Applications of Mathematics

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Applications of Mathematics, Vol. 52 (2007), No. 4, 303--319

Persistent URL: http://dml.cz/dmlcz/134677

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THREE-PARAMETRIC ROBOT MANIPULATORS WITH PARALLEL ROTATIONAL AXES

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(Received June 27, 2005, in revised version June 22, 2006)

Abstract. The paper deals with asymptotic motions of 3-parametric robot manipulators with parallel rotational axes. To describe them we use the theory of Lie groups and Lie algebras. An example of such motions are motions with the zero Coriolis accelerations. We will show that there are asymptotic motions with nonzero Coriolis accelerations. We introduce the notions of the Klein subspace, the Coriolis subspace and show their relation to asymptotic motions of robot manipulators. The asymptotic motions are introduced without explicit use of the Levi-Civita connection.

Keywords: local differential geometry, robotics, Lie algebra, asymptotic motion

MSC 2000: 53A17

1. Introduction

In this paper we deal with the properties of 3-parametric robot manipulators (in short robots) with parallel rotational axes. We describe motions of the robot effector by using the theory of Lie groups and Lie algebras which is applied to the Lie group $E(3)$ of all orientation preserving congruences of the Euclidean space E_3. By the concept of an n-parametric robot we will understand the map $\Upsilon_{A_n}\colon \mathbb{R}^n \to E(3)$, see [3], where the robot Υ_{A_n} is viewed as an immersed submanifold Υ_{A_n} of the Lie group $E(3)$. We classify 3-parametric robots into four classes. The classification criterion is the *spherical rank* of the robot, which is the number of independent directions of revolute joints axes. Robots of the spherical rank 1 are robots whose axes of revolute joints are mutually parallel and different.

The main aim of the paper is to introduce *asymptotic robot motions*. The notion of asymptotic motions is connected with the theory of connections. On a pseudo-Riemannian manifold ($E(3)$ has pseudo-Riemannian structure), there is a canonical

connection called the Levi-Civita connection. As a connection on the tangent bundle, it provides a well defined method for differentiating all kinds of tensors. The Levi-Civita connection is a torsion-free connection on the tangent bundle and it can be used to describe many intrinsic geometric objects. For instance, a geodesic path, a parallel transport for vector fields, a curvature and so on.

On the Lie group $E(3)$ there is the Levi-Civita connection ∇ induced by the Klein form KL. If the restriction $\mathrm{KL}|_{\Upsilon_{A_n}}$ is regular then there is the Levi-Civita connection $\widetilde{\nabla}$ on Υ_{A_n} such that $\nabla_{\dot\gamma}\dot\gamma = \widetilde{\nabla}_{\dot\gamma}\dot\gamma + V$, where V lies in the KL-orthogonal complement to the tangent bundle $T\Upsilon_{A_n}$. If $V = \overline{0}$ then the motion on Υ_{A_n} is *asymptotic*, see [5]. We will introduce asymptotic robot motions without explicit use of the Levi-Civita connection. A robot motion is asymptotic, if the Coriolis acceleration is tangential to $T_e\Upsilon_{A_n}$. Obviously, robot motions with zero Coriolis accelerations are asymptotic. We will introduce the notion of the Coriolis and Klein subspaces and show that they are closely associated with asymptotic motions.

2. Assumptions and definitions

Common commercial industrial robots are serial robot manipulators consisting of a sequence of links connected by joints, see Fig. 1. Each joint has one degree of freedom, either prismatic or revolute. For a robot with n joints, numbered from 1 to n, there are $n+1$ links, numbered from 0 to n. The link 0 or n will be called respectively the base or the effector of the robot. The base will be fixed (non movable). Joint i connects links i and $i+1$. We view a link as a rigid body defining the relationship between two neighbouring joints. In the concept of the Denavit-Hartenberg conventions, see [1], the base coordinate system S_0 is firmly connected with the base. The base axis z_0 is the axis o_1 of the 1st joint. The effector begins in the nth joint and is firmly connected to the coordinate system S_n.

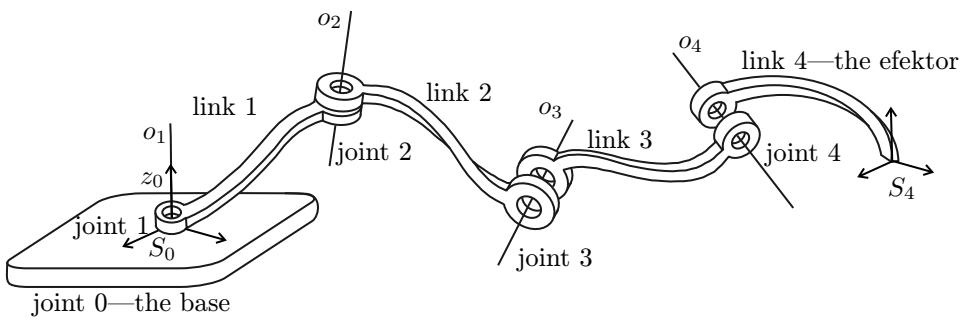

Figure 1. n-parametric robot, $n = 4$.

A congruence in the Euclidean space E_3 is determined by the base coordinate system S_0 and by the effector coordinate system S_n in each position of the robot (i.e., at time t). Therefore a motion of the effector determines a curve on the Lie group $E(3)$. We assume a fixed choice of the base orthonormal coordinate system $S_0 = \{O; \overline{i}_0, \overline{j}_0, \overline{k}_0\}$ with respect to which we will relate all elements.

Let us recall basic facts about the Lie group $E(3)$ and its Lie algebra $e(3)$. Elements of the Lie group $E(3)$ will be considered in the matrix form 4×4, which will be written in the form $\begin{pmatrix} A & P \\ 0 & 1 \end{pmatrix}$, where A is an orthogonal matrix of the form 3×3, $\det A = 1$ and P is a column matrix of the form 3×1 (a translation vector).

Let V_3 be the vector space associated with the Euclidean space E_3 and let $\gamma(t) = H(t)$ be a curve on $E(3)$ which is going through the unit element I of the group $E(3)$; i.e., $H(t_0) = I$, where I is the unit matrix. Then the motion of the effector point $\mathcal{L}$ determined by the curve $\gamma(t)$ can be expressed by

$$\begin{pmatrix} x(t) \\ y(t) \\ z(t) \\ 1 \end{pmatrix} = \begin{pmatrix} A(t) & P(t) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix},$$

where $(x, y, z, 1)^T$ are the homogeneous coordinates of the point $\mathcal{L}$ at t_0 and $(x(t), y(t), z(t), 1)^T$ are the homogeneous coordinates of the point $\mathcal{L}$ at any t. The coordinates of the point $\mathcal{L}$ are related to the base coordinate system S_0. $A(t)$ is an orthogonal matrix; i.e., $A(t)A^T(t) = I$, where $A^T(t)$ is the transposed matrix to the matrix $A(t)$. The inverse matrix to the matrix $H(t)$ is $H^{-1}(t) = \begin{pmatrix} A^T(t) & -A^T(t)P(t) \\ 0 & 1 \end{pmatrix}$. We suppose $A(t_0) = I$. The derivative of the equation $A(t)A^T(t) = I$ at $t = t_0$ is $\dot{A}(t_0) = -\dot{A}^T(t_0)$; i.e., $\dot{A}(t_0)$ is a skew-symmetric matrix. All skew-symmetric matrices have the form $\dot{A}(t_0) = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}$ and we can associate them with vectors $\overline{\omega} := (\omega_1, \omega_2, \omega_3) \in V_3$. If we denote $\dot{P}^T(t_0) := \overline{b} = (\beta_1, \beta_2, \beta_3)$, then the tangent vector

$$\dot{\gamma}(t_0) = \dot{H}(t_0) = \begin{pmatrix} \dot{A}(t_0) & \dot{P}(t_0) \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\omega_3 & \omega_2 & \beta_1 \\ \omega_3 & 0 & -\omega_1 & \beta_2 \\ -\omega_2 & \omega_1 & 0 & \beta_3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{1}$$

of the curve $\gamma(t)$ at $t = t_0$ can be associated with the element $(\overline{\omega}, \overline{b}) \equiv X \in V_3 \times V_3$ and we call it the twist, see [7]. Hence the Lie algebra $e(3)$ can be represented in

the matrix form (1) or by twists in $V_3 \times V_3$, where addition and the Lie bracket are defined as follows:

$$k_1(\overline{\omega}_1, \overline{b}_1) + k_2(\overline{\omega}_2, \overline{b}_2) = (k_1\overline{\omega}_1 + k_2\overline{\omega}_2, k_1\overline{b}_1 + k_2\overline{b}_2),$$
$$[(\overline{\omega}_1, \overline{b}_1), (\overline{\omega}_2, \overline{b}_2)] = (\overline{\omega}_1 \times \overline{\omega}_2, \overline{\omega}_1 \times \overline{b}_2 - \overline{\omega}_2 \times \overline{b}_1),$$

where $(\overline{\omega}_i, \overline{b}_i) \in V_3 \times V_3$, $k_i \in \mathbb{R}$, $i = 1, 2$ and $\times$ denotes the vector product in V_3. The line p determined by the point C, $\overline{OC} = (1/\overline{\omega}^2)\overline{\omega} \times \overline{b}$ and by the direction $\overline{\omega}$ will be called the axis of the twist $X = (\overline{\omega}, \overline{b})$, $\overline{\omega} \neq \overline{0}$. If $\overline{\omega} = \overline{0}$, then the axis of the element $X = (\overline{0}, \overline{b})$ is the line at infinity of the plane in the projective space P_3 (P_3 is E_3 together with the points at infinity) which is perpendicular to the vector $\overline{b}$.

