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ROBUST POLE PLACEMENT FOR SECOND–ORDER SYSTEMS: AN LMI APPROACH[1]

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Based on recently developed sufficient conditions for stability of polynomial matrices, an LMI technique is described to perform robust pole placement by proportional-derivative feedback on second-order linear systems affected by polytopic or norm-bounded uncertainty. As illustrated by several numerical examples, at the core of the approach is the choice of a nominal, or central quadratic polynomial matrix.

Keywords: polynomial matrix, second-order linear systems, LMI, pole placement, robust control

AMS Subject Classification: 93E12, 62A10, 62F15

1. INTRODUCTION

In this paper we study linear systems described by the second-order dynamical equations

$$\begin{aligned}(A_0 + A_1 s + A_2 s^2)x &= Bu \\ y &= Cx\end{aligned} \tag{1}$$

where s denotes indifferently the Laplace variable for continuous-time systems or the backward shift operator for discrete-time systems. In the above equation x is the state vector, u is the input vector, and y is the output vector. Dynamical system (1) is controlled by a proportional and derivative (PD) output-feedback controller of the form

$$u = -(F_0 + F_1 s)y \tag{2}$$

so that the closed-loop system behavior is captured by the quadratic polynomial matrix

$$N(s) = (A_0 + BF_0C) + (A_1 + BF_1C)s + A_2 s^2. \tag{3}$$

Closed-loop system poles are zeros of polynomial matrix $N(s)$, i.e. roots of its determinant.

Dynamical system (1) arises naturally in a wide range of applications, including:

- control of large flexible space structures

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- earthquake engineering
- control of mechanical multi-body systems
- stabilization of damped gyroscopic systems
- robotics control
- vibration control in structural dynamics
- linear stability of flows in fluid mechanics
- electrical circuit simulation

see e. g. [16, 24] and the many references therein. Matrices A_0, A_1 and A_2 are usually referred to as stiffness, damping and mass matrices, respectively.

We aim at designing controller (2), i. e. finding feedback matrices F_0 and F_1, such that zeros of closed-loop matrix $N(s)$ lie within a specified stability region $\mathcal{D}$, a subset of the complex plane. Typical choices for $\mathcal{D}$ are the left half-plane (continuous-time systems) or the unit disk (discrete-time systems) but more elaborate choices are also sometimes required, such as shifted half-planes, strips, shifted disks, sectors or their intersections.

In addition, we suppose that entries in system matrix $A(s)$ are not known exactly. Due to approximate knowledge of the physical parameters, or neglected parasitic dynamics, some uncertainty affects the system. Controller (2) must be insensitive, or robust, to this uncertainty in the sense that closed-loop properties, namely stability or pole location in $\mathcal{D}$, are preserved. We assume that two kinds of uncertainty can affect open-loop quadratic system matrix, namely additive norm-bounded (unstructured) uncertainty and polytopic (structured) uncertainty. We will elaborate further on the uncertainty model later in the paper.

Typically, when performing analysis or design, system (1) is transformed into first-order (pencil) state-space representation. However, as pointed out in [8] or [24], retaining the model in matrix second-order form has many advantages:

- physical insight of the original problem is preserved
- system matrix sparsity and structure are preserved
- uncertainty structure is preserved
- PD feedback can be used directly, entailing easier implementation.

The main drawback when keeping the second-order form is that a very few design methods are available, most of them being developed for first-order forms. Some attempts to fill up the gap are reported in e. g. [6, 7, 8, 15] or more recently in [9, 16].

The objective of this paper is to provide a new approach to robust stabilization of second-order systems with PD compensators. We believe that the main reason why there is so few design methods for second-order systems is that most of the design methods currently available are based either on numerical linear algebra (Lyapunov or algebraic Riccati equations) or convex optimization (linear matrix inequality, or LMI) algorithms for which tractability (linearity, or at least convexity) is ensured due to the particular first-order system structure, see e. g. [4, 21]. For higher order (including second-order) systems, convexity cannot be ensured for design namely because stability conditions are highly non-convex in the space of system coefficients, hence in the space of design parameters [1].

