance in computer arithmetic) can be derived from those contained here simply by replacing "10" by "2". In this way the formula (7) takes on the form

$$\alpha = \log_2(0.5 \cdot \|A_{(d)}^{-1}\|_{\infty,1}) = \log_2 \|A_{(d)}^{-1}\|_{\infty,1} - 1,$$

and similarly for the formulae (21), (22) and (23) of Theorem 12.

2. The norm $\|A\|_{\infty,1}$

The norm $\|A\|_{\infty,1}$ is defined by the usual formula for subordinate matrix norms (see Higham [2] or Golub and van Loan [1]) by

$$\|A\|_{\infty,1} = \max_{\|x\|_\infty=1} \|Ax\|_1,$$

where $\|x\|_\infty = \max_i |x_i|$ and $\|x\|_1 = \sum_i |x_i|$. As proved in [9], the norm $\|A\|_{\infty,1}$ can be expressed by finite closed-form formulae as

$$\|A\|_{\infty,1} = \max_{z,y\in Y} z^T Ay = \max_{y\in Y} \|Ay\|_1, \tag{8}$$

where $Y = \{y \in \mathbb{R}^n : \ y_j \in \{-1, 1\}$ for each $j\}$ is the set of all ±1-vectors in $\mathbb{R}^n$. Since the cardinality of the set Y is 2^n, both the formulae in (8) involve an exponential number of operations. And indeed, the following proposition proved in [9] (building on a result from Poljak and Rohn [4]) shows that unless P=NP holds (which is nowadays expected not to be the case), exponentiality cannot be removed from computation of $\|A\|_{\infty,1}$:

Proposition 3. *The following problem is* NP-*complete:*
Instance. *A symmetric rational M-matrix A.*
Question. *Is $\|A\|_{\infty,1} \geqslant 1$?*

Thus even for M-matrices (i.e., matrices $A = (a_{ij})$ satisfying $A^{-1} \geqslant 0$ and $a_{ij} \leqslant 0$ for $i \neq j$) it is hard to check whether $\|A\|_{\infty,1} \geqslant 1$ holds, hence it is even harder to compute the value of $\|A\|_{\infty,1}$ itself. This preliminary result sheds bad light on the results using $\|A\|_{\infty,1}$ to follow, but unfortunately such is the nature of the problems under consideration.

3. Nonsingularity

In this section we shall address the question of condition(s) under which nonsingularity of $A_{(d)}$ would imply nonsingularity of A. It follows from Proposition 1 that for $A \in \mathbb{R}^n$ and an integer $d \geqslant 0$ we have

$$A_{(d)} - \delta ee^T \leqslant A \leqslant A_{(d)} + \delta ee^T, \tag{9}$$

where δ is given by (2) and $e = (1, 1, \ldots, 1)^T \in \mathbb{R}^n$. We can also write (9) as

$$A \in [A_{(d)} - \delta ee^T, A_{(d)} + \delta ee^T],$$

where we have employed the common notation $[\underline{A}, \bar{A}] = \{A\colon \underline{A} \leqslant A \leqslant \bar{A}\}$, the set $[\underline{A}, \bar{A}]$ being called an interval matrix. The connection of our problem with the matrix norm $\|\cdot\|_{\infty,1}$ defined in Section 2 will become clear from the following result:

Proposition 4. *Each matrix contained in an interval matrix $[A_{(d)} - \delta ee^T, A_{(d)} + \delta ee^T]$ is nonsingular if and only if*

$$\delta\|A_{(d)}^{-1}\|_{\infty,1} < 1 \tag{10}$$

holds.