In the algebra $V_3 \times V_3$ we have the *Klein form* given by

$$\mathrm{KL}(X_1, X_2) \stackrel{\mathrm{def}}{:=} \overline{\omega}_1 \cdot \overline{b}_2 + \overline{\omega}_2 \cdot \overline{b}_1,$$

where $X_1 = (\overline{\omega}_1, \overline{b}_1)$, $X_2 = (\overline{\omega}_2, \overline{b}_2)$ are twists from $V_3 \times V_3$ and the dot $\cdot$ denotes the scalar product in V_3. If $\mathrm{KL}(X_1, X_2) = 0$, then the twists X_1, X_2 will be called KL-*orthogonal*. The Klein form is a symmetric regular bilinear form.

A subspace $A \subset V_3 \times V_3$ is called KL-orthogonal to a subspace $B \subset V_3 \times V_3$, if $\mathrm{KL}(X, Y) = 0$ for every $X \in A$ and every $Y \in B$. There is a unique subspace $A^K \subset V_3 \times V_3$ which is KL-orthogonal to the subspace $A \subset V_3 \times V_3$; i.e., if any arbitrary vector subspace B is KL-orthogonal to A, then $B \subset A^K$.

Definition 1. Let $A \subset V_3 \times V_3$. The subspace $K \stackrel{\mathrm{def}}{:=} A \cap A^K$ will be called the *Klein subspace* of the space A. If $K = A$, then A is isotropic.

Let us recall that the matrix form of the exponential map from the Lie algebra $e(3)$ to the Lie group $E(3)$, $\exp\colon e(3) \to E(3)$, is given by $\exp(S) = \sum\limits_{n=0}^{\infty} \frac{1}{n!} S^n$, where $S \in e(3)$ is the matrix of the form (1) and S^n is the nth power of the matrix S. The matrix $\exp(S)$ is a regular matrix, $(\exp(S))^{-1} = \exp(-S)$, for further properties see [2]. For the motion determined by the curve $\gamma(t) = \exp(t(\overline{\omega}, \overline{b}))$, where $(\overline{\omega}, \overline{b}) \in e(3)$ and exp is the exponential map, see [3], we have:

(1) If $\overline{\omega} = \overline{0}$, then the curve $\gamma(t) = \exp(t(\overline{0}, \overline{b}))$ determines a translation with velocity $\overline{b}$.

(2) If $\overline{\omega} \neq \overline{0}$ then the curve $\gamma(t) = \exp(t(\overline{\omega}, \overline{b}))$ determines a uniform screw motion in E_3 with the axis p of the twist $(\overline{\omega}, \overline{b})$, the angular velocity $\overline{\omega}$ and with the translation $h\overline{\omega}$, where $h = (\overline{\omega} \cdot \overline{b})/\overline{\omega}^2$, see Fig. 2.

If $h = 0$ (i.e., $\overline{\omega} \cdot \overline{b} = 0$) then it is a rotational motion.

From the mathematical point of view, we can define a robot by the exponential map which is applied to the elements of the Lie algebra $e(3)$, see [3], as follows:

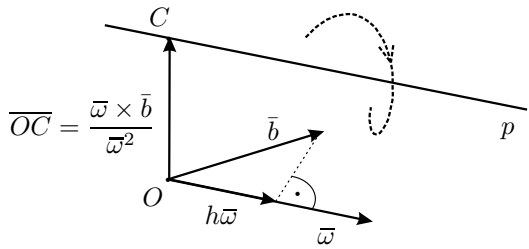

Figure 2. Screw motion determined by $\exp t(\overline{\omega}, \overline{b})$.

Definition 2. Let $X_i \in e(3)$, $i = 1, 2, \ldots, n$. Then a robot with n degrees of freedom is a map $\Upsilon_{X_1,\ldots,X_n} \colon \mathbb{R}^n \to E(3)$ given by

$$\Upsilon_{X_1,\ldots,X_n}(u_1, u_2, \ldots, u_n) = \exp u_1 X_1 \exp u_2 X_2 \ldots \exp u_n X_n.$$

Let us deal with the velocity and the acceleration of an effector point $\mathcal{L}$. Let $\Upsilon_{X_1,\ldots,X_n}$ be any n-parametric robot given by twists $X_1, X_2, \ldots, X_n$. Let the motion of the effector be given by a curve $\gamma(t) = \exp u_1(t) X_1 \exp u_2(t) X_2 \ldots \exp u_n(t) X_n \equiv H(t)$ and let $L(t_0)$ be the homogeneous coordinates of the effector point $\mathcal{L}$ at t_0. Then the homogeneous coordinates $L(t)$ of the point $\mathcal{L}$ at any t are given by $L(t) = H(t)L(t_0)$. So its velocity is given by $\dot{L}(t) = \dot{H}(t)H^{-1}(t)L(t)$. The element $\dot{H}(t)$ determines the tangent vector at $H(t)$ and $\dot{H}(t)H^{-1}(t)$ is a right translation by $H^{-1}(t)$. Then $Y(t) \overset{\text{def}}{:=} \dot{H}(t)H^{-1}(t)$ belongs to the Lie algebra $e(3)$. The velocity of the motion $L(t) = H(t)L(t_0)$ determined by $H(t)$ at t_0 and the velocity of the motion $L(s) = \exp(sY(t_0))L(t_0)$ determined by $\exp(sY(t_0))$ at $s = 0$ are the same. The twist $Y(t)$ is called the *velocity operator* or shortly the *velocity twist*.

Remark 1. For simplicity we will use u instead of $u(t)$.

As $H = \exp u_1 X_1 \ldots \exp u_n X_n$ we get

$$Y = \dot{H}H^{-1} = \dot{u}_1 Y_1 + \dot{u}_2 Y_2 + \ldots + \dot{u}_n Y_n, \tag{2}$$

see [6], where $Y_1 = X_1$, $Y_i = g_{i-1} X_i g_{i-1}^{-1}$, $g_{i-1} = \exp u_1 X_1 \ldots \exp u_{i-1} X_{i-1}$ and g_{i-1}^{-1} is the inverse element of g_{i-1}, $i = 2, \ldots, n$. Elements Y_i belong to the Lie algebra $e(3)$. The space $A_n(u) \overset{\text{def}}{:=} \operatorname{span}(Y_1, Y_2, \ldots, Y_n)$ will be called the *space of velocity twists*, where $(u) = (u_1, u_2, \ldots, u_n)$. If $\dim A_n(u) = n$ then we call the point (u) *regular* or we say that the robot is in a *regular* position. If $\dim A_n(u) < n$ then we call the point (u) *singular* or we say that the robot is in a *singular* position. If every point of the curve $\gamma(t) = H(t) \subset \Upsilon_{X_1,\ldots,X_n}$ is singular then this motion of the robot determined by the curve $\gamma(t) = H(t)$ will by called *singular*. A robot $\Upsilon_{X_1,\ldots,X_n}$ is of *rank* m if m is the maximal dimension of the velocity twists spaces; i.e., $m = \max\limits_{(u)}\{\dim A_n(u)\}$.

Remark 2. In what follows we confine ourselves to $n = m$. Without loss of generality we will assume that $A_n := \operatorname{span}(X_1, X_2, \ldots, X_n)$ and $\dim A_n = n = m$ is the rank of the robot. Then there is a neighborhood $\Omega_0 \subset \mathbb{R}^n$ of the point $O = (0, 0, \ldots, 0) \in \mathbb{R}^n$ that $\Upsilon_{X_1,\ldots,X_n}|_{\Omega_0} =: \Upsilon_{A_n}$ is an immersed submanifold of the Lie group $E(3)$ and $(u_1, u_2, \ldots, u_n)$ is a local coordinate system of Υ_{A_n}.