The basic idea lying behind our design approach is then to use convex optimization over LMIs to relax the stability conditions via introduction of additional decision variables, in the spirit of the inspiring work [17]. Following our research endeavor initiated in [12], we formulate a sufficient condition for stability of quadratic polynomial matrices with the help of the emerging theory of positive polynomial matrices, see e. g. [11] for a recent overview. Due to the linearity of the LMI condition in the system matrices, structural constraints on the controller can be easily incorporated as soon as they are convex, and parametric system uncertainty can also be handled. Several numerical examples illustrate that the approach, although potentially conservative, may prove efficient, in addition to be very easy to implement with off-the-shelf software. The routines described in this paper will be implemented in the release 3.0 of the Polynomial Toolbox for Matlab [20].

2. LMI CONDITION FOR ROBUST STABILITY

Let $N(s) = N_0 + N_1 + N_2s^2$ and $D(s) = D_0 + D_1 + D_2s^2$ be two square quadratic polynomial matrices of dimension n with coefficient matrices

$$\begin{aligned} N &= \begin{bmatrix} N_0 & N_1 & N_2 \end{bmatrix} \\ D &= \begin{bmatrix} D_0 & D_1 & D_2 \end{bmatrix}. \end{aligned}$$

Let $\mathcal{D} = \{s : H_{11} + H_{12}s + H_{12}^\star s^\star + H_{22}ss^\star < 0\}$ be a stability region in the complex plane such that 2×2 Hermitian matrix

$$H = \begin{bmatrix} H_{11} & H_{12} \\ H_{12}^\star & H_{22} \end{bmatrix}$$

has one positive eigenvalue and one negative eigenvalue, where the star denotes the complex conjugate. Standard choices are $H_{11} = H_{22} = 0$, $H_{12} = 1$ for the left half-plane and continuous-time systems, or $H_{11} = -H_{22} = -1$, $H_{12} = 0$ for the unit disk and discrete-time systems. Following the terminology of [14], we say that a square polynomial matrix is $\mathcal{D}$-stable when all its zeros (i. e. the roots of its determinant) lie within region $\mathcal{D}$. Let

$$H(P) = \Pi'(H \otimes P)\Pi, \quad \Pi = \begin{bmatrix} I_n & 0 & 0 \\ 0 & I_n & 0 \\ 0 & I_n & 0 \\ 0 & 0 & I_n \end{bmatrix}$$

be a linear matrix function of a square matrix P of dimension $2n$, where $\otimes$ denotes the Kronecker product and Π is a projection matrix of size $4n \times 3n$. Finally, for a symmetric matrix A, the notation $A \succ 0$ means that A is positive definite, i.e. all its eigenvalues are positive.

Lemma 1. Polynomial matrix $N(s)$ is $\mathcal{D}$-stable if and only if there exists a $\mathcal{D}$-stable polynomial matrix $D(s)$ and a constant matrix P satisfying

$$D'N + N'D - H(P) \succ 0, \quad P = P'. \tag{4}$$

Proof. The complete proof can be found in [12], see also [13] and [14]. Matrix inequality (4) holds if and only if rational matrix $G(s) = N(s)D^{-1}(s)$ is strictly positive real (SPR) along the one-dimensional boundary of $\mathcal{D}$. Indeed, the theory of positive polynomial matrices [11] can be invoked to show that matrix P in inequality (4) captures all the degrees of freedom available to make $G(s)$ SPR. Then, if $D(s)$ is given and $\mathcal{D}$-stable, then $N(s)$ is also $\mathcal{D}$-stable from the SPRness of $G(s)$. Conversely, if $N(s)$ is $\mathcal{D}$-stable then the choice $D(s) = N(s)$ makes $G(s) = I_n$ obviously SPR. □

The main idea is then to freeze $D(s)$ to some given $\mathcal{D}$-stable polynomial matrix. As a result, matrix inequality (4) becomes affine both in N and P. In other words, once polynomial matrix $D(s)$ is fixed in Lemma 1, we obtain a sufficient LMI condition for $\mathcal{D}$-stability of polynomial matrix $N(s)$:

Lemma 2. Given a $\mathcal{D}$-stable polynomial matrix $D(s)$, polynomial matrix $N(s)$ is $\mathcal{D}$-stable if LMI (4) is feasible.