P r o o f. In [6, Theorem 5.2] it is proved that an interval matrix $[A_c - pq^T, A_c + pq^T]$, where $A_c \in \mathbb{R}^{n\times n}$ and $p, q \in \mathbb{R}^n$ are nonnegative (column) vectors, consists of nonsingular matrices if and only if

$$\max_{z,y\in Y} z^T \operatorname{diag}(q) A_c^{-1} \operatorname{diag}(p) y < 1$$

holds (diag(p) denotes the diagonal matrix with the diagonal vector p), which in view of (8) is equivalent to

$$\|\operatorname{diag}(q) A_c^{-1} \operatorname{diag}(p)\|_{\infty,1} < 1 \tag{11}$$

(the formula (8) was not known to the author when the paper [6] was being written). In our case of an interval matrix $[A_{(d)} - \delta ee^T, A_{(d)} + \delta ee^T]$ we have $p = \delta e$, $q = e$, thus $\operatorname{diag}(p) = \delta I$ and $\operatorname{diag}(q) = I$, so that (11) reduces to (10). □

This proposition leads us to the first main result of this paper.

Theorem 5. *Let A be square and let $A_{(d)}$ be nonsingular for some integer $d \geqslant 0$. Then we have:*

(i) *if*

$$d > \log_{10}(0.5 \cdot \|A_{(d)}^{-1}\|_{\infty,1}), \tag{12}$$

then A is nonsingular,

(ii) *if*

$$d < \log_{10}(0.5 \cdot \|A_{(d)}^{-1}\|_{\infty,1}), \tag{13}$$

then there exists a singular matrix A' satisfying $A'_{(d)} = A_{(d)}$.

P r o o f. (i) If (12) holds, then

$$0.5 \cdot 10^{-d} \|A_{(d)}^{-1}\|_{\infty,1} = \delta \|A_{(d)}^{-1}\|_{\infty,1} < 1,$$

hence, by virtue of Proposition 4, the interval matrix $[A_{(d)} - \delta ee^T, A_{(d)} + \delta ee^T]$ consists of nonsingular matrices only. Since A belongs to this interval matrix by (9), it follows that A is nonsingular.

(ii) If (13) holds, then

$$\delta \|A_{(d)}^{-1}\|_{\infty,1} > 1.$$

Let us choose a $\delta' \in (0, \delta)$ such that $\delta' \|A_{(d)}^{-1}\|_{\infty,1} > 1$. Then by Proposition 4 there exists a singular matrix $A' \in [A_{(d)} - \delta' ee^T, A_{(d)} + \delta' ee^T]$. Since $\delta' < \delta$, this singular matrix satisfies

$$A_{(d)} - \delta ee^T < A' < A_{(d)} + \delta ee^T,$$

and Proposition 2 gives that $A'_{(d)} = A_{(d)}$, which was to be proved. □

In case (ii) a singular matrix can be given explicitly:

Proposition 6. *Let* (13) *hold and let* $z, y \in Y$ *be any two vectors satisfying*

$$d < \log_{10}(0.5(z^T A_{(d)}^{-1} y)) \tag{14}$$

(they exist due to (8)*). Then the matrix*

$$A' = A_{(d)} - \frac{yz^T}{z^T A_{(d)}^{-1} y} \tag{15}$$

is singular and satisfies $A'_{(d)} = A_{(d)}$.

P r o o f. Indeed,

$$A'(A_{(d)}^{-1} y) = y - \frac{y(z^T A_{(d)}^{-1} y)}{z^T A_{(d)}^{-1} y} = 0,$$

hence A' is singular. From (14) we have

$$\frac{1}{z^T A_{(d)}^{-1} y} < \delta,$$

which implies $|A' - A_{(d)}| < \delta ee^T$, so that

$$A_{(d)} - \delta ee^T < A' < A_{(d)} + \delta ee^T$$

and $A'_{(d)} = A_{(d)}$ by Proposition 2. $\square$

As we have already mentioned in Section 1, (12) and (13) handle all possibilities except the case $d = \log_{10}(0.5 \cdot \|A_{(d)}^{-1}\|_{\infty,1})$ which can occur only scarcely because d is integer whereas the right-hand side is a real number. Nevertheless, the result, although enjoying almost full generality, is of limited use only because of the NP-hardness of computing the norm $\|\cdot\|_{\infty,1}$ stated in Proposition 3. But, using an upper estimation of the norm, we can arrive at an easily verifiable sufficient nonsingularity condition:

Corollary 7. *Let A be square and let $A_{(d)}$ be nonsingular for some nonnegative integer d satisfying*

$$d > \log_{10}(0.5 \cdot \||A_{(d)}^{-1}|e\|_1). \tag{16}$$

Then A is nonsingular.