Let us consider the acceleration of any effector point $\mathcal{L}$. The velocity of the point $\mathcal{L}$ at a time t is determined by $\dot{L}(t) = \dot{H}(t)H^{-1}(t)L(t) = Y(t)L(t)$. Let us differentiate the last equation. We get the relation for the acceleration of the effector point $\mathcal{L}$ at the time t: $\ddot{L}(t) = \dot{Y}(t)L(t) + Y(t)\dot{L}(t) = (\dot{Y}(t) + Y(t)Y(t))L(t)$. The derivative of the equation (2) is $\dot{Y}(t) = \sum\limits_{i=1}^{n} \dot{u}_i\dot{Y}_i + \sum\limits_{i=1}^{n} \ddot{u}_i Y_i$, where $\dot{Y}_i = \sum\limits_{k=1}^{i-1} (\partial Y_i/\partial u_k)\dot{u}_k$. All elements Y_i are in the matrix form, therefore we can use the Lie bracket in the matrix form: $[A, B] = AB - BA$. Then we get $\dot{Y}_1 = \overline{0}$, $\dot{Y}_i = [Y_1, Y_i]\dot{u}_1 + [Y_2, Y_i]\dot{u}_2 + \ldots + [Y_{i-1}, Y_i]\dot{u}_{i-1}$, $i = 2, \ldots, n$. The acceleration of the point $\mathcal{L}$ at any t is of the form

$$\ddot{L}(t) = \left(\sum_{k<i} \dot{u}_k\dot{u}_i[Y_k, Y_i] + \sum_{i=1}^{n} \ddot{u}_i Y_i + YY\right)L(t), \quad k, i = 1, \ldots, n,$$

see [6]. The expression $\sum\limits_{i=1}^{n} \ddot{u}_i Y_i$ represents the acceleration caused by joint accelerations $\ddot{u}_i$, the expression YY represents centrifugal or centripetal components of acceleration and the expression $\dot{Y}_c \overset{\text{def}}{:=} \sum\limits_{k<i} \dot{u}_k\dot{u}_i[Y_k, Y_i]$ stands for the so-called Coriolis acceleration.

Definition 3. The subspace $CA \overset{\text{def}}{:=} \operatorname{span}([Y_1, Y_2], [Y_1, Y_3], \ldots, [Y_{n-1}, Y_n])$ of the space $e(3)$ will be called the *Coriolis subspace*.

Definition 4.

(1) If $CA \subset A_n(u)$ then the point (u) of the robot $\Upsilon_{X_1,\ldots,X_n}$ will be called *flat*.
(2) If every point of the robot is flat then the robot will be called *flat*.
(3) If $\dot{Y}_c(u(t_0)) \in A_n(u(t_0))$ for a point $u(t_0)$ then the motion of the robot determined by the curve $\gamma(t) = H(t)$ will by called *asymptotic at the point* $u(t_0)$.
(4) If $\dot{Y}_c(t) \in A_n(u)$ for every t then this motion of the robot determined by the curve $\gamma(t) = H(t)$ will by called *asymptotic*.

Examples.

I. Robot motions with zero Coriolis acceleration are asymptotic. Then:
   a) If only prismatic joints work then the robot motion is asymptotic.
   b) If only one joint works then the robot motion is asymptotic.

II. If $A_n(u)$ is a subalgebra; i.e., iff $CA \subset A_n(u)$ then (u) is flat and thus every motion is asymptotic at (u).

If $A_n(u)$ is a subalgebra or only translational joints work or just one joint works then these motions will be called *trivial* asymptotic motions. In the next part we will deal only with nontrivial asymptotic motions.

The base coordinate system S_0 is connected with the 1st joint. Its axis z_0 is the axis o_1 of the first joint of the robot. The axis of the twist $Y_i = (\overline{\omega}_i, \overline{b}_i)$ is the axis o_i of the ith joint. If the ith joint is revolute or helical then its axis is the axis of Y_i. If it is prismatic then the direction of its axis is $\overline{b}_i$, $Y_i = (\overline{0}, \overline{b}_i)$.

Remark 3. If $X = (\overline{0}, \overline{b})$ is translational then $\overline{b}$ is orthogonal to the plane (O, o) where O is the origin of the coordinate system S_0 and o is the axis of X.

We will deal with robots which have no helical joints. The capital letter R will indicate a revolute joint, T a translational (prismatic) joint. Then, for example, RRT denotes a 3-parametric robot the first and second joints of which are revolute and the third is prismatic.

Definition 5. A robot Υ_{A_n} is the robot of *spherical rank* r if r is the maximal number of linearly independent directions of revolute joint axes.

It is obvious that 3-parametric robots have spherical rank 0, 1, 2 or 3. In the next part we will deal with 3-parametric robots of spherical rank 1.

Remark 4. It is interesting to deal with the problem whether 3-parametric robots Υ_{A_3} of spherical rank 1 lie within any subgroup of the Lie group $E(3)$ the dimension of which is less than 6. Common knowledge is that there is only one connected 4-dimensional Lie subgroup H_4 of $E(3)$ (up to conjugacy), see [4], [8]. The subgroup H_4 is the group generated by a one parametric rotation around the straight line $(SO(2))$ and all translations $(\mathbb{R}^3)$, (i.e., $H_4 = SO(2) \rtimes \mathbb{R}^3$). Hence three parametric robot manipulators of spherical rank 1 belong to 4-dimensional Lie subgroups. This fact does not affect our own work except that we know that all Coriolis space elements of the 3-parametric robots of spherical rank 1 belong to the Lie algebra h_4 of the Lie group H_4.

3. Three parametric robots of spherical rank 1

Now $\dim A_3 = 3$ and the revolute joint axes are parallel at any position u. Therefore Y_i is of the form $Y_i = (k_i\overline{\omega}, \overline{m}_i)$ where at least one of the k_i is not zero. We assume $\|\overline{\omega}\| = 1$ and $k_i \in \{0, 1\}$. We can always choose twists $B_1 = (\overline{\omega}, \overline{b}_1)$, $B_2 = (\overline{0}, \overline{b}_2)$, $B_3 = (\overline{0}, \overline{b}_3)$, $\overline{\omega} \cdot \overline{b}_1 = 0$ in the space $A_3(u) = \operatorname{span}(Y_1, Y_2, Y_3)$ such that $A_3(u) = \operatorname{span}(B_1, B_2, B_3)$.

R e m a r k 5. The robot Υ_{A_3} is in a singular position iff $\dim A_3(u) < 3$ and this occurs iff $\overline{b}_2 \times \overline{b}_3 = \overline{0}$.

Let us determine conditions for an arbitrary twist $B = t_1B_1 + t_2B_2 + t_3B_3$ of the space $A_3(u)$, $t_1, t_2, t_3 \in \mathbb{R}$ to be a rotational or translational twist. A twist B is rotational or translational iff $\mathrm{KL}(B, B) = 0$ and this is equivalent to $t_1(t_2(\overline{\omega} \cdot \overline{b}_2) + t_3(\overline{\omega} \cdot \overline{b}_3)) = 0$.

A *twist B is translational* iff $t_1 = 0$; i.e., iff $B \in \mathrm{span}(B_2, B_3) \stackrel{\mathrm{def}}{=:} \tau$. Therefore $\dim \tau \leqslant 2$. In singular positions, $\dim \tau = 1$.

A *twist B is rotational* if and only if $t_2(\overline{\omega} \cdot \overline{b}_2)+t_3(\overline{\omega} \cdot \overline{b}_3) = 0$, $t_1 \neq 0$. If $\overline{\omega} \cdot \overline{b}_2 = 0$, $\overline{\omega} \cdot \overline{b}_3 = 0$ then there are no screw elements in $A_3(u)$; i.e., $A_3(u) - \tau$ is the space of all rotational twists. If at least one number of $\overline{\omega} \cdot \overline{b}_2$, $\overline{\omega} \cdot \overline{b}_3$ is not equal to 0 then there is a two-dimensional space of rotational twists.

For example, if $\overline{\omega} \cdot \overline{b}_2 \neq 0$ then $B = (t_1\overline{\omega}, t_1\overline{b}_1 + (\overline{b}_3 - (\overline{\omega} \cdot \overline{b}_3/\overline{\omega} \cdot \overline{b}_2)\overline{b}_2)t_3)$. The axes of rotational twists generate a bundle of parallel lines with the direction $\overline{\omega}$.