In the special case of first-degree polynomial matrices $D(s) = D_0 + sD_1$ and $N(s) = N_0 + sN_1$ this is precisely the idea found in [17] and later on generalized in [18]. Coefficients of polynomial matrix $D(s)$ play the role of additional decision variables decoupling system coefficients N and a Lyapunov-like matrix P ensuring SPRness, hence $\mathcal{D}$-stability.

Based on Lemma 2, one can easily obtain sufficient conditions for robust stability of an uncertain polynomial matrix $N(s)$. Assume first that matrix $N(s)$ is subject to additive norm-bounded (unstructured) uncertainty

$$N(s) + \Delta M(s), \quad \sigma_{\max}(\Delta) \leq \delta \tag{5}$$

where Δ is a uncertainty matrix of arbitrary column dimension, $\sigma_{\max}$ denotes the maximum singular value, δ is a given positive scalar and

$$M(s) = M_0 + M_1 s + M_2 s^2$$

is a given polynomial matrix whose coefficients are stored in matrix

$$M = \left[\begin{array}{ccc} M_0 & M_1 & M_2 \end{array}\right].$$

Inequality (4) reads

$$D'(N + \Delta M) + (N + \Delta M)'D - H(P) \succ 0, \quad P = P'$$

which is an uncertain LMI depending on matrix Δ. Applying results on robust convex optimization found e.g. in [10], the above LMI is feasible for all admissible uncertainty Δ norm-bounded by δ if and only if the LMI

$$\begin{bmatrix} D'N + N'D - H(P) - \gamma D'D & \delta M' \\ \delta M & \gamma I \end{bmatrix} \succ 0, \quad P = P' \tag{6}$$

is feasible for some scalar γ.

Corollary 1. Given a $\mathcal{D}$-stable polynomial matrix $D(s)$, norm-bounded uncertain polynomial matrix (5) is robustly $\mathcal{D}$-stable if LMI (6) is feasible.

Similarly, when matrix $N(s)$ is subject to (structured) polytopic uncertainty

$$N(s) = \sum_i \lambda^i N^i(s), \quad \sum_i \lambda^i = 1, \quad \lambda^i \geq 0, \quad i = 1, 2, \ldots \tag{7}$$

where $N^i(s)$ are given polynomial matrix vertices with matrix coefficients

$$N^i = \begin{bmatrix} N_0^i & N_1^i & N_2^i \end{bmatrix}, \quad i = 1, 2, \ldots$$

we can write inequality (4) at each vertex

$$D'N^i + (N^i)'D - H(P^i) \succ 0, \quad P^i = (P^i)' \tag{8}$$

and derive the following result:

Corollary 2. Given a $\mathcal{D}$-stable polynomial matrix $D(s)$, polytopic uncertain polynomial matrix (7) is robustly $\mathcal{D}$-stable if LMI (8) is feasible for all vertex indices $i = 1, 2, \ldots$

Combining the vertex LMIs, we obtain

$$\begin{aligned} & \sum_i \lambda^i (D'N^i + (N^i)'D - H(P^i)) \\ = \; & D'\left(\sum_i \lambda^i N^i\right) + \left(\sum_i \lambda^i N^i\right)' D - H\left(\sum_i \lambda^i P^i\right) \succ 0 \end{aligned}$$

which proves stability of polytopic matrix (7) with a parameter-dependent matrix

$$P = \sum_i \lambda^i P^i,$$

in virtue of Lemma 2.

Note finally that the approach can naturally handle intersections of stability regions

$$\mathcal{D} = \cap_j \mathcal{D}^j$$

as soon as one selects one common polynomial matrix $D(s)$ and one distinct Lyapunov-like matrix P^j for each region $\mathcal{D}^j$. Intersections of region include vertical strips, truncated disks, or even sectors if one allows complex-valued LMIs. More complicated LMI stability regions (parabolae, ellipses) as described in [5] can also be handled similarly, see [14] for further details.

