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A NOTE ON A CLASS OF EQUILIBRIUM PROBLEMS WITH EQUILIBRIUM CONSTRAINTS[1]

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This paper is dedicated to Prof. Dr. Dr.h.c. František Nožička on the occasion of his 85th birthday.

The paper concerns a two-level hierarchical game, where the players on each level behave noncooperatively. In this way one can model e. g. an oligopolistic market with several large and several small firms. We derive two types of necessary conditions for a solution of this game and discuss briefly the possibilities of its computation.

Keywords: hierarchical game, Nash equilibrium, stationarity conditions

AMS Subject Classification: 49J40, 49J52, 90C

1. INTRODUCTION

The behaviour of firms on an oligopolistic market is usually modeled via the Cournot–Nash equilibrium concept ([13, 18]). In fact, it is the classical Nash equilibrium, where each player (firm) maximizes its profit subject to production constraints. Assuming that one from the firms has a temporal advantage over the others, this firm can increase its profit by replacing a Cournot–Nash strategy by a Stackelberg strategy ([6, 17, 18]). This firm (called Leader) computes its new strategy under the assumption that the remaining firms (Followers) will share the rest of the market again in the noncooperative way. One obtains a bilevel structure with a Cournot–Nash equilibrium (parametrized by the Leader's strategy) on the lower level. To compute the Leader's strategy, one has thus to solve a so-called mathematical program with equilibrium constraints (MPEC) ([8]). It might happen, however, that the standard Cournot–Nash strategy is simultaneously deserted by two or more firms. In such a case each of them has to make some assumptions not only about the behaviour of the Followers, but also about the behaviour of the remaining Leaders. Concerning this behaviour, two "extreme" situations can be distinguished:

(i) All Leaders cooperate;

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(ii) The Leaders act noncooperatively, i.e. their strategies belong to the Cournot–Nash equilibrium on the upper level.

In the former case one has to do with a special multiobjective MPEC. Concerning necessary optimality conditions, this case has been carefully investigated in [11, 12], using the tools of the generalized differential calculus of B. Mordukhovich. The latter case arises in [7], where the authors model deregulated electricity markets under an independent system operator regime. One obtains a rather complex equilibrium problem generated by a number of coupled MPECs. In both cases one can speak, in accordance with [20], about equilibrium problems with equilibrium constraints (EPECs). This terminology could also be used in other situations where, for example, the Leaders build some coalitions.

The aim of this note is to investigate the situation (ii), also using the above mentioned generalized differential calculus. The organization is as follows:

In the next section we give a rigorous definition of a noncooperative solution to EPEC. Thereby our model is not necessarily associated with an oligopolistic market; we will consider a general game-theoretical framework. We will also pay a small attention to the existence of such solutions, which seems to be a very difficult question. Section 3 is then devoted to necessary conditions for a vector of strategies to be a noncooperative solution to EPEC. We also apply the so-called implicit programming approach ([18]) in this context, which leads to another type of necessary conditions. At the end we discuss briefly two possible approaches to the computation of a noncooperative solution.

Our notation is basically standard. For a multifunction $Q[\mathbb{R}^n \rightsquigarrow \mathbb{R}^m]$, $\operatorname{Gph} Q := \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^m \mid y \in Q(x)\}$. If D is a cone with vertex at the origin, then D^0 is its negative polar cone. For a single-valued locally Lipschitz map $f[\mathbb{R}^n \to \mathbb{R}^m]$, $\bar{\partial} f(x)$ denotes the Clarke's generalized Jacobian of f at x ([2, Def. 2.6.1]).

For the reader's convenience, we close this section with three fundamental definitions from Mordukhovich's generalized differential calculus used throughout the paper.

Consider a set $\Pi \subset \mathbb{R}^p$.