The matrix of the Klein form has the form

$$\mathrm{KL}|_{A_3(u)} = \begin{pmatrix} 0 & \overline{\omega} \cdot \overline{b}_2 & \overline{\omega} \cdot \overline{b}_3 \\ \overline{\omega} \cdot \overline{b}_2 & 0 & 0 \\ \overline{\omega} \cdot \overline{b}_3 & 0 & 0 \end{pmatrix}$$

in the basis B_1, B_2, B_3. The rank of the Klein form is 0 or 2; i.e., $\mathrm{KL}|_{A_3(u)}$ is singular.

The rank is 0 if and only if $\overline{\omega} \cdot \overline{b}_2 = 0$, $\overline{\omega} \cdot \overline{b}_3 = 0$; i.e., if the vector $\overline{\omega}$ is perpendicular to $\tau_2 \stackrel{\mathrm{def}}{:=} \mathrm{span}(\overline{b}_2, \overline{b}_3)$; i.e., if there are no screw elements in $A_3(u)$. In this case the Klein space is $K = A_3(u)$; i.e., $A_3(u)$ is isotropic and the robot is planar.

The rank is 2 iff the direction of the revolute joints axes is not perpendicular to τ_2; i.e., at least one number $\overline{\omega} \cdot \overline{b}_2$, $\overline{\omega} \cdot \overline{b}_3$ is not equal to 0. Let us determine the Klein subspace K. The twist $B = t_1B_1 + t_2B_2 + t_3B_3$, $t_1, t_2, t_3 \in \mathbb{R}$ is KL-orthogonal to $A_3(u)$ if and only if $\mathrm{KL}(B, A_3(u)) = 0$ and this is equivalent to $t_1 = 0$, $t_2 = k(\overline{\omega} \cdot \overline{b}_3)$, $t_3 = -k(\overline{\omega} \cdot \overline{b}_2)$, $k \in \mathbb{R}$. Therefore the Klein subspace is determined by the element $\hat{Y} = (0, (\overline{\omega} \cdot \overline{b}_3)\overline{b}_2 - (\overline{\omega} \cdot \overline{b}_2)\overline{b}_3)$ (i.e., $K = \mathrm{span}(\hat{Y}) \subset \tau$), its direction is perpendicular to $\overline{\omega}$ and it belongs to τ_2.

Let us summarize the above considerations.

Proposition 1. *Let Υ_{A_3} be a robot of spherical rank* 1. *Then*

a) *if the direction $\overline{\omega}$ of revolute joints is perpendicular to the space τ_2 of translational elements directions then the rank of $\mathrm{KL}|_{A_3(u)}$ is 0, the Klein space K is $K = A_3(u)$; i.e., $A_3(u)$ is isotropic and the robot is planar;*

b) *if $\overline{\omega}$ is not perpendicular to the space τ_2 then the rank of* $\mathrm{KL}|_{A_3(u)}$ *is* 2 *and the Klein space K is $K = \mathrm{span}((0, (\overline{\omega} \cdot \overline{b}_3)\overline{b}_2 - (\overline{\omega} \cdot \overline{b}_2)\overline{b}_3))$, $K \subset \tau$ and its direction is perpendicular to $\overline{\omega}$.*

The Coriolis subspace is $CA = \mathrm{span}([B_1, B_2], [B_1, B_3], [B_2, B_3])$, where $[B_1, B_2] = (\overline{0}, \overline{\omega} \times \overline{b}_2)$, $[B_1, B_3] = (\overline{0}, \overline{\omega} \times \overline{b}_3)$, $[B_2, B_3] = (\overline{0}, \overline{0})$. It means that the elements of CA are translational and their directions are perpendicular to $\overline{\omega}$ and $\dim CA \leqslant 2$. The following cases are possible:

(1) $\dim CA \leqslant 1$ if and only if $(\overline{\omega} \times \overline{b}_2) \times (\overline{\omega} \times \overline{b}_3) = ((\overline{\omega} \times \overline{b}_2) \cdot \overline{b}_3)\overline{\omega} = \overline{0}$ and this is equivalent to $\overline{\omega} \cdot (\overline{b}_2 \times \overline{b}_3) = 0$. We have the following cases:

a) $\overline{b}_2 \times \overline{b}_3 = \overline{0}$; i.e., the robot is in a singular position, $\dim \tau = 1$.
Let $\overline{b}_3 = k\overline{b}_2$, $k \in \mathbb{R}$. Then $A_3(u) \cap CA \neq \overline{0}$ if and only if $\overline{\omega} \times \overline{b}_2 = c\overline{b}_2$ and $\overline{\omega} \times \overline{b}_2 \neq \overline{0}$, $c \in \mathbb{R}$. It is impossible. Therefore $A_3(u) \cap CA = \overline{0}$.

b) $\overline{b}_2 \times \overline{b}_3 \neq \overline{0}$ and $\overline{\omega} \in \mathrm{span}(\overline{b}_2, \overline{b}_3)$; i.e., the robot is in a regular position and the vectors ω, $\overline{b}_2$, $\overline{b}_3$ are linearly dependent. We can write $\overline{\omega} = c_2\overline{b}_2 + c_3\overline{b}_3$, $c_2, c_3 \in \mathbb{R}$. Then $CA = \mathrm{span}((\overline{0}, \overline{b}_2 \times \overline{b}_3))$. The vector $\overline{b}_2 \times \overline{b}_3$ does not belong to τ_2, therefore $CA \cap A_3(u) = \overline{0}$ and $\dim CA = 1$.

(2) $\dim CA = 2$ if and only if the position of the robot is regular (i.e., $\overline{b}_2 \times \overline{b}_3 \neq \overline{0}$) and the direction $\overline{\omega}$ of the revolute joints axes is not complanar with the space τ_2. It means that the vectors $\overline{\omega} \times \overline{b}_2$, $\overline{\omega} \times \overline{b}_3$ are linearly independent. The twists $[B_1, B_2]$, $[B_1, B_3]$ determine the basis of the Coriolis space CA. This basis will be called the *canonical basis* of the space CA. In this case the space CA is the space of all translational elements, whose directions are perpendicular to the direction of the revolute joints $\overline{\omega}$. We have the following cases:

a) $CA = \tau$. In this case, the vector $\overline{\omega}$ is perpendicular to the space τ_2; i.e., $\overline{\omega} \cdot \overline{b}_2 = 0$, $\overline{\omega} \cdot \overline{b}_3 = 0$ and the rank of the Klein form is 0. If $CA = \tau$ then $CA \subset A_3(u)$; i.e., $A_3(u)$ is a Lie subalgebra. A reverse assertion is also valid. If $A_3(u)$ is a subalgebra then $CA \subset A_3(u)$, $\dim CA = 2$ and the elements of CA are translational, therefore $CA = \tau$.

b) $CA \neq \tau$. In this case at least one of the numbers $\overline{\omega} \cdot \overline{b}_2$, $\overline{\omega} \cdot \overline{b}_3$ is not equal to 0 and the rank of the Klein form is 2. Now $K = \mathrm{span}((\overline{\omega} \cdot \overline{b}_3)\overline{B}_2 - (\overline{\omega} \cdot \overline{b}_2)B_3) = CA \cap A_3(u)$.

Let us summarize the above reflections.

Proposition 2. *If a robot of spherical rank* 1 *is in a singular position or the direction of the revolute joint axes is complanar with the space τ_2 in a regular position then* $\dim CA = 1$ *and* $CA \cap A_3(u) = \overline{0}$. *There are asymptotic motions with zero Coriolis acceleration only in these positions.*

If a robot of spherical rank 1 *is in a regular position and $\overline{\omega} \notin \tau_2$ then* $\dim CA = 2$ *and there are two cases.*

a) *If* $CA = \tau_2 \subset A_3(u)$*; i.e., if* $\overline{\omega}$ *is perpendicular to* τ_2*; i.e., if* $A_3(u)$ *is a subalgebra of* $e(3)$ *then all motions are asymptotic in this position. The point* (u) *is flat.*
b) *If* $CA \neq \tau_2$ *then* $CA \cap A_3(u) = \operatorname{span}(\hat{Y}) = K$ *and there are asymptotic motions with nonzero Coriolis acceleration in this regular position.*

Revolute joints axes of spherical rank 1 robots are in all positions parallel, therefore perpendicularity of revolute joints axes and prismatic joints is preserved. Therefore if $A_3(u)$ is a subalgebra in one regular position then it is a subalgebra in all regular positions. Then $A_3(u)$ is a subalgebra in a regular position iff $A_3 = \operatorname{span}(X_1, X_2, X_3)$ is a subalgebra.