3. ROBUST DESIGN

The robust analysis results obtained so far can be easily extended to robust design. Indeed, second-order system (1) controlled with PD feedback (2) gives rise to the closed-loop system matrix $N(s)$ given in equation (3), where feedback coefficient matrices F_0 and F_1 enter linearly. Since the design matrix inequalities obtained in the previous section are all affine in system matrix $N(s)$, one can straightforwardly extend analysis results to incorporate structural design constraints.

For example, one may seek to minimize the 2-norm (maximum singular value) of feedback matrix

$$F = \begin{bmatrix} F_0 & F_1 \end{bmatrix}$$

since inequality constraint $\sigma_{\max}(F) \leq f$ can be written as the LMI

$$\begin{bmatrix} fI & F \\ F' & I \end{bmatrix} \succeq 0. \tag{9}$$

Minimizing the 2-norm of the feedback matrix entails minimizing the control effort, which is desirable to avoid saturations and nasty side-effects.

One can also try to maximize the unstructured uncertainty radius δ in LMI (6), the resulting optimization problem remaining a convex LMI problem. The obtained controller is then robustified in the sense that it is robust to the largest (worst-case) uncertainty for which the sufficient LMI condition of Corollary 1 can ensure robust stability.

From Corollaries 1 and 2 it is apparent that the crucial point in the design process resides in the choice of polynomial matrix $D(s)$. Indeed, robust stability is ensured via robust SPRness of rational matrix $G(s) = N(s)D^{-1}(s)$ where numerator (closed-loop system) matrix $N(s)$ is uncertain and denominator matrix $D(s)$ is fixed. For this reason, $D(s)$ can be referred to as the central (or nominal) polynomial matrix. Success in solving robust design LMIs, as well as the whole conservatism of the approach, will strongly depend on the choice of central polynomial matrix $D(s)$. A good policy could be to set $D(s)$ to some nominal system matrix obtained by some standard design algorithm (pole placement, LQ, H_2 or H_∞), and then to use the LMI algorithm to robustify the controller. This point is illustrated in the following numerical examples.

4. NUMERICAL EXAMPLES

All the numerical examples were carried out with Matlab 6.1 and SeDuMi 1.05 [22] combined with the user-friendly LMI interface 1.01 [19] running on a 1.7 GHz Pentium IV PC with 512MB RAM.

4.1. Mechanical structure

We consider the mechanical system shown in Figure 1, consisting of five material points linked by elastic springs [2]. The points can slide without friction along their

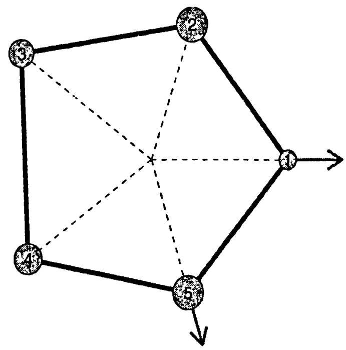

Fig. 1. Five masses linked by elastic springs.

Table 1. System data.

Point	Mass	Distance	Spring	Stiffness
1	0.5093	0.8034	1-2	1.461
2	0.9107	0.7430	2-3	1.369
3	0.7224	0.9456	3-4	1.088
4	0.8077	0.8810	4-5	1.203
5	0.8960	0.7282	5-1	1.468

respective axes. Mass, distance to the origin at the equilibrium, and spring stiffness are given for each point in Table 1.

The system is controlled by two external forces acting at masses 1 and 5. Dynamical system equations are given by equations (1) where

$$A_0 = \begin{bmatrix} 2.565 & 1.080 & 0 & 0 & 1.089 \\ 0.6038 & 0.8206 & 0.4766 & 0 & 0 \\ 0 & 0.6009 & 1.504 & 0.4808 & 0 \\ 0 & 0 & 0.4300 & 1.114 & 0.5131 \\ 0.6190 & 0 & 0 & 0.4626 & 0.8352 \end{bmatrix}, \quad A_1 = 0_5, \quad A_2 = I_5$$

and

$$B = \begin{bmatrix} 0 & 1.964 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1.116 & 0 \end{bmatrix}, \quad C = I_5.$$

Open-loop poles are all purely imaginary and located at $\pm i1.783$, $\pm i1.380$, $\pm i1.145$, $\pm i0.5675$ and $\pm i0.3507$.

