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NECESSARY AND SUFFICIENT CONDITIONS FOR THE OSCILLATION OF FORCED NONLINEAR SECOND ORDER DELAY DIFFERENCE EQUATION

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In this paper the authors give necessary and sufficient conditions for the oscillation of solutions of nonlinear delay difference equations of Emden–Fowler type in the form $\Delta^2 y_{n-1} + q_n y^{\gamma}_{\sigma(n)} = g_n$, where γ is a quotient of odd positive integers, in the superlinear case ($\gamma > 1$) and in the sublinear case ($\gamma < 1$).

1. INTRODUCTION

In several recent papers [1, 5, 6, 9, 10, 11, 13, 15] oscillation and nonoscillation of solutions of second order nonlinear difference equations have been investigated. Difference equations appear as natural description of observed evolution phenomena as well as in the study of discretization methods for differential equations. The application of the theory of difference equations is rapidly growing to various fields such as numerical analysis, economics, chemistry, population dynamics, queueing theory, control theory and computer science, in particular, the connection between the theory of difference equations and computer science has become more important in recent years, because of the successful use of computers to solve difficult problems arising in applications. For general background on difference equations and for applications to many diverse fields, one can refer to [1, 7].

This paper deals with the study of the oscillation problem for the solutions of the forced nonlinear difference equation

$$\Delta^2 y_{n-1} + q_n y^{\gamma}_{\sigma(n)} = g_n, \quad n \in \mathcal{N} = \{1, 2, 3, \ldots\} \tag{1}$$

where Δ denotes the forward difference operator defined by $\Delta y_n = y_{n+1} - y_n$, γ is a quotient of odd positive integers, $\{q_n\}$, $\{g_n\}$ are non-negative real sequences and $\{\sigma(n)\}$ is an increasing sequence of integers with $\sigma(n) \le n$ and $\lim \sigma(n) \to \infty$ as $n \to \infty$. When $\sigma(n) = n$, equation (1) takes the form

$$\Delta^2 y_{n-1} + q_n y^{\gamma}_n = g_n, \quad \gamma > 0 \tag{2}$$

which is the forced discrete Emden–Fowler equation.

By a solution of equation (1), we mean a real sequence $\{y_n\}$ satisfying equation (1) so that $\sup_{n \geq m} |y_n| > 0$ for any $m \in \mathcal{N}$. We always assume that such solutions of equation (1) exist. A solution of equation (1) is called oscillatory if there is no end of n_1 and n_2 ($n_1 < n_2$) in $\mathcal{N}$ such that $y_{n_1} y_{n_2} \leq 0$; otherwise it is called nonoscillatory. Clearly a nonoscillatory solution of equation (1) must be eventually of fixed sign.

Although several results regarding oscillation theory in the discrete case are similar to those of already known in the continuous case, the adaptation from the continuous to the discrete is not direct, but it requires some special devices. Further it has been shown in [4] that there exist some properties of differential equations which do not carry over directly to the corresponding difference equations. Therefore, it is useful to study the oscillatory and nonoscillatory behavior of solutions of difference equations.

In [1, 2, 3, 12, 14], one can find different conditions sufficient for the oscillation of all solutions of equation of type (2). The technique of the proof mainly depends on the assumption that there exists an oscillatory sequence $\{h_n\}$ such that $\Delta^2 h_{n-1} = g_n$. Other results on forced oscillations on nonlinear second order equations can be found in [1, 11, 12] and for linear equations in [9].

In the present paper, for nonnegative $\{q_n\}$ and bounded $\{h_n\}$ ($\Delta^2 h_{n-1} = g_n$), we give necessary and sufficient conditions for equation (1) to be oscillatory in the sublinear and the superlinear cases. The conditions resemble those obtained earlier for the unforced equation [1, 5].

All our results here could be obtained equally well for the difference equation

$$\Delta^2 y_{n-1} + q_n |y_{\sigma(n)}|^\gamma \operatorname{sgn} y_{\sigma(n)} = g_n, \quad \gamma > 0$$

with no essential change in the proofs given. For simplicity of notation, we instead restrict γ to be a quotient of odd positive integers and discuss equation (1).

2. SUPERLINEAR EQUATIONS

Let $\gamma > 1$ in equation (1) and assuming the following conditions:

(i) $q_n \geq 0$ for all $n \geq 1$, and for every $N \geq 1$, $q_n > 0$ for some $n > N$,

(ii) there exists a bounded sequence $\{h_n\}$ such that $\Delta^2 h_{n-1} = g_n$ and let $|h_n| \leq M$ for all n.

Theorem 1. Assume conditions (i) and (ii) are satisfied. If

$$\sum_{n=1}^{\infty} n\, q_n < \infty \tag{3}$$

then equation (1) has a nonoscillatory solution.