3a. Robots with 2 prismatic and 1 revolute joints

There are the following possibilities with respect to the configuration.

a$_1$) For RTT we have $Y_1 = (\overline{\omega}, \overline{0})$, $Y_2 = (\overline{0}, \overline{m}_2)$, $Y_3 = (\overline{0}, \overline{m}_3)$. Now $B_i = Y_i$, $[Y_i, Y_j] = [B_i, B_j]$, $i, j = 1, 2, 3$ and $\tau_2 = \operatorname{span}(\overline{m}_2, \overline{m}_3)$.

a$_2$) For TRT we have $Y_1 = (\overline{0}, \overline{m}_1)$, $Y_2 = (\overline{\omega}, \overline{m}_2)$, $Y_3 = (\overline{0}, \overline{m}_3)$ and $B_1 = Y_2$, $B_2 = Y_1$, $Y_3 = B_3$. Therefore $[Y_1, Y_2] = -[B_1, B_2]$, $[Y_1, Y_3] = [B_2, B_3]$, $[Y_2, Y_3] = [B_1, B_3]$ and $\tau_2 = \operatorname{span}(\overline{m}_1, \overline{m}_3)$.

a$_3$) For TTR we have $Y_1 = (\overline{0}, \overline{m}_1)$, $Y_2 = (\overline{0}, \overline{m}_2)$, $Y_3 = (\overline{\omega}, \overline{m}_3)$ and $B_1 = Y_3$, $B_2 = Y_2$, $B_3 = Y_1$. Now $[Y_1, Y_2] = -[B_2, B_3]$, $[Y_1, Y_3] = -[B_1, B_3]$, $[Y_2, Y_3] = -[B_1, B_3]$ and $\tau_2 = \operatorname{span}(\overline{m}_1, \overline{m}_2)$.

A singular position exists only in the case TRT provided there is $u_2(t_0) = \tilde{u}_2$ such that $o_1(t_0) \parallel o_3(t_0)$, i.e., $\overline{b}_2(t_0) \times \overline{b}_3(t_0) = \overline{0}$. This is possible iff $\angle(o_1, o_2) = \angle(o_2, o_3)$. $A_3(u)$ is a subalgebra iff $CA = \tau_2$ in a regular position is valid and this is possible iff $\overline{\omega} \cdot \overline{b}_2 = 0$, $\overline{\omega} \cdot \overline{b}_3 = 0$ in a regular position, so we have the following statements.

Proposition 3. *All positions of robots* RTT, TTR *are regular.*

There are singular positions in the case of TRT *iff* $\angle(o_1, o_2) = \angle(o_2, o_3)$.

$A_3(u)$ *is an algebra iff the axis of the revolute joint is perpendicular to the axes of the prismatic joints in a regular position.*

Remark 6. Robots RTT, TTR are homogeneous spaces, see [4]. In the case TRT, this robot is a homogeneous space if A_3 is a subalgebra (the planar robot).

Let us investigate asymptotic motions of robots RTT, TRT, TTR. In the case when $A_3(u)$ is a subalgebra then all motions through the point (u) are asymptotic. We have the following cases:

(1) In a singular position; i.e., only for TRT, when $\overline{m}_3 = c\overline{m}_1$, $c \in \mathbb{R}$, the subspace CA is defined by $[Y_1, Y_2] = (\overline{0}, -\overline{\omega} \times \overline{m}_1)$, $[Y_1, Y_3] = (\overline{0}, \overline{0})$, $[Y_2, Y_3] =$

$(\overline{0}, c\overline{\omega} \times \overline{m}_1)$ and the Coriolis acceleration is

$$\dot{Y}_c = \sum_{i<j} \dot{u}_i \dot{u}_j [Y_i, Y_j] = (-\dot{u}_1 \dot{u}_2 + c\dot{u}_2 \dot{u}_3)(\overline{0}, \overline{\omega} \times \overline{m}_1),$$

where $\overline{\omega} \times \overline{m}_1 \neq \overline{0}$.

A motion is asymptotic at a singular point $u(t_0)$ if and only if

$$\dot{u}_2(t_0)(-\dot{u}_1(t_0) + c\dot{u}_3(t_0)) = 0;$$

i.e., $\dot{u}_2(t_0) = 0$ so that the revolute joint is not working at t_0 or the joint velocities of the prismatic joints satisfy the relationship $\dot{u}_1(t_0)\colon\ \dot{u}_3(t_0) = c$.

If every position of the robot motion is singular (i.e., $u_2(t) = \hat{u}_2 = \text{const}$, $\dot{u}_2 = 0$) then this motion is the trivial asymptotic motion (only prismatic joints work).

Proposition 4. *A motion of the robot* TRT *is nontrivial asymptotic in singular position $u(t_0)$ iff all joints work and the joint velocities of the prismatic joints satisfy the relationship $\dot{u}_1(t_0) : \dot{u}_3(t_0) = c$. The singular motion of the robot* TRT *is trivial asymptotic.*

(2) Let us investigate asymptotic motions of the robot in a regular position when the subspace CA is one-dimensional. We know that $\overline{\omega} = c_2\overline{b}_2 + c_3\overline{b}_3$ and $CA \cap A_3(u) = \overline{0}$. A motion is asymptotic when the Coriolis acceleration $\dot{Y}_c = \sum \dot{u}_i \dot{u}_j [Y_i, Y_j] = \overline{0}$ and this occurs

a$_1$) if $\dot{u}_1\dot{u}_2(\overline{0}, -c_3\overline{m}_2 \times \overline{m}_3) + \dot{u}_1\dot{u}_3(\overline{0}, c_2\overline{m}_2 \times \overline{m}_3) = 0$; i.e., $\dot{u}_1(-\dot{u}_2 c_3 + \dot{u}_3 c_2) = 0$ in the case of RTT,

a$_2$) if $\dot{u}_1\dot{u}_2(\overline{0}, c_3\overline{m}_1 \times \overline{m}_3) + \dot{u}_2\dot{u}_3(\overline{0}, c_2\overline{m}_1 \times \overline{m}_3) = 0$; i.e., $\dot{u}_2(\dot{u}_1 c_3 + \dot{u}_3 c_2) = 0$ in the case of TRT,

a$_3$) if $\dot{u}_1\dot{u}_3(\overline{0}, -c_3\overline{m}_1 \times \overline{m}_2) + \dot{u}_2\dot{u}_3(\overline{0}, c_2\overline{m}_1 \times \overline{m}_2) = 0$; i.e., $\dot{u}_3(\dot{u}_1 c_3 - \dot{u}_2 c_2) = 0$ in the case of TTR.

In the cases of RTT, TTR, if the equation $\overline{\omega} = c_2\overline{b}_2 + c_3\overline{b}_3$ is valid in one position then it is valid for all positions.

In the case TRT, the equation $\overline{\omega} = c_2\overline{b}_2 + c_3\overline{b}_3$, $c_2 \cdot c_3 \neq 0$ is valid only if the 3rd axis turns around the axis o_2 to the position complanar with the axes o_1, o_2 (i.e., the directions of the joint axes are linearly dependent). If $c_2 \cdot c_3 = 0$; i.e., $o_3 \parallel o_2$ or $o_1 \parallel o_2$ then the equation $\overline{\omega} = c_2\overline{b}_2 + c_3\overline{b}_3$ is valid for all positions of the axes.

Let us recall that we are interested only in nontrivial asymptotic motions. Then the Coriolis acceleration is zero in the case RTT if $-\dot{u}_2 c_3 + \dot{u}_3 c_2 = 0$, in the case TRT if $\dot{u}_1 c_3 + \dot{u}_3 c_2 = 0$ and in the case TTR if $-\dot{u}_1 c_3 + \dot{u}_2 c_2 = 0$. We have the following cases:

a) Let $c_2 \cdot c_3 \neq 0$ at (u). Then a motion through the point (u) is nontrivial asymptotic iff all joints work and the joint velocities of the prismatic joints satisfy the relationship $c_2 : c_3$ in the cases RTT, TTR and $-c_2 : c_3$ in the case TRT.

b) Let $c_2 \cdot c_3 = 0$ at (u). Then a motion through the point (u) is nontrivial asymptotic iff the revolute joint and only the prismatic joint whose axis is parallel to the axis of the revolute joint, work.