A stabilizing PD controller (2) is obtained in [2] with a nearly optimal linear-quadratic robust design method:

$$F_0^0 = \begin{bmatrix} 0.03960 & -0.02200 & 0.3685 & 0.8069 & 0.4099 \\ 0.3993 & 0.6453 & 0.4886 & 0.2269 & 0.03220 \end{bmatrix},$$

$$F_1^0 = \begin{bmatrix} 0.01520 & -0.3694 & 0.06470 & -0.04980 & 1.317 \\ 1.186 & -0.5896 & -0.2165 & -0.3263 & 0.02680 \end{bmatrix}.$$

Poles of closed-loop quadratic matrix polynomial

$$D(s) = (A_0 + BF_0^0C) + (A_1 + BF_1^0C)s + A_2s^2$$

are located at $-0.1067 \pm i1.406$, -0.1405, $-0.1809 \pm i0.5350$, $-0.2174 \pm i1.099$, $-0.8157 \pm i1.450$ and -1.016. Feedback matrix $F^0 = [F_0^0 \quad F_1^0]$ has norm $f^0 = 1.859$.

In view of the closed-loop poles, we choose

$$\mathcal{D} = \{s : \text{Re}\, s < -0.1\}$$

as the stability region. With the above $\mathcal{D}$-stable polynomial matrix $D(s)$ as central system matrix, minimizing f subject to LMI (4) combined with LMI (9) yields after 0.5 seconds of CPU time feedback matrices

$$F_0 = \begin{bmatrix} -0.1673 & -0.1209 & -0.04443 & 0.07785 & -0.05071 \\ -0.2680 & 0.04915 & 0.1401 & 0.05788 & -0.1366 \end{bmatrix},$$

$$F_1 = \begin{bmatrix} -0.1205 & -0.3140 & 0.2210 & 0.1788 & 0.5398 \\ 0.5330 & -0.2232 & -0.0002223 & -0.2126 & 0.06163 \end{bmatrix}.$$

Poles of the new closed-loop quadratic matrix polynomial

$$N(s) = (A_0 + BF_0C) + (A_1 + BF_1C)s + A_2s^2$$

are located at $-0.1069 \pm i1.403$, $-0.1342 \pm i0.5443$, $-0.1441 \pm i1.1117$, $-0.1827 \pm i0.2323$ and $-0.2568 \pm i1.462$, well inside region $\mathcal{D}$. Feedback matrix $F = [F_0 \quad F_1]$ has largest singular value $f = 0.7537 < f^0$. Consequently, new feedback F requires less control effort and is less prone to saturation than original feedback F^0.

4.2. Vibrating rod

We consider as in [7] the finite difference model of an axially vibrating non-conservative rod. The model is parametrized by the number of nodes n, and system matrices in equation (1) are given by $A_0 = 1000FF'$, $A_1 = FGF'$ and $A_2 = 2(I + SS') + S + S'$ where $S = [\delta_{i+1,j}]$ is a shift matrix of size n , δ_{ij} is the Kronecker delta, $F = I_n - S$, $G = 0.01 \,\text{diag}\{\sin \frac{i\pi}{2n}\}$ for $i = 1, \ldots, n$. We assume that all the inputs and the outputs are available for feedback. For example, when $n = 4$, system matrices are:

$$A_0 = 1000 \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix},$$

$$A_1 = 0.01 \begin{bmatrix} 1.090 & -0.7071 & 0 & 0 \\ -0.7071 & 1.631 & -0.9239 & 0 \\ 0 & -0.9239 & 1.924 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

Table 2. CPU time to solve the LMI problem vs. number of nodes n.

n	5	7	10	12	15
Time (sec)	2.0	5.2	49	143	540

and

$$A_2 = \begin{bmatrix} 4 & 1 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

Open-loop system poles are located at $-7.681 \cdot 10^{-5} \pm i5.102$, $-9.644 \cdot 10^{-4} \pm i16.10$, $-3.259 \cdot 10^{-3} \pm i29.24$ and $-8.195 \cdot 10^{-3} \pm i42.28$.