Proof. Choose $N \in \mathcal{N}$ sufficiently large so that

$$\sum_{n=N}^{\infty} n\, q_n < \frac{1}{2} \min\left\{ \frac{1}{(2M+1)^\gamma}, \frac{1}{\gamma(2M+1)^{\gamma-1}} \right\}. \tag{4}$$

Consider the complete metric space S consisting of all real sequences $y = \{y_n\}$, $n \in \mathcal{N}$ and satisfying the inequalities

$$\frac{1}{2} \le y_n \le 2M + 1 \tag{5}$$

endowed with the metric

$$\rho(x, y) = \sup_{n \in \mathcal{N}} |x_n - y_n|.$$

The operator T defined by

$$\begin{aligned} (Ty)_n &= (M+1) + h_n - \sum_{s=n+1}^{\infty} (s-n)\, q_s\, y^{\gamma}_{\sigma(s)}, \quad n \ge N, \\ &= Ty_N, \qquad 1 \le n \le N \end{aligned}$$

maps S into itself. In fact, if $y \in S$ then $(Ty)_n \le 2M + 1$ since $h_n \le M$. Moreover from (4) and (5) we see that

$$(Ty)_n \ge 1 - \sum_{s=N}^{\infty} s\, q_s\, y^{\gamma}_{\sigma(s)} \ge 1 - (2M+1)^{\gamma} \sum_{s=N}^{\infty} s\, q_s \ge \frac{1}{2}.$$

Now, we shall show that T has a fixed point. For this,

$$|(Ty)_n - (Tz)_n| \le \sum_{s=N}^{\infty} (s-n)\, q_s \left| y^{\gamma}_{\sigma(s)} - z^{\gamma}_{\sigma(s)} \right| \le \sum_{s=N}^{\infty} s\, q_s \left| y^{\gamma}_{\sigma(s)} - z^{\gamma}_{\sigma(s)} \right|.$$

Using the Mean value theorem applied to the function $f(x) = x^{\gamma}$, we see that

$$\rho(Ty, Tz) \le \gamma \rho(y, z) \sum_{s=N}^{\infty} s\, q_s x^{\gamma-1}_{\sigma(s)}$$

where $x_{\sigma(s)}$ lies between $y_{\sigma(s)}$ and $z_{\sigma(s)}$, $s \ge N$, that is, satisfies the inequalities (5). So, we have

$$\rho(Ty, Tz) \le \gamma \rho(y, z)\, (2M+1)^{\gamma-1} \sum_{s=N}^{\infty} s\, q_s.$$

From (4), we see that $\rho(Ty, Tz) \le \frac{1}{2}\rho(y, z)$. Thus T is a contraction on S, so by the known Banach contraction mapping theorem, T has a unique fixed point $y \in S$, that is

$$y_n = M + 1 + h_n - \sum_{s=n+1}^{\infty} (s-n)\, q_s\, y^{\gamma}_{\sigma(s)}, \quad n \ge N.$$

Taking difference twice, we see that $\{y_n\}$ is a nonoscillatory solution of equation (1). The proof is now complete. □

Next we prove that the condition

$$\sum_{n=1}^{\infty} n\, q_n = \infty \tag{6}$$

is sufficient for all solutions of equation (1) to be oscillatory assuming that $\{h_n\}$ is oscillatory and satisfies the condition:

(iii) $\{h_n\}$ is oscillatory and there exist two sequences $\{n_j\}$, $\{\overline{n}_j\}$ tending to infinity such that for all j

$$\begin{aligned} h_{n_j} &= \inf\{h_n : n \geq n_j\} \\ h_{\overline{n}_j} &= \sup\{h_n : n \geq \overline{n}_j\}. \end{aligned}$$

Theorem 2. Assume that conditions (i), (ii) and (iii) are satisfied. If condition (6) holds then all solutions of equation (1) are oscillatory.

Proof. Suppose $\{y_n\}$ is a nonoscillatory solution of equation (1), and assume without loss of generality that $y_n > 0$, $y_{\sigma(n)} > 0$ for all $n \geq N$, for some $N > 0$. Put $y_n = z_n + h_n$. Then z_n satisfies the equation

$$\Delta^2 z_{n-1} + q_n\, y^{\gamma}_{\sigma(n)} = 0. \tag{7}$$

From this we see that $\Delta^2 z_{n-1} \leq 0$. Hence z_n is of one sign and, definitely, it is positive otherwise $\{h_n\}$ will not be oscillatory. Further, if $\Delta z_n \leq 0$ for $n \geq N$, then there exists an integer $N_1 > N$ such that $\Delta z_n \leq \Delta z_{N_1} < 0$. Summing the last inequality from N_1 to $n-1$ and then taking $n \to \infty$, we see that $z_n \to -\infty$, a contradiction. Thus $\Delta z_n > 0$ and we have