Definition 1.1. Let $a \in \operatorname{cl} \Pi$. The nonempty cone

$$T_\Pi(a) := \limsup_{t \downarrow 0} \frac{\Pi - a}{t}$$

is called the *contingent* cone to Π at a. The *limiting* normal cone to Π at a, denoted $N_\Pi(a)$, is defined by

$$N_\Pi(a) = \limsup_{a' \xrightarrow{\operatorname{cl} \Pi} a} (T_\Pi(a'))^0.$$

If Π is convex, $N_\Pi(a)$ amounts to the standard normal cone to Π at a in the sense of convex analysis. The cone $N_\Pi(a)$ is generally nonconvex, but the multifunction $N_\Pi(\cdot)$ is upper semicontinuous at each point of $\operatorname{cl} \Pi$ (with respect to $\operatorname{cl} \Pi$), which is essential in the generalized differential calculus of B. Mordukhovich ([9, 10]).

Definition 1.2. Let $\varphi[\mathbb{R}^p \to \overline{\mathbb{R}}]$ be an arbitrary extended real-valued function and $a \in \operatorname{dom} \varphi$. The set

$$\partial\varphi(a) := \{a^* \in \mathbb{R}^p \mid (a^*, -1) \in N_{\operatorname{epi} \varphi}(a, \varphi(a))\}$$

is called the *limiting* subdifferential of φ at a.

Definition 1.3. Let $\Phi[\mathbb{R}^p \rightsquigarrow \mathbb{R}^q]$ be an arbitrary multifunction and $(a,b) \in \operatorname{cl} \operatorname{Gph} \Phi$. The multifunction $D^*\Phi(a,b)[\mathbb{R}^q \rightsquigarrow \mathbb{R}^p]$ defined by

$$D^*\Phi(a,b)(b^*) := \{a^* \in \mathbb{R}^p \mid (a^*, -b^*) \in N_{\operatorname{Gph} \Phi}(a,b)\}, \quad b^* \in \mathbb{R}^q,$$

is called the *coderivative* of Φ at (a,b).

2. PROBLEM FORMULATION

Consider a game of l players, where the ith player aims to minimize his objective f^i, $i = 1, 2, \ldots, l$, by using a strategy x^i from his *set of admissible strategies* $U^i \subset \mathbb{R}^n$. Assume that the set $\{1, 2, \ldots, l\}$ splits into two subsets I_1 and I_2. If $i \in I_1$, we will call the ith player a *Leader*, otherwise a *Follower*. It is assumed that for each vector of admissible Leaders' strategies $\bar{x}^i$, $i \in I_1$, the Followers will compute a noncooperative (Nash) equilibrium in their own "reduced" game, where the Leaders' strategies arise as parameters. To be able to describe mathematically this behaviour in a simple form, we will assume that for all $i \in I_2$ the objectives f^i are continuously differentiable and convex in x^i for all feasible strategies of the remaining players and that the sets U^i are nonempty, closed and convex. Then one can describe the Followers' behaviour by generalized equations (GEs)

$$0 \in \nabla_{x^j} f^j(\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^l) + N_{U^j}(\bar{x}^j), \quad j \in I_2, \tag{1}$$

where the Leaders' strategies x^i, $i \in I_1$, arise only as parameters ([18]). From the point of view of a Leader, relations (1) represent a constraint; the above game belongs thus to so-called equilibrium problems with equilibrium constraints (EPECs), introduced in [20]. In what follows, we will consequently refer to the above described game as to EPEC. As already mentioned in the Introduction, EPEC admits various solution concepts, dependent on the behaviour of the Leaders.

For the sake of simplicity, let us reorganize the players in such a way that $I_1 = \{1, 2, \ldots, k\}$ and $I_2 = \{k+1, k+2, \ldots, l\}$. To unburden the notation, let $x_L := (x^1, x^2, \ldots, x^k) \in \mathbb{R}^{nk}, x_F := (x^{k+1}, x^{k+2}, \ldots, x^l) \in \mathbb{R}^{n(l-k)}$ and, for $i \in I_1$, x_L^{-i} be the subvector of x_L from which the strategy x^i has been removed. Thus, $x_L^{-i} \in \mathbb{R}^{n(k-1)}$. Further, we denote by $F(x_L, x_F)$ the vector composed from the partial gradients $\nabla_{x^j} f^j(x^1, x^2, \ldots, x^l), j \in I_2$, and put $\Omega := \times_{j \in I_2} U^j$. For $i \in I_1$, the notation $f^i(x_L^{-i}, y, z)$ means the value of the objective of the ith player at the point (x_L, x_F) with $x^i = y$ and $x_F = z$. Correspondingly, $F(x_L^{-i}, y, z)$ is composed from the partial gradients of $f^j(x_L^{-i}, y, z)$, $j \in I_2$, with respect to the appropriate components of z.