We choose the strip

$$\mathcal{D} = \{s : -2 < \operatorname{Re} s < -0.5\}$$

as the intersection of two basic stability regions and

$$D(s) = (s+1)^2 I_n$$

as an (arbitrary) $\mathcal{D}$-stable central system matrix. When $n = 4$ our LMI algorithm returns after 1.4 seconds of CPU time a stabilizing PD compensator (3) with feedback matrices

$$F_0 = 1000 \begin{bmatrix} -1.989 & 0.9992 & 0.002612 & 0.0002172 \\ 0.9992 & -1.987 & 0.9982 & -0.0003341 \\ 0.002611 & 0.9982 & -1.988 & 1.003 \\ 0.0002169 & -0.0003337 & 1.003 & -0.9956 \end{bmatrix},$$

$$F_1 = \begin{bmatrix} 10.31 & 1.426 & 0.8742 & 0.07307 \\ 1.427 & 11.18 & 1.081 & -0.1125 \\ 0.8741 & 1.081 & 10.84 & 2.524 \\ 0.07278 & -0.1121 & 2.523 & 4.833 \end{bmatrix}.$$

Closed-loop system poles are then located at $-1.209 \pm i0.8601$, $-1.209 \pm i0.8604$, $-1.210 \pm i0.8612$ and $-1.883 \pm i1.636$, well inside the assigned stability region.

In Table 2 we report CPU times required to compute a stabilizing feedback for various values of n, the number of nodes of the finite difference model of the vibrating rod. Stability region $\mathcal{D}$ and central system matrix $D(s)$ are as above.

4.3. Wing in airstream

In [23] the authors consider an eigenvalue problem arising from the analysis of the oscillations of a wing in an airstream. Quadratic system matrix coefficients are given by:

$$A_0 = \begin{bmatrix} 121.0 & 18.90 & 15.90 \\ 0 & 2.700 & 0.1450 \\ 11.90 & 3.640 & 15.50 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 7.660 & 2.450 & 2.100 \\ 0.2300 & 1.040 & 0.2230 \\ 0.6000 & 0.7560 & 0.6580 \end{bmatrix}$$

and

$$A_2 = \begin{bmatrix} 17.60 & 1.280 & 2.890 \\ 1.280 & 0.8240 & 0.4130 \\ 2.890 & 0.4130 & 0.7250 \end{bmatrix}.$$

The system is open-loop unstable since its poles are located at $0.09427 \pm i2.553$, $-0.8848 \pm i8.442$ and $-0.9180 \pm i1.761$.

Choosing

$$\mathcal{D} = \{s : -2 < \operatorname{Re} s < 0\}$$

as the stability region and

$$D(s) = (s+1)^2 I_3$$

as the $\mathcal{D}$-stable central closed-loop matrix, our LMI algorithm returns

$$F_0 = \begin{bmatrix} -4.867 & -13.12 & -2.449 \\ 1.988 & -1.033 & 0.8636 \\ 1.549 & -1.346 & -12.94 \end{bmatrix}, \quad F_1 = \begin{bmatrix} 14.10 & -3.915 & 0.01323 \\ 1.507 & 0.6210 & 0.6726 \\ 1.082 & -0.7586 & -0.2070 \end{bmatrix}$$

as stabilizing feedback matrices assigning closed-loop poles at $-0.5662 \pm i0.5042$, $-0.8351 \pm i1.528$ and $-1.054 \pm i2.659$.

Now suppose that a failure affects the second actuator, i. e. let

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad C = I_3.$$

The LMI algorithm is still able to compute $\mathcal{D}$-stabilizing feedback matrices

$$F_0^B = \begin{bmatrix} -4.395 & -15.11 & -0.4276 \\ 1.954 & -1.585 & -12.84 \end{bmatrix}, \quad F_1^B = \begin{bmatrix} 14.98 & -3.983 & 0.7889 \\ 2.119 & -0.4108 & 0.4697 \end{bmatrix}$$

now assigning closed-loop poles to $-0.3394 \pm i1.777$, $-1.033 \pm i2.615$ and $-1.828 \pm i0.4991$.