$$z_n > 0, \quad \Delta z_n > 0, \quad \Delta^2 z_{n-1} \leq 0. \tag{8}$$

From the increase of $\{z_n\}$ and condition (iii) on $\{h_n\}$, we easily see that there exists an integer $N_1 > N$ such that $z_n + h_n \geq \beta_0 > 0$ for all $n > N_1$, that is,

$$y_n \geq \beta_0 > 0 \quad \text{for } n > N_1. \tag{9}$$

This implies that there exists a positive number β such that

$$y_n \geq \beta\, z_n. \tag{10}$$

If this is not true, then there exists a sequence $\{n_j\}$ tending to infinity such that

$$y_{n_j} = z_{n_j} + h_{n_j} \leq \frac{1}{j}\, z_{n_j}. \tag{11}$$

So $\left(1 - \frac{1}{j}\right) z_{n_j} + h_{n_j} \leq 0$. If $z_{n_j} \to \infty$, then h_{n_j} will tend to $-\infty$ which contradicts the fact $\{h_n\}$ is bounded. If, on the other hand, $z_{n_j} \to$ constant, then $y_{n_j} \to 0$ which contradicts (9). Hence (10) is true. Now, put

$$w_n = \frac{-n\Delta z_{n-1}}{z^{\gamma}_{\sigma(n-1)}}, \quad n > N_1.$$

From equation (7), we obtain

$$\Delta w_n = \frac{n\, q_n y^{\gamma}_{\sigma(n)}}{z^{\gamma}_{\sigma(n)}} - \frac{\Delta z_n}{z^{\gamma}_{\sigma(n)}} + \frac{n \Delta z_{n-1} \Delta z^{\gamma}_{\sigma(n)}}{z^{\gamma}_{\sigma(n-1)}\, z^{\gamma}_{\sigma(n)}}.$$

From (8) and (10) and the Mean value theorem, we have

$$\Delta w_n \geq \beta^{\gamma}\, n\, q_n - \frac{\Delta z^{\gamma}_{\sigma(n)}}{z^{\gamma}_{\sigma(n)}} + \frac{c\, n\, w^2_{n+1}}{(n+1)^2},$$

where $c = \gamma\, z^{\gamma-1}_{\sigma(N_1-1)}$. Summing the last inequality from N_1 to $n-1$, we get

$$\begin{aligned} w_n &\geq w_{N_1} + \beta^{\gamma} \sum_{s=N_1}^{n-1} s\, q_s - \int_{z_{\sigma(N_1)}}^{z_{\sigma(n)}} \frac{\mathrm{d}s}{s^{\gamma}} + c \sum_{s=N_1}^{n-1} \frac{s\, w^2_{s+1}}{(s+1)^2} \\ &= w_{N_1} + \beta^{\gamma} \sum_{s=N_1}^{n-1} s\, q_s + \frac{1}{\gamma-1}\left[z^{-\gamma+1}_{\sigma(n)} - z^{-\gamma+1}_{\sigma(N_1)}\right] + c \sum_{s=N_1}^{n-1} \frac{s}{(s+1)^2}\, w^2_{s+1}. \end{aligned}$$

From (6), we see that there exists an integer $N_2 > N_1$ such that

$$w_n \geq c \sum_{s=N_1}^{n-1} \frac{s}{(s+1)^2}\, w^2_{s+1}, \quad n > N_2. \tag{12}$$

Letting $R_n = c \sum_{s=N_1}^{n-1} \frac{s}{(s+1)}\, w^2_{s+1}$, we see that $\Delta R_n = \left[\frac{cn}{(n+1)^2}\right] w^2_{n+1}$. From (12), we get $\Delta R_n = \left[\frac{cn}{(n+1)^2}\right] R^2_{n+1}$. Dividing by R^2_{n+1} and summing from N_2 to $n-1$, we get

$$\int_{R_{N_2}}^{R_n} \frac{\mathrm{d}s}{s^2} \geq \sum_{s=N_2}^{n-1} \frac{\Delta R_s}{R^2_{s+1}} > c \sum_{s=N_2}^{n-1} \frac{s}{(s+1)^2}.$$

Thus

$$c \sum_{s=N_2}^{n-1} \frac{s}{(s+1)^2} \leq \frac{1}{R_{N_2}}$$

which proves a contradiction as $n \to \infty$. This completes the proof of the theorem. □

3. SUBLINEAR EQUATIONS

Assume $0 < \gamma < 1$ in equation (1).

Theorem 3. Assume that conditions (i) and (ii) are satisfied. If

$$\sum_{n=1}^{\infty} n^{\gamma}\, q_n < \infty$$

then equation (1) has a nonoscillatory solution.