Proposition 5. *Let Υ_{A_3} be a robot of spherical rank 1 with two prismatic joints and let the directions of the joint axes be linearly dependent at $u(t_0)$; i.e., $\overline{\omega} = c_2\overline{b}_2 + c_3\overline{b}_3$. Then:*

a) *The zero Coriolis acceleration is a necessary condition for the motion to be asymptotic at $u(t_0)$.*

b) *In the case that no two axes of joints are parallel at (u): a motion through the point (u) is nontrivial asymptotic iff all joints work and the joint velocities of the prismatic joints satisfy the relationship $c_2 : c_3$ in the cases* RTT, TTR *and $-c_2 : c_3$ in the case* TRT.

c) *In the case that the axis of the revolute joint is parallel to one axis of a prismatic joint: a motion is nontrivial asymptotic iff the revolute joint and only the prismatic joint whose axis is parallel to the axis of the revolute joint work.*

(3) Let us investigate asymptotic robot motions in a regular position, when $\dim CA = 2$ and $A_3(u)$ is not a subalgebra. Then $CA \cap A_3(u) = K = \operatorname{span}(((\overline{\omega}\cdot\overline{b}_3)B_2 - (\overline{\omega}\cdot\overline{b}_2)B_3))$; i.e., the equation $(\overline{\omega}\cdot\overline{b}_3)\overline{b}_2 - (\overline{\omega}\cdot\overline{b}_2)\overline{b}_3 = k_2(\overline{\omega} \times \overline{b}_2) + k_3(\overline{\omega} \times \overline{b}_3)$, $k_2, k_3 \in \mathbb{R}$, is valid.

In this case the motion is asymptotic at the point (u) if and only if

a$_1$) for RTT $\dot{u}_1\dot{u}_2(\overline{0}, \overline{\omega}\times\overline{m}_2) + \dot{u}_1\dot{u}_3(\overline{0}, \overline{\omega}\times\overline{m}_3) = \lambda(\overline{0}, k_2(\overline{\omega}\times\overline{m}_2) + k_3(\overline{\omega}\times\overline{m}_3))$, $\lambda \in \mathbb{R}$; i.e., $\dot{u}_1\dot{u}_2 = k_2\lambda$, $\dot{u}_1\dot{u}_3 = k_3\lambda$,

a$_2$) for TRT $\dot{u}_1\dot{u}_2(\overline{0}, \overline{\omega}\times\overline{m}_1) + \dot{u}_2\dot{u}_3(\overline{0}, \overline{\omega}\times\overline{m}_3) = \lambda(\overline{0}, k_2(\overline{\omega}\times\overline{m}_1) + k_3(\overline{\omega}\times\overline{m}_3))$, $\lambda \in \mathbb{R}$; i.e., $\dot{u}_1\dot{u}_2 = k_2\lambda$, $\dot{u}_2\dot{u}_3 = k_3\lambda$,

a$_3$) for TTR $\dot{u}_1\dot{u}_3(\overline{0}, \overline{\omega}\times\overline{m}_1) + \dot{u}_2\dot{u}_3(\overline{0}, \overline{\omega}\times\overline{m}_2) = \lambda(\overline{0}, k_2(\overline{\omega}\times\overline{m}_1) + k_3(\overline{\omega}\times\overline{m}_2))$, $\lambda \in \mathbb{R}$; i.e., $\dot{u}_1\dot{u}_3 = k_2\lambda$, $\dot{u}_2\dot{u}_3 = k_3\lambda$.

We summarize the previous results.

Proposition 6. *Let Υ_{A_3} be a robot of spherical rank 1 with two prismatic joints and let the directions of the joint axes be independent at t_0; i.e., $\overline{\omega} \neq c_2\overline{b}_2 + c_3\overline{b}_3$. Then:*

A motion is nontrivial asymptotic at t_0 iff joint velocities at t_0 satisfy $\dot{u}_1\dot{u}_2 = k_2\lambda$, $\dot{u}_1\dot{u}_3 = k_3\lambda$ for RTT, *$\dot{u}_1\dot{u}_2 = k_2\lambda$, $\dot{u}_2\dot{u}_3 = k_3\lambda$ for* TRT *and $\dot{u}_1\dot{u}_3 = k_2\lambda$, $\dot{u}_2\dot{u}_3 = k_3\lambda$ for* TTR, *where $\lambda \in \mathbb{R}$ and k_2, k_3 are the coefficients of the linear combination of*

$\hat{Y} = (0, (\overline{\omega} \cdot \overline{b}_3)\overline{b}_2 - (\overline{\omega} \cdot \overline{b}_2)\overline{b}_3)$ *in the canonical basis of the Coriolis space. If these relations are true for any admissible t then the motion is asymptotic.*

In this case there are nontrivial asymptotic motions with nonzero Coriolis acceleration.

3b. Robots with 1 prismatic and 2 revolute joints

Let ξ be the plane determined by the axes of the revolute joints. There are three possibilities with respect to the configuration.

b$_1$) RRT: then $Y_1 = (\overline{\omega}, \overline{0})$, $Y_2 = (\overline{\omega}, \overline{m}_2)$, $Y_3 = (\overline{0}, \overline{m}_3)$, where $\overline{m}_2 \neq \overline{0}$ and $\overline{\omega} \cdot \overline{m}_2 = 0$. Now $B_1 = Y_1 = (\overline{\omega}, \overline{0})$, $B_2 = Y_2 - Y_1 = (\overline{0}, \overline{b}_2 = \overline{m}_2)$, $B_3 = Y_3 = (\overline{0}, \overline{b}_3 = \overline{m}_3)$. We know (see Remark 3), that the vector $\overline{m}_2$ is perpendicular to the plane ξ. We have $[Y_1, Y_2] = [B_1, B_2]$, $[Y_1, Y_3] = [B_1, B_3]$, $[Y_2, Y_3] = [B_1, B_3]$ and $\tau_2 = \operatorname{span}(\overline{m}_2, \overline{m}_3)$.

b$_2$) RTR: then $Y_1 = (\overline{\omega}, \overline{0})$, $Y_2 = (\overline{0}, \overline{m}_2)$, $Y_3 = (\overline{\omega}, \overline{m}_3)$, where $\overline{m}_3 \neq \overline{0}$ and $\overline{\omega} \cdot \overline{m}_3 = 0$. Now $B_1 = Y_1 = (\overline{\omega}, \overline{0})$, $B_2 = Y_2 = (\overline{0}, \overline{m}_2)$, $B_3 = Y_3 - Y_1 = (\overline{0}, \overline{b}_3 = \overline{m}_3)$. The vector $\overline{m}_3$ is perpendicular to the plane ξ. We have $[Y_1, Y_2] = [B_1, B_2]$, $[Y_1, Y_3] = [B_1, B_3]$, $[Y_2, Y_3] = -[B_1, B_2]$ and $\tau_2 \operatorname{span}(\overline{m}_2, \overline{m}_3)$.

b$_3$) TRR: then $Y_1 = (\overline{0}, \overline{m}_1)$, $Y_2 = (\overline{\omega}, \overline{m}_2)$, $Y_3 = (\overline{\omega}, \overline{m}_3)$, $\overline{m}_2 \neq \overline{m}_3$, $\overline{\omega} \cdot \overline{m}_2 = 0$, $\overline{\omega} \cdot \overline{m}_3 = 0$. Now $B_1 = Y_2$, $B_2 = Y_1$, $B_3 = Y_3 - Y_2$. It is easy to show that the vector $\overline{m}_3 - \overline{m}_2$ is perpendicular to the plane ξ. We have $[Y_1, Y_2] = -[B_1, B_2]$, $[Y_1, Y_3] = -[B_1, B_2]$, $[Y_2, Y_3] = [B_1, B_3]$ and $\tau_2 = \operatorname{span}(\overline{m}_1, \overline{m}_3 - \overline{m}_2)$.

So we have

Proposition 7. *Let ξ be the plane determined by the axes of the revolute joints. The space τ_2 of the directions of the translational velocity elements is generated by the direction of the prismatic joint and the normal vector of the plane ξ. If the axis of the prismatic joint is perpendicular to the plane ξ then the robot is in the singular position. The robot has a singular position iff A_3 is a subalgebra.*

The subspace $A_3(u)$ is a subalgebra iff the axes of the revolute joints are perpendicular to the axis of the prismatic joint in a regular position.