Similarly, assuming that all the actuators are available, but that a failure affects the second sensor, i. e.

$$B = I_3, \quad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

the LMI algorithm returns

$$F_0^C = \begin{bmatrix} -4.790 & 3.061 \\ 13.52 & 2.430 \\ 5.265 & -11.80 \end{bmatrix}, \quad F_1^C = \begin{bmatrix} 15.76 & 1.562 \\ 4.591 & 0.7364 \\ 3.425 & 0.7564 \end{bmatrix}$$

and closed-loop poles are located at $-0.3577 \pm i0.4310$, $-0.9133 \pm i2.600$, -1.469 and -1.864.

Finally, we suppose that the damping matrix is subject to additive norm-bounded uncertainty, i. e. $M(s) = sI_3$ in equation (5). Using Corollary 1, we were then able to robustly stabilize the system with feedback matrices

$$F_0^R = \begin{bmatrix} -80.25 & -15.93 & -9.200 \\ 2.967 & -0.8435 & 0.8139 \\ -5.200 & -2.681 & -13.88 \end{bmatrix}, \quad F_1^R = \begin{bmatrix} 22.99 & -0.2247 & 2.940 \\ 1.995 & 0.4010 & 0.5176 \\ 4.440 & -0.01539 & 0.5585 \end{bmatrix}$$

for all uncertainty with worst-case norm $\delta = 0.1918$. In Figure 2 we represent the closed-loop robust root locus for 10000 randomly chosen systems in the admissible uncertainty range. Nominal closed-loop poles are represented by stars.

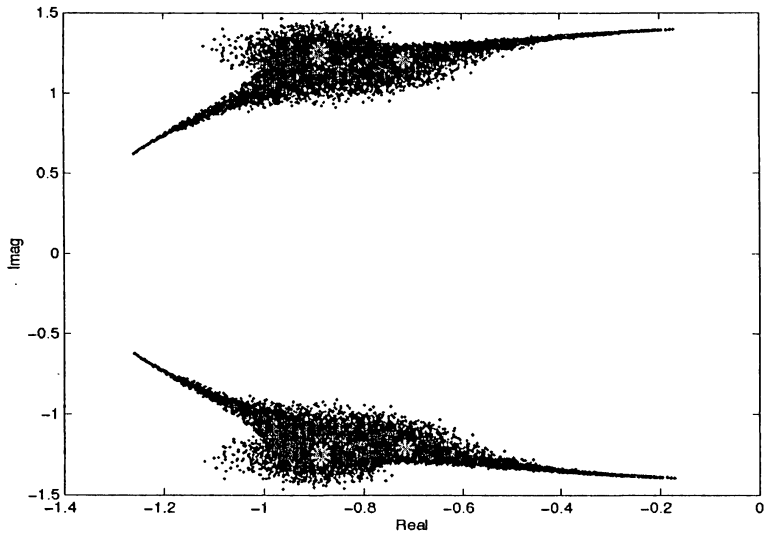

Fig. 2. Robust root-locus of the wing.

4.4. Mass-spring system

Consider the undamped mass-spring example [16, Example 1], taken from [6, Example 2], where system matrices are given by:

$$A_0 = \begin{bmatrix} 40 & -40 & 0 \\ -40 & 80 & -40 \\ 0 & -40 & 80 \end{bmatrix}, \quad A_1 = 0_3, \quad A_2 = 10I_3, \quad B = \begin{bmatrix} 1 & 2 \\ 3 & 2 \\ 3 & 4 \end{bmatrix}, \quad C = I_3.$$

Following [16], a choice of nominally stabilizing PD controller matrices assigning closed-loop poles to -1, -2, -3, -4, -5 and -6 is as follows:

$$F_0^0 = \begin{bmatrix} 1.257 & 44.62 & -120.2 \\ -56.18 & -42.28 & 227.7 \end{bmatrix}, \quad F_1^0 = \begin{bmatrix} -86.18 & 27.23 & 16.52 \\ 85.49 & -13.02 & 4.992 \end{bmatrix}.$$