Proof. Choose $N \in \mathcal{N}$ large enough so that

$$\sum_{n=N}^{\infty} n^{\gamma} q_n < \frac{1}{8}, \quad N > 8M. \tag{13}$$

Let B_N be the Banach space of all real sequences $y = \{y_n\}$, $n \geq N$, with norm $\|y\| = \sup_{n \geq N} |y_n|$. We define a partial ordering on B_N as follows: for given x, $y \in B_N$, $x \leq y$ means $x_n \leq y_n$ for $n \geq N$. Let $S = \{y \in B_N : \frac{n}{2} \leq y_n \leq n,\, n \in N\}$. Define the operator T acting in S by

$$(Ty)_n = \frac{n}{2} + M + h_n + \sum_{s=N}^{n} s\, q_s\, y^{\gamma}_{\sigma(s)} + n \sum_{s=n+1}^{\infty} q_s\, y^{\gamma}_{\sigma(s)}, \quad n \geq N.$$

Now we will show that T maps S into itself. In fact, for $y \in S$, $(Ty)_n \geq \frac{n}{2}$, and from (13) we have

$$(Ty)_n \leq \frac{n}{2} + 2M + n \sum_{s=N}^{n} s^{\gamma} q_s + n \sum_{s=N}^{\infty} s^{\gamma} q_s \leq \frac{n}{2} + \frac{n}{4} + \frac{n}{8} + \frac{n}{8} = n.$$

From the hypotheses one can easily see that T is an increasing mapping. Hence, by the Knaster–Tarski fixed point theorem [8] there exist $y \in S$ such that $Ty = y$, that is,

$$y_n = \frac{n}{2} + M + h_n + \sum_{s=N}^{n} s\, q_s\, y^{\gamma}_{\sigma(s)} + n \sum_{s=n+1}^{\infty} q_s\, y^{\gamma}_{\sigma(s)}.$$

Taking difference twice, we see that $\{y_n\}$ is a nonoscillatory solution of equation (1). This completes the proof. □

Finally, we give sufficient conditions for the oscillation of all solutions of equation (1) when $0 < \gamma < 1$.

Theorem 4. Assume that conditions (i), (ii) and (iii) are satisfied. If

$$\sum_{n=1}^{\infty} n^{\gamma} q_n = \infty \tag{14}$$

then all solutions of equation (1) are oscillatory.

Proof. Suppose $\{y_n\}$ is a nonoscillatory solution of equation (1), and assume without loss of generality that $y_n > 0$, $y_{\sigma(n)} > 0$ for all $n \geq N$, for some $N \in \mathcal{N}$. Put $y_n = z_n + h_n$. Then z_n satisfies equation (7). Further from the proof of Theorem 2, we obtain

$$\Delta^2 z_n \leq 0, \quad \Delta z_n > 0, \quad z_n > 0, \quad y_{\sigma(n)} > \beta\, z_{\sigma(n)} \qquad \text{for } n \geq N \tag{15}$$

and from the Lemma 4.1 of Hooker and Patula [5], there exists some constant $b > 0$ such that

$$z_n < bn. \tag{16}$$

Now define $w_n = \frac{-n^\gamma \Delta z_{n-1}}{z^\gamma_{\sigma(n-1)}}$ and we obtain

$$\Delta w_n = \frac{n^\gamma q_n y^\gamma_{\sigma(n)}}{z^\gamma_{\sigma(n)}} - \frac{\Delta n^\gamma \, \Delta z_n}{z^\gamma_{\sigma(n)}} + \frac{n^\gamma \Delta z_{n-1} \, \Delta z^\gamma_{\sigma(n-1)}}{z^\gamma_{\sigma(n-1)} \, z^\gamma_{\sigma(n)}}.$$

From (15) and (16) and the Mean value theorem yields

$$\Delta w_n \geq \beta^\gamma \, n^\gamma \, q_n - \gamma(n+1)^{\gamma-1} \frac{\Delta z_{\sigma(n)}}{z^\gamma_{\sigma(n)}} + \frac{\gamma n^{2\gamma-1}}{(n+1)^{2\gamma}} b^{\gamma-1} \, w^2_{n+1}.$$

Summing from N to $n-1$, using the assumption (14) of the theorem and that $0 < \gamma < 1$, we get for sufficiently large n

$$w_n \geq \gamma \, b^{\gamma-1} \sum_{s=N}^{n-1} \frac{s^{2\gamma-1}}{(s+1)^{2\gamma}} \, w^2_{s+1}.$$

Rest of the proof is similar to that of Theorem 2 and hence the details are omitted. □

Remark. When $g_n = 0$, the theorems of Hooker and Patula [5] for the oscillation of unforced equation (2) follow as consequences of Theorems 1 – 4. Further, the proof given here for the Theorems 2 and 4 are different from that of Hooker and Patula [5].

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