Proof. For each $z, y \in Y$ we have

$$z^T A_{(d)}^{-1} y \leqslant |z^T A_{(d)}^{-1} y| \leqslant e^T |A_{(d)}^{-1}| e = \||A_{(d)}^{-1}|e\|_1,$$

hence $\|A_{(d)}^{-1}\|_{\infty,1} \leqslant \||A_{(d)}^{-1}|e\|_1$ and from (16) we obtain

$$d > \log_{10}(0.5 \cdot \||A_{(d)}^{-1}|e\|_1) \geqslant \log_{10}(0.5 \cdot \|A_{(d)}^{-1}\|_{\infty,1}),$$

which is the condition (12), and nonsingularity of A is verified. $\square$

Both the conditions (12) and (16) support the intuitive idea that nonsingularity of A can be checked if the precision, represented by the number of decimal places d, is sufficiently large. Condition (16) can be used in practical computations.

4. Positive definiteness

In this section we shall show that the previous results on nonsingularity can be literally carried over to positive definiteness due to a result linking both the topics in case of a symmetric interval matrix, stated in the following proposition proved in [8, Theorem 3]:

Proposition 8. *Let $[\underline{A}, \bar{A}]$ be an interval matrix whose both bounds $\underline{A}$, $\bar{A}$ are symmetric. Then each symmetric $A \in [\underline{A}, \bar{A}]$ is positive definite if and only if $[\underline{A}, \bar{A}]$ consists solely of nonsingular matrices and contains at least one positive definite matrix.*

Theorem 9. *Let A be symmetric and let $A_{(d)}$ be positive definite for some integer $d \geqslant 0$. Then we have:*

(i) *if*

$$d > \log_{10}(0.5 \cdot \|A_{(d)}^{-1}\|_{\infty,1}), \tag{17}$$

then A is positive definite,

(ii) *if*

$$d < \log_{10}(0.5 \cdot \|A_{(d)}^{-1}\|_{\infty,1}), \tag{18}$$

then there exists a symmetric matrix A' satisfying $A'_{(d)} = A_{(d)}$ which is not positive definite.

P r o o f. (i) Since $A_{(d)}$ is positive definite by assumption and since (17) guarantees nonsingularity of all matrices contained in $[A_{(d)} - \delta ee^T, A_{(d)} + \delta ee^T]$ (proof of Theorem 5), Proposition 8 gives that each symmetric matrix in $[A_{(d)} - \delta ee^T, A_{(d)} + \delta ee^T]$ is positive definite, thus also A is positive definite.

(ii) If (18) holds, then we know from the proof of Theorem 5 that there exists a singular matrix A'' satisfying

$$A_{(d)} - \delta ee^T < A'' < A_{(d)} + \delta ee^T, \tag{19}$$

i.e., $A''x = 0$ for some $x \neq 0$. Because both the matrices $A_{(d)} - \delta ee^T$ and $A_{(d)} + \delta ee^T$ are symmetric (symmetry of A implies symmetry of $A_{(d)}$), from (19) we have

$$A_{(d)} - \delta ee^T < 0.5(A'' + A''^T) < A_{(d)} + \delta ee^T.$$

Then the matrix

$$A' = 0.5(A'' + A''^T)$$

is symmetric, satisfies $A'_{(d)} = A_{(d)}$ by Proposition 2, and

$$x^T A' x = x^T A'' x = 0,$$

so that A' is not positive definite. □

A symmetric matrix A' from (ii) can be constructed along the lines of the proof: first we construct a singular matrix A'' according to Proposition 6, and then we symmetrize it by setting $A' = 0.5(A'' + A''^T)$.