In the next part we will investigate asymptotic robot motions of RRT, RTR, TRR. If $A_3(u)$ is a subalgebra then all motions through the point (u) are asymptotic. Let $\overline{n}_\xi$ be the normal vector of the plane ξ. By our previous considerations we have the following cases:

(1) Let $u(t_0)$ be a singular position (A_3 is a subalgebra). Then $\tau_2 = \operatorname{span}(\overline{n}_\xi)$ and $A_3(u(t_0))$ is not a subalgebra. We have at t_0: for RRT, $\overline{m}_3 = c\overline{m}_2$, $0 \neq c \in \mathbb{R}$, $\dot{Y}_c = (\dot{u}_1\dot{u}_2 + c(\dot{u}_1\dot{u}_3 + \dot{u}_2\dot{u}_3))(\overline{0}, \overline{\omega} \times \overline{m}_2)$, $\overline{\omega} \cdot \overline{m}_2 = 0$ for RTR, $\overline{m}_3 = c\overline{m}_2$,

$0 \neq c \in \mathbb{R}$, $\dot{Y}_c = (\dot{u}_1\dot{u}_2 + c\dot{u}_1\dot{u}_3 - \dot{u}_2\dot{u}_3)(\overline{0}, \overline{\omega} \times \overline{m}_2)$, $\overline{\omega} \cdot \overline{m}_2 = 0$; and for TRR, $\overline{m}_3 - \overline{m}_2 = c\overline{m}_1$, $0 \neq c \in \mathbb{R}$, $\dot{Y}_c = (\dot{u}_1\dot{u}_2 + \dot{u}_1\dot{u}_3 - c\dot{u}_2\dot{u}_3)(\overline{0}, \overline{\omega} \times \overline{m}_1)$, $\overline{\omega} \cdot \overline{m}_1 = 0$. We know that a motion is asymptotic at a singular position $u(t_0)$ only if the Coriolis acceleration is zero. A singular motion ($u_2(t) = u_2(t_0) = \text{const}$, $\dot{u}_2(t) = 0$) can be only trivial asymptotic when only one joint works. Thus we get

Proposition 8. *Let Υ_{A_3} be a robot of spherical rank 1 with two revolute joints. Then a motion is nontrivial asymptotic at the singular position $u(t_0)$ iff at t_0 all joints work and for* RRT, RTR, TRR *we have $(\dot{u}_1\dot{u}_2 + c(\dot{u}_1\dot{u}_3 + \dot{u}_2\dot{u}_3)) = 0$, $(\dot{u}_1\dot{u}_2 + c\dot{u}_1\dot{u}_3 - \dot{u}_2\dot{u}_3) = 0$, $(\dot{u}_1\dot{u}_2 + \dot{u}_1\dot{u}_3 - c\dot{u}_2\dot{u}_3) = 0$ at t_0, respectively. The singular motion is trivial asymptotic.*

(2) Let us assume that $u(t_0)$ is a regular position, $\overline{\omega} \in \tau_2$ and $A_3(u)$ is not a subalgebra. Then $\overline{\omega} = c_1\overline{m} + c_2\overline{n}_\xi$, c_1, $c_2 \in \mathbb{R}$, $c_1 \neq 0$, where $\overline{m}$ is the direction of the axis of the prismatic joint and $\overline{n}_\xi$ is the normal vector of the plane ξ. The axis of the prismatic joint is parallel to the axes of the revolute joints iff $c_2 = 0$. This position does not vary to time. If the axis of the prismatic joint is not parallel to the axes of the revolute joints then always $u_2 = \tilde{u}_2$, when $\overline{\omega} = c_1\overline{m} + c_2\overline{n}_\xi$.

b1) For RRT: if $\overline{\omega} = \overline{m}_3$ then $\dot{Y}_c = \dot{u}_1\dot{u}_2(\overline{0}, \overline{\omega} \times \overline{m}_2)$, $\overline{\omega} \cdot \overline{m}_2 = 0$ for every (u). If $\overline{\omega} \neq \overline{m}_3$ then there is the position so that the axis o_3 turning around the axis o_2 gets into the position complanar with the space $\text{span}(\overline{\omega}, \overline{m}_2)$; i.e., $\overline{m}_3 = c_1\overline{\omega} + c_2\overline{m}_2$. Then $\dot{Y}_c = (\dot{u}_1\dot{u}_2 + c_2\dot{u}_1\dot{u}_3 + c_2\dot{u}_2\dot{u}_3)(\overline{0}, \overline{\omega} \times \overline{m}_2)$.

b2) For RTR: if $\overline{\omega} = \overline{m}_2$ then $\dot{Y}_c = \dot{u}_1\dot{u}_3(\overline{0}, \overline{\omega} \times \overline{m}_3)$, $\overline{\omega} \cdot \overline{m}_3 = 0$ for every (u). If $\overline{\omega} \neq \overline{m}_2$ then there is the position so that the normal $\overline{m}_3$ of the plane ξ is complanar with the space $\text{span}(\overline{\omega}, \overline{m}_2)$; i.e., $\overline{m}_3 = c_1\overline{\omega} + c_2\overline{m}_2$. Then $\dot{Y}_c = (\dot{u}_1\dot{u}_2 + c_2\dot{u}_1\dot{u}_3 - \dot{u}_2\dot{u}_3)(\overline{0}, \overline{\omega} \times \overline{m}_2)$.

b3) For TRR: if $\overline{m}_1 = \overline{\omega}$ then $\dot{Y}_c = \dot{u}_2\dot{u}_3(\overline{0}, \overline{\omega} \times (\overline{m}_3 - \overline{m}_2))$, for every (u). If $\overline{m}_1 \neq \overline{\omega}$ then there is the position so that the normal $\overline{m}_3 - \overline{m}_2$ of the plane ξ is complanar with the space $\text{span}(\overline{\omega}, \overline{m}_1)$; i.e., $\overline{m}_3 - \overline{m}_2 = c_1\overline{\omega} + c_2\overline{m}_1$. Then $\dot{Y}_c = (\dot{u}_1\dot{u}_2 + \dot{u}_1\dot{u}_3 - c_2\dot{u}_2\dot{u}_3)(\overline{0}, \overline{\omega} \times \overline{m}_1)$. We know, see Proposition 2, that in the case when $\overline{\omega} \in \tau_2$ the motion is asymptotic iff $\dot{Y}_c = 0$. We get

Proposition 9. *Let Υ_{A_3} be a robot of spherical rank 1 with two revolute joints and let the axis of the prismatic joint is complanar with the space* $\text{span}(\overline{\omega}, \overline{n}_\xi)$ *at t_0, i.e. $\overline{m} = c_1\overline{\omega} + c_2\overline{n}_\xi$. Then we have:*

a) *The zero Coriolis acceleration is a necessary condition for a motion to be asymptotic at t_0.*

b) *A motion of the robot* Υ_{A_3} *is nontrivial asymptotic at the point* $u(t_0)$ *iff in the cases of* RRT, RTR, TRR *the equalities* $(\dot{u}_1\dot{u}_2 + c_2\dot{u}_1\dot{u}_3 + c_2\dot{u}_2\dot{u}_3) = 0$, $(\dot{u}_1\dot{u}_2 + c_2\dot{u}_1\dot{u}_3 - \dot{u}_2\dot{u}_3) = 0$, $(\dot{u}_1\dot{u}_2 + \dot{u}_1\dot{u}_3 - c_2\dot{u}_2\dot{u}_3) = 0$ *are valid at* t_0, *respectively.*

c) *A motion of the robot* Υ_{A_3}, *whose all axes are parallel to each other* $(c_2 = 0)$, *is nontrivial asymptotic iff the prismatic joint and only one revolute joint work.*

(3) Let $\dim CA = 2$ and let $A_3(u)$ be not a subalgebra. Then $CA \cap A_3(u) = K$ is the Klein subspace, $K = \mathrm{span}(\hat{Y})$, $\hat{Y} \in \tau$ and the direction of $\hat{Y}$ is perpendicular to $\overline{\omega}$. A motion is asymptotic at the point (u) iff $\dot{u}_1\dot{u}_2[Y_1, Y_2] + \dot{u}_1\dot{u}_3[Y_1, Y_3] + \dot{u}_2\dot{u}_3[Y_2, Y_3] = \lambda\hat{Y}$, $\lambda \in \mathbb{R}$. We get

b$_1$) for RRT: $[Y_2, Y_3] = [Y_1, Y_3]$ and $[Y_1, Y_2]$, $[Y_1, Y_3]$ are the basis elements of the space CA and $\hat{Y} = k_2[Y_1, Y_2] + k_3[Y_1, Y_3]$, $k_2, k_3 \in \mathbb{R}$. Then the motion is asymptotic iff $\dot{u}_1\dot{u}_2[Y_1, Y_2] + (\dot{u}_1\dot{u}_3 + \dot{u}_2\dot{u}_3)[Y_1, Y_3] = \lambda(k_2[Y_1, Y_2] + k_3[Y_1, Y_3])$ and this occurs and only if $\dot{u}_1\dot{u}_2 = \lambda k_2$, $(\dot{u}_1 + \dot{u}_2)\dot{u}_3 = \lambda k_3$.