Definition 2.1. ([20]) The vector of admissible strategies $\hat{x} := (\hat{x}_L, \hat{x}_F)$, is declared a *noncooperative* solution to EPEC, provided for all $i \in I_1$ the pair $(\hat{x}^i, \hat{x}_F)$ belongs to the set of (local) solutions to the MPEC

$$\begin{aligned} \text{minimize} \quad & f^i(\hat{x}_L^{-i}, y, z) \\ \text{subject to} \quad & \\ & 0 \in F(\hat{x}_L^{-i}, y, z) + N_\Omega(z) \\ & y \in U^i \end{aligned} \tag{2}$$

in variables y, z.

Each problem (2) is a standard MPEC so that the computation of a noncooperative solution to EPEC amounts to solving of k coupled MPECs. Problems (2) are generally nonconvex even if the Leaders' objectives and sets of admissible strategies satisfy the convexity assumptions, imposed on f^i and U^i for $i \in I_2$. This prevents to apply the existence theory of Nash ([14]). To grasp the existence question associated with a noncooperative solution to EPEC, consider the multifunctions P^i which assign to each admissible vector x_L^{-i} the set of (local) solutions to (2) with $\hat{x}_L^{-i}$ replaced by x_L^{-i}. Evidently, $\operatorname{Gph} P^i \subset X_{i=1}^l U^i$. Moreover, $(\hat{x}_L, \hat{x}_F)$ is a noncooperative solution to EPEC iff

$$(\hat{x}_L^{-i}, \hat{x}^i, \hat{x}_F) \in \operatorname{Gph} P^i \quad \text{for all } i \in I_1.$$

To ensure the existence of a noncooperative solution to EPEC, we have thus to analyze the structure of the maps P^i. The local behaviour of these maps has been studied in [21] in case of MPECs with equilibria governed by (generalized) complementarity problems. A globalized version of [21, Theorem 11] together with the Brouwer's Fixed-Point Theorem create a basis for proving the existence of noncooperative solutions to EPEC. This investigation goes, however, beyond the aims of this note and so we turn our attention to conditions which must necessarily be fulfilled by each noncooperative solution to EPEC.

3. NECESSARY CONDITIONS

Let us posit the following simplifying technical assumptions:

(A1) For $i = 1, 2, \ldots, k$ the objectives f^i are continuously differentiable on an open set containing $X_{i=1}^l U^i$.

(A2) For $j = k+1, k+2, \ldots, l$ the partial gradients $\nabla_{x^j} f^j(\cdot)$ are continuously differentiable on an open set containing $X_{i=1}^l U^i$.

This implies in particular that the map

$$F(x_L, x_F) = \begin{bmatrix} \nabla_{x^{k+1}} f^{k+1}(x_L, x_F) \\ \vdots \\ \nabla_{x^l} f^l(x_L, x_F) \end{bmatrix} \tag{1}$$

possesses continuous partial derivatives with respect to x^i, $i \in I_1$, and x_F, whenever the pair (x_L, x_F) is admissible. Observe that $\nabla_{x_F} F(x_L, x_F)$ amounts to the square matrix

$$\begin{bmatrix} \nabla^2_{x^{k+1}x^{k+1}} f^{k+1}(x_L, x_F) & \cdots\cdots\cdots & \nabla^2_{x^{k+1}x^l} f^{k+1}(x_L, x_F) \\ \vdots & & \vdots \\ \vdots & & \vdots \\ \vdots & & \vdots \\ \nabla^2_{x^l x^{k+1}} f^l(x_L, x_F) & \cdots\cdots\cdots & \nabla^2_{x^l x^l} f^l(x_L, x_F) \end{bmatrix}.$$