Now suppose that each diagonal entry in mass matrix A_2 belongs to an independent uncertainty interval [9, 11]. As a result, the system matrix belongs to a polytope with $2^3 = 8$ vertices, as in equation (6). Let

$$\mathcal{D} = \{s : \operatorname{Re} s < -0.5\}$$

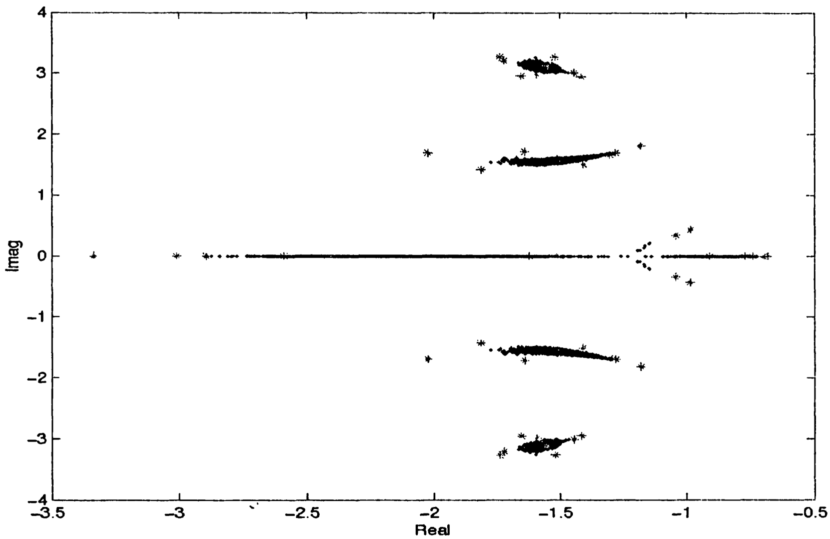

Fig. 3. Robust root-locus of the mass-spring system.

be the stability region, and let

$$D(s) = (A_0 + BF_0^0C) + (A_1 + BF_1^0C)s + A_2s^2$$

be the $\mathcal{D}$-stable central system matrix when diagonal entries in A_2 are equal to their nominal value 10. After 2.2 seconds of CPU time, solving the LMI problem of Corollary 2 yields

$$F_0^R = \begin{bmatrix} -7.992 & -11.22 & 22.39 \\ -3.851 & 3.637 & 12.42 \end{bmatrix}, \quad F_1^R = \begin{bmatrix} -21.55 & 12.41 & 7.180 \\ 22.23 & -10.93 & 7.767 \end{bmatrix}$$

as robustly stabilizing feedback matrices. In Figure 3 we represent the closed-loop robust root locus for 10000 randomly chosen systems in the admissible uncertainty range. Closed-loop poles of the 8 polytope vertices are represented by red stars.

5. CONCLUSION

In this paper, we describe a simple but efficient approach to design robust proportional-derivative feedback controller for second-order systems. A sufficient LMI condition for robust stabilizability is obtained as an extension of the results of [17]. Additional decision variables are introduced to allow decoupling between closed-loop system matrices and Lyapunov-like matrices ensuring stability. Thanks to this decoupling, the design LMIs remain linear in the controller matrices and structural constraints can be readily incoporated. Moreover, recent results on strict positive realness and positivity of polynomial matrices can be invoked to give a clear interpretation of the additional decision variables as a central, or nominal closed-loop polynomial matrix. The whole degrees of conservatism of the approach are captured by the choice of this matrix.

Further research must be devoted to the choice of the central polynomial matrix, since it is the crucial step in the whole design procedure. Numerical examples illustrate that the nominal closed-loop polynomial matrix obtained by some standard design algorithm is often a good choice. If no closed-loop polynomial matrix is available, then an arbitrary choice (e. g. a diagonal matrix with entries $(s + \alpha)^2$ where α is a given real) is also sometimes suitable, provided it has zeros within the stability region where closed-loop poles must be located.

As far as first-order systems are concerned, necessary and sufficient LMI conditions for quadratic stability (i. e. robust stability guaranteed by an uncertainty-independent Lyapunov matrix) were obtained via a linearizing change of variables first proposed in [3]. It is open question to know whether such a technique can also be applied to second-order systems, and, by extension, to higher-order systems described by polynomial matrices.

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