Again, (17) can be turned into a verifiable sufficient condition.

Corollary 10. *Let A be symmetric and let $A_{(d)}$ be positive definite for some nonnegative integer d satisfying*

$$d > \log_{10}(0.5 \cdot \||A_{(d)}^{-1}|e\|_1). \tag{20}$$

Then A is positive definite.

P r o o f. As we have seen in the proof of Corollary 7, (20) implies (17), hence A is positive definite. □

5. Positive invertibility

As is well known, a square matrix A is called positive invertible if $A^{-1} > 0$ (componentwise). For the purpose of checking positive invertibility of A by means of properties of the rounded matrix $A_{(d)}$, we will again employ a theorem from interval analysis ([5], Theorem 1, assertion (iii)); ϱ denotes the spectral radius.

Proposition 11. *All matrices contained in an interval matrix $[\underline{A}, \bar{A}]$ are positive invertible if and only if $\bar{A}^{-1} > 0$ and $\varrho(\bar{A}^{-1}(\bar{A} - \underline{A})) < 1$.*

The result has again the form of an alternative ($d > \beta$ or $d < \beta$), but this time the right-hand side β does not contain the norm $\|\cdot\|_{\infty,1}$, and can be easily computed.

Theorem 12. *Let A be square and let*

$$(A_{(d)} + 0.5 \cdot 10^{-d} ee^T)^{-1} > 0 \tag{21}$$

hold for some integer $d \geqslant 0$. Then we have:

(i) *if*

$$d > \log_{10} \|(A_{(d)} + 0.5 \cdot 10^{-d} ee^T)^{-1} e\|_1, \tag{22}$$

then A is positive invertible,

(ii) *if*

$$d < \log_{10} \|(A_{(d)} + 0.5 \cdot 10^{-d} ee^T)^{-1} e\|_1, \tag{23}$$

then there exists a matrix A' satisfying $A'_{(d)} = A_{(d)}$ which is not positive invertible.

Proof. Consider again the interval matrix $[\underline{A}, \bar{A}] = [A_{(d)} - \delta ee^T, A_{(d)} + \delta ee^T]$, where $\delta = 0.5 \cdot 10^{-d}$ as before. Then $\bar{A}^{-1} > 0$ by the assumption (21), and

$$\varrho(\bar{A}^{-1}(\bar{A} - \underline{A})) = \varrho(2\delta \bar{A}^{-1} ee^T) = 2\delta e^T \bar{A}^{-1} e = 10^{-d} \|\bar{A}^{-1} e\|_1.$$

Hence, $\varrho(\bar{A}^{-1}(\bar{A} - \underline{A})) < 1$ if and only if

$$d > \log_{10} \|\bar{A}^{-1} e\|_1$$

holds. This means that if this condition is met, then, by Proposition 11, all matrices contained in $[\underline{A}, \bar{A}]$ are positive invertible and thus also A is positive invertible, which proves (i). If (23) holds, then

$$2\delta \|(A_{(d)} + \delta ee^T)^{-1} e\|_1 > 1.$$

Because of continuity there exists a $\delta' \in (0,\delta)$ such that $(A_{(d)} + \delta' ee^T)^{-1} > 0$ and

$$2\delta' \|(A_{(d)} + \delta' ee^T)^{-1} e\|_1 > 1.$$

Then, by Proposition 11, the interval matrix $[A_{(d)} - \delta' ee^T, A_{(d)} + \delta' ee^T]$ contains a matrix A' which is not positive invertible. Since

$$A_{(d)} - \delta ee^T < A_{(d)} - \delta' ee^T \leqslant A' \leqslant A_{(d)} + \delta' ee^T < A_{(d)} + \delta ee^T,$$

we have $A'_{(d)} = A_{(d)}$ by Proposition 2, which concludes the proof of (ii). □

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