Theorem 3.1. Assume that $\hat{x} = (\hat{x}_L, \hat{x}_F)$ is a noncooperative solution to EPEC and assumptions (A1),(A2) are fulfilled. Further suppose that for all $i \in I_1$ the qualification conditions

$$\left.\begin{array}{lll} 0 & \in & (\nabla_{x^i} F(\hat{x}_L, \hat{x}_F))^T v + N_{U^i}(\hat{x}^i) \\ 0 & \in & (\nabla_{x_F} F(\hat{x}_L, \hat{x}_F))^T v + D^* N_\Omega(\hat{x}_F, -F(\hat{x}_L, \hat{x}_F))(v) \end{array}\right\} \Rightarrow v = 0 \qquad (2)$$

hold true.

Then for all $i \in I_1$ there exist Karush–Kuhn–Tucker (KKT) vectors $\hat{v}^i \in \mathbb{R}^{n(l-k)}$ such that

$$\begin{array}{lll} 0 & \in & \nabla_{x^i} f^i(\hat{x}_L, \hat{x}_F) + (\nabla_{x^i} F(\hat{x}_L, \hat{x}_F))^T \hat{v}^i + N_{U^i}(\hat{x}^i) \\ 0 & \in & \nabla_{x_F} f^i(\hat{x}_F, \hat{x}_F) + (\nabla_{x_F} F(\hat{x}_L, \hat{x}_F))^T \hat{v}^i + D^* N_\Omega(\hat{x}_F, -F(\hat{x}_L, \hat{x}_F))(\hat{v}^i). \end{array} \qquad (3)$$

P r o o f. It suffices to apply [22, Thm. 3.2] to each MPEC (2) separately. Thereby x^i is the control, x_F is the state variable and x_L^{-i} is a parameter vector through which these MPECs are coupled. Conditions (2) ensure the so-called calmness (pseudo-upper Lipschitz continuity) of the multifunctions

$$p \mapsto \left\{(y, z) \in U^i \times \mathbb{R}^{n(l-k)} \,|\, p \in F(\hat{x}_L^{-i}, y, z) + N_\Omega(z)\right\}, \quad i \in I_1 \qquad (4)$$

at $(\hat{x}^i, \hat{x}_F)$, which is the property needed for relations (3) to hold. □

It is easy to see that the multifunctions (4) possess the required calmness property, provided F is affine and all sets U^i are convex polyhedral, cf. [19]. In such a case conditions (2) can be omitted.

In what follows, in accordance with the MPEC literature, the points $(\hat{x}_L, \hat{x}_F)$ satisfying conditions (3) will be termed *M(ordukhovich)-stationary*. The statement of Theorem 3.1 can be applied only in the case, if we are able to compute the coderivative of the normal cone mapping $N_\Omega(\cdot)$. Let $(\bar{z}, \bar{w}) \in \operatorname{Gph} N_\Omega$ and assume that Ω is a convex polyhedron. Then by Definition 1.1 and due to the properties of $N_\Omega(\cdot)$ one has

$$N_{\mathrm{Gph}\, N_\Omega}(\bar{z},\bar{w}) = \bigcup_{(z,w)\in O\cap \mathrm{Gph}\, N_\Omega} (T_{\mathrm{Gph}\, N_\Omega}(z,w))^0 \tag{5}$$

where O is a sufficiently small neighborhood of $(\bar{z},\bar{w})$ ([3]). Formula (5) enables among others to compute the required coderivative in the case, where $U^i, i \in I_2$, are given by box constraints ([16]). If Ω is not a convex polyhedron, one has to apply directly Definition 1.1, but this can be a difficult task.