b$_2$) for RTR: $[Y_2, Y_3] = -[Y_1, Y_2]$ and $[Y_1, Y_2]$, $[Y_1, Y_3]$ are the basis elements of the space CA and $\hat{Y} = k_2[Y_1, Y_2] + k_3[Y_1, Y_3]$, $k_2, k_3 \in \mathbb{R}$. Then the motion is asymptotic iff $(\dot{u}_1\dot{u}_2 - \dot{u}_2\dot{u}_3)[Y_1, Y_2] + \dot{u}_1\dot{u}_3[Y_1, Y_3] = \lambda(k_2[Y_1, Y_2] + k_3[Y_1, Y_3])$ and this occurs if and only if $\dot{u}_2(\dot{u}_1 - \dot{u}_3) = \lambda k_2$, $\dot{u}_1\dot{u}_3 = \lambda k_3$.

b$_3$) for TRR: $[Y_1, Y_3] = [Y_1, Y_2]$ and $[Y_1, Y_2]$, $[Y_2, Y_3]$ are the basis elements of the space CA and $\hat{Y} = k_2[Y_1, Y_3] + k_3[Y_2, Y_3]$, $k_2, k_3 \in \mathbb{R}$. Then the motion is asymptotic iff $(\dot{u}_1\dot{u}_2 + \dot{u}_1\dot{u}_3)[Y_1, Y_2] + \dot{u}_2\dot{u}_3[Y_2, Y_3] = \lambda(k_2[Y_1, Y_3] + k_3[Y_2, Y_3])$ and this occurs if and only if $\dot{u}_1(\dot{u}_2 + \dot{u}_3) = \lambda k_2$, $\dot{u}_2\dot{u}_3 = \lambda k_3$.

So we have

Proposition 10. *Let* Υ_{A_3} *be a robot of spherical rank* 1 *with two revolute joints and let the axis of the prismatic joint be not complanar with the space* $\mathrm{span}(\overline{\omega}, \overline{n}_\xi)$ *at* t_0, *i.e.* $\overline{m} \neq c_1\overline{\omega} + c_2\overline{n}_\xi$. *Then a motion is asymptotic at* t_0 *iff the joint velocities at* t_0 *satisfy* $\dot{u}_1\dot{u}_2 = \lambda k_2$, $(\dot{u}_1 + \dot{u}_2)\dot{u}_3 = \lambda k_3$ *for* RRT, $\dot{u}_2(\dot{u}_1 - \dot{u}_3) = \lambda k_2$, $\dot{u}_1\dot{u}_3 = \lambda k_3$ *for* RTR *and* $\dot{u}_1(\dot{u}_2 + \dot{u}_3) = \lambda k_2$, $\dot{u}_2\dot{u}_3 = \lambda k_3$ *for* TRR, *where* $\lambda \in \mathbb{R}$ *and* k_2, k_3 *are the coefficients of the linear combination of* $\hat{Y} = (0, (\overline{\omega} \cdot \overline{b}_3)\overline{b}_2 - (\overline{\omega} \cdot \overline{b}_2)\overline{b}_3)$ *in the canonical basis of the Coriolis space* CA. *If these relations are true for any admissible* t *then the motion is asymptotic.*

In this case there are nontrivial asymptotic motions with nonzero Coriolis acceleration.

3c. Robots with 3 revolute joints

These robots have the axes of the joints parallel and different from each other (the robots are planar). The elements Y_i satisfy $Y_1 = (\overline{\omega}, \overline{0})$, $Y_2 = (\overline{\omega}, \overline{m}_2)$, $Y_3 = (\overline{\omega}, \overline{m}_3)$, $\overline{\omega} \cdot \overline{m}_2 = 0$, $\overline{\omega} \cdot \overline{m}_3 = 0$, $\overline{m}_3 \neq \overline{m}_2 \neq \overline{0}$. Let us denote the planes $\xi_2 = (o_1, o_2)$ and

$\xi_3 = (o_1, o_3)$. Then $\overline{m}_2$ is the normal vector to the plane ξ_2 and $\overline{m}_3$ is the normal vector to the plane ξ_3. For the elements B_i we have $B_1 = Y_1$, $B_2 = Y_2 - Y_1 = (\overline{0}, \overline{b}_2 = \overline{m}_2)$, $B_2 = Y_3 - Y_1 = (\overline{0}, \overline{b}_3 = \overline{m}_3)$. Because $\tau_2 = \mathrm{span}(\overline{m}_2, \overline{m}_3)$, $\overline{\omega} \cdot \tau_2 = 0$ and $[Y_1, Y_2] = (\overline{0}, \overline{\omega} \times \overline{m}_2)$, $[Y_1, Y_3] = (\overline{0}, \overline{\omega} \times \overline{m}_3)$, $[Y_1, Y_2] = (\overline{0}, \overline{\omega} \times \overline{m}_3 - \overline{\omega} \times \overline{m}_2)$ we conclude that $A_3(u)$ is a subalgebra in a regular position. If the plane ξ_3 turning around the axis o_2 coincides with the plane ξ_2 then the robot is in a singular position at t_0; i.e., $\overline{m}_3 = c\overline{m}_2$, $c \in \mathbb{R}$. In regular positions we have $\dim CA = 2$ and all motions are asymptotic while $\dim CA = 1$ in singular positions and the Coriolis acceleration satisfies $\dot{Y}_C = \dot{u}_1(\dot{u}_2 + c\dot{u}_3)(\overline{0}, \overline{\omega} \times \overline{m}_2)$.

Proposition 11. *Let* RRR *be a robot the revolute joint axes of which are parallel. Then its position $u(t_0)$ is singular if all axes of the joints lie in a plane. $A_3(u)$ is a subalgebra in the regular position and $K = A_3(u)$. $A_3(u)$ is not a subalgebra in the singular position. A motion through the singular position $u_2(t_0) = \hat{u}_2$ is asymptotic at $u_2(t_0) = \hat{u}_2$*

a) *if the* 1*st joint does not work or*

b) *the ratio of the joint velocities of the* 2*nd and* 3*rd joints at t_0 is $-c$.*

A singular motion $(u_2(t) = \hat{u}_2)$ can be only trivial asymptotic.

Let us present a survey of all nontrivial asymptotic motions of the robots of spherical rank one.

1. The robots with one revolute joint (RTT, TRT, TTR).
   a) Let the directions of the joint axes be dependent (i.e., $\overline{\omega} = c_2\overline{b}_2 + c_3\overline{b}_3$) and let $c_2c_3 \neq 0$ in the cases RTT, TTR. Then a robot motion is nontrivial asymptotic iff all joints work and the ratio of the joint velocities of the prismatic joints is $c_2 : c_3$.
   b) Let the axis of the revolute joint be parallel to one axis of the prismatic joint (i.e., $c_2c_3 = 0$). Then a robot motion is nontrivial asymptotic iff the revolute joint and only the prismatic joint whose axis is parallel to the revolute joint axis work.
   c) Let the directions of the joint axes be independent (i.e., $\overline{\omega} \neq c_2\overline{b}_2 + c_3\overline{b}_3$). Then a robot motion is nontrivial asymptotic iff the joint velocities satisfy for any admissible t: $\dot{u}_1\dot{u}_2 = k_2\lambda$, $\dot{u}_1\dot{u}_3 = k_3\lambda$ for RTT, $\dot{u}_1\dot{u}_2 = k_2\lambda$, $\dot{u}_2\dot{u}_3 = k_3\lambda$ for TRT, $\dot{u}_1\dot{u}_3 = k_2\lambda$, $\dot{u}_2\dot{u}_3 = k_3\lambda$ for TTR, where k_2, k_3 are the coefficients of the linear combination of the Klein direction in the canonical basis of the Coriolis space.
2. The robots with two revolute joints (RRT, RTR, TRR).
   a) Let the joint axes be parallel. Then a robot motion is nontrivial asymptotic iff one revolute joint does not work.

b) Let the axis of the prismatic joint be not complanar with the space $\operatorname{span}(\overline{\omega}, \overline{n}_\xi)$. Then a robot motion is nontrivial asymptotic iff for the joint velocities and any admissible t we have: $\dot{u}_1\dot{u}_2 = \lambda k_2$, $(\dot{u}_1 + \dot{u}_2)\dot{u}_3 = \lambda k_3$ for RRT, $\dot{u}_2(\dot{u}_1 - \dot{u}_3) = \lambda k_2$, $\dot{u}_1\dot{u}_3 = \lambda k_3$ for RTR and $\dot{u}_1(\dot{u}_2 + \dot{u}_3) = \lambda k_2$, $\dot{u}_2\dot{u}_3 = \lambda k_3$ for TRR, where k_2, k_3 are the coefficients of the linear combination of $\hat{Y} = (0, (\overline{\omega} \cdot \overline{b}_3)\overline{b}_2 - (\overline{\omega} \cdot \overline{b}_2)\overline{b}_3)$ in the basis of the Coriolis space.

3. The robots with three revolute joints (RRR).

   In this case, there are only trivial asymptotic motions.

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