The monograph [18] deals with MPECs in which the equilibrium constraint exhibits some special properties. They enable to apply the so-called implicit programming approach both to the derivation of optimality conditions as well as to the numerical solution of the considered MPECs. Let us investigate the possibilities of this approach in the context of EPECs. To this purpose, we denote by S the multifunction which assigns an admissible vector x_L the set of solutions x_F to the GE

$$0 \in F(x_L, x_F) + N_\Omega(x_F). \tag{6}$$

Moreover, for a given $i \in I_1$ and a fixed admissible vector $\tilde{x}_L^{-i}$, we define the multifunction $S_{\tilde{x}_L^{-i}}[U^i \rightsquigarrow \mathbb{R}^{(l-k)n}]$ by

$$S_{\tilde{x}_L^{-i}}(x^i) := S(\tilde{x}^1, \dots, \tilde{x}^{i-1}, x^i, \tilde{x}^{i+1}, \dots, \tilde{x}^k).$$

By this definition, for all $i \in I_1$ one has

$$S_{\tilde{x}_L^{-i}}(\tilde{x}^i) = S(\tilde{x}_L).$$

The essential assumption for the application of the implicit programming approach now reads as follows:

(A3) For all $i \in I_1$ and for all admissible vectors $\tilde{x}_L^{-i}$ the map $S_{\tilde{x}_L^{-i}}$ is single-valued and locally Lipschitz on an open set containing U^i.

Under (A3) we can now rewrite the single MPECs (2) to the form

$$\begin{aligned} &\text{minimize} \quad \Theta_{\hat{x}_L^{-i}}(y) \\ &\text{subject to} \\ &\qquad\qquad y \in U^i, \quad i \in I_1, \end{aligned} \tag{7}$$

where

$$\Theta_{\hat{x}_L^{-i}}(y) := f^i(\hat{x}_L^{-i}, y, S_{\hat{x}_L^{-i}}(y)).$$

We face a new game only among the Leaders without any hierarchical structure. Its local Nash equilibria (cf. [7]) amount to the Leaders' components of noncooperative solutions to EPEC, which facilitates the formulation of corresponding necessary conditions.

Theorem 3.2. Let the assumptions of Theorem 3.1 be fulfilled with the qualification conditions (2) replaced by (A3). Then $\hat{x}_F = S(\hat{x}_L)$ and

$$0 \in \nabla_{x^i} f^i(\hat{x}_L, \hat{x}_F) + D^* S_{\hat{x}_L^{-i}}(\hat{x}^i)(\nabla_{x_F} f^i(\hat{x}_L, \hat{x}_F)) + N_{U^i}(\hat{x}^i), \quad i \in I_1. \tag{8}$$

Proof. By virtue of the assumptions, the functions $\Theta_{\hat{x}_L^{-i}}$ are locally Lipschitz on the respective open sets containing U^i. It suffices thus to apply the optimality conditions in [9, Thm. 7.1] combined with the chain rule in [10, Cor. 5.3] to problems (7). □

The coderivatives of the maps $S_{\hat{x}_L^{-i}}$ are generally not easy to compute. If, however, Ω is given e.g. by means of equalities and inequalities, then one can replace the GE (6) by a complementarity problem and invoke the implicit (multi)function theorem [10, Thm. 6.10] along with the results from [15].

Due to the relation between coderivatives and Clarke's generalized Jacobians, relations (8) imply that

$$0 \in \nabla_{x^i} f^i(\hat{x}_L, \hat{x}_F) + (\bar{\partial} S_{\hat{x}_L^{-i}}(\hat{x}^i))^T \nabla_{x_F} f^i(\hat{x}_L, \hat{x}_F) + N_{U^i}(\hat{x}^i), \quad i \in I_1. \tag{9}$$

In this way we obtain another, less stringent necessary conditions for a noncooperative solution to EPEC. Following the MPEC terminology, the points satisfying (9) will be called *C(larke)-stationary.*

Assumptions (A3) can be ensured via the following, more easily verifiable requirement, cf. [18]:

(A3)' S is single-valued and locally Lipschitz on an open set containing $\omega := \times_{i \in I_1} U^i$.

Under (A3)' we can now state an existence result at least for Clarke stationary points.

Theorem 3.3. Let assumptions (A1), (A2), (A3)' be fulfilled and suppose that for $i \in I_1$

(i) The sets U^i are convex and compact;

(ii) the multifunctions $\Gamma^i : x_L \mapsto \bar{\partial} S_{x_L^{-i}}(x^i)$ are upper semicontinuous on ω.

Then the considered EPEC possesses a Clarke stationary point.

Proof. We have to show that the GE

$$0 \in C(x_L) + N_\omega(x_L) \tag{10}$$

with

$$C(x_L) = \begin{bmatrix} \nabla_{x^1} f^1(x_L, S(x_L)) \\ \vdots \\ \nabla_{x^k} f^k(x_L, S(x_L)) \end{bmatrix} + \begin{bmatrix} (\bar{\partial} S_{x_L^{-1}}(x^1))^T \\ \vdots \\ (\bar{\partial} S_{x_L^{-k}}(x^k))^T \end{bmatrix} \nabla_{x_F} f(x_L, S(x_L))$$

satisfies the assumptions of [1, Thm. 9.9]. Since C is apparently nonempty-, convex- and compact-valued, it suffices to add assumptions (i),(ii) and we are done. □

If the multifunctions Γ^i are not upper semicontinuous on ω, then one can modify the definition of the generalized Jacobian following the idea from [2, Section 2.8]. In case of the map $S_{\hat{x}_L^{-i}}(\cdot)$ at the point $\hat{x}^i$ we arrive at the notion

$$\tilde{\partial} S_{\hat{x}_L^{-i}}(\hat{x}^i)$$
$$:= \left\{ A \in \mathcal{L}[\mathbb{R}^n, \mathbb{R}^{(l-k)n}] \,|\, A = \lim_{j\to\infty} A_j, \text{where}\, A_j \in \bar{\partial} S_{(x_L^{-i})_j}(x_j^i), (x_L)_j \xrightarrow{\omega} \hat{x}_L \right\}.$$

If we now define Γ^i by means of $\tilde{\partial} S_{x_L^{-i}}(x^i)$ (instead of $\bar{\partial} S_{x_L^{-i}}(x^i)$), assumption (ii) of Theorem 3.3 is fulfilled. Nevertheless, in this way we prove only the existence of stationary points in a weaker sense (with respect to Clarke).

Concerning the computation of noncooperative solutions to EPEC, the procedure can be organized in two steps:

(i) Computation of a (Mordukhovich or Clarke) stationary point;

(ii) Verification of the minimizing properties required in Definition 2.1.

We finish the paper with a few concluding remarks regarding mainly the first step. Unfortunately, the conditions of Theorem 3.1 do not seem to be suitable for construction of a numerical method to the computation of stationary points. So, if the implicit programming approach cannot be applied, then it is probably better to follow [7] and look at problems (2) as standard nonlinear programs (the equilibrium constraint is then replaced by a complementarity problem). In this way one arrives at l coupled KKT systems, solvable possibly by existing solvers. The noncooperative solution to EPEC, however, need not satisfy these coupled KKT systems (since the equilibrium constraint violates standard constraint qualifications), and so the whole procedure may fail. Nevertheless, if we obtain a candidate for stationarity in this way, the conditions of Theorem 3.1 can be used as a stationarity test.

Under assumptions of Theorem 3.3 one can invoke an idea from [5] and rewrite the GE (10) to the form of a fixed-point problem. Indeed, the GE (10) amounts then to the relations

$$\hat{x}_L = \mathrm{Proj}_\omega(\hat{x}_L - \hat{y}), \quad \hat{y} \in C(\hat{x}_L).$$

In another words. $\hat{x}_L$ is a fixed point of the multifunction $\Psi := \mathrm{Proj}_\omega \circ (Id - C)$. Since Proj_ω is continuous and C is upper semicontinuous with convex and compact values, the multifunction Ψ is also upper semicontinuous and convex- and compact-valued. Consequently, there is a number of available numerical methods which compute the fixed points of Ψ, cf. [4] and the references therein.

Concerning the step (ii), it requires generally rather advanced tools from the 2nd-order nonsmooth analysis. The situation becomes, however, substantially simpler provided the assumptions of Theorem 3.3 are fulfilled, Ω is given by inequalities, and the strict complementarity holds for the GE (6) at $(\hat{x}_L, \hat{x}_F)$. In such a case all maps $S_{\hat{x}_L^{-i}}$ are differentiable at $\hat{x}^i$, which facilitates the analysis of programs (7). Another simplifying assumptions are discussed in [7].

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