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THE MULTISAMPLE VERSION OF THE LEPAGE TEST

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The two-sample Lepage test, devised for testing equality of the location and scale parameters against the alternative that at least for one of the parameters the equality does not hold, is extended to the general case of $k > 1$ sampled populations. It is shown that its limiting distribution is the chi-square distribution with $2(k-1)$ degrees of freedom. This k-sample statistic is shown to yield consistent test and a formula for its noncentrality parameter under Pitman alternatives is derived. For some particular alternatives, the power of the k-sample test is compared with the power of the Kruskal–Wallis test or with the power of the Ansari–Bradley test by means of simulation estimates. Multiple comparison methods for detecting differing populations, based on this multisample version of the Lepage test or on the multisample version of the Ansari–Bradley test, are also constructed.

Keywords: multisample rank test for location and scale, Lepage statistic, consistency, noncentrality parameter, multiple comparisons for location and scale parameters

AMS Subject Classification: 62G10

1. INTRODUCTION

Perhaps the most widely used two-sample rank test of equality of location parameters is the Wilcoxon–Mann–Whitney test, constructed by Wilcoxon in [24] and by Mann and Whitney in [16]. Its practical use is explained in currently used monographs like [3] or [10]. When it is desirable to test the equality of the scale parameters of two underlying populations by means of a rank test, then in the case of the equal medians usually the Ansari–Bradley test, constructed in [1] is used, the formulas and tables for this test can be found also in [10]. However, while the Wilcoxon–Mann–Whitney test does not react well to changes in the scale parameters when the location remains constant, analogously the Ansari–Bradley test has not good sensitivity to changes in the location parameters when the scale remains constant. For testing the hypothesis of equality both of the location and scale parameters of two populations against the alternative that at least for one of the parameters the equality does not hold, the monograph [10] recommends to use the Lepage test constructed in [14]. The Lepage test statistic is a combination of the Wilcoxon–Mann–Whitney and the Ansari–Bradley test statistics; a version of this two-sample test, based on general scores, has been studied also in [5].

Since the Lepage test statistic has under the null hypothesis asymptotically the chi-square distribution with 2 degrees of freedom, it can be used either by means of tables from [15] and [10], or with the help of the asymptotic approximation by means of the critical values of the chi-square distribution. A multisample version of the Ansari–Bradley test statistic was proposed by Puri in [18] (more detailed computational formulas for this multisample statistic are given in [23]). Further, it is well known that the multisample extension of the Wilcoxon–Mann–Whitney test is the Kruskal–Wallis test (described in [12] and [13]), because in the two sample case both the tests yield the same critical region for testing the equality of the location parameters. In an analogy with the two-sample case, the multisample version of the Lepage statistic is in the Section 2 proposed to be the sum of the Kruskal–Wallis and the Ansari–Bradley statistics. Consistency of the resulting test, limiting distribution of this multisample statistic and its behaviour under Pitman alternatives are the topic of Theorem 2.1 of the mentioned section.

A general assertion on multiple comparison procedure is in Theorem 2.2, a procedure for detecting populations differing in scale parameters (when the location parameters are assumed to be equal but can be unknown) is derived by means of Theorem 2.3 (II) and labelled as (2.42) and (2.43). A multiple comparison procedure for detecting difference either in the location or in the scale parameter (and aimed also at the use in conjunction with the multisample version of the Lepage test statistic) is derived by means of Theorem 2.4 and labelled as (2.46) – (2.47). In principle, all the comparison rules constructed in this paper can be used with their exact critical constants based on the uniform distribution of the ranks of the pooled random sample, but similarly as in the case of the multisample tests, only the critical constants of the constructed rules based on the asymptotic approximation are mentioned, because the tables of the exact constants including all possible sample sizes with values in a chosen bounded range would be very extensive.

The Section 2 contains also some simulation results on the power of the Kruskal–Wallis, the Ansari–Bradley and the multivariate extension of the Lepage test. The proofs of the assertions of the Section 2, as well as some assertions on limiting distribution of location-scale problem test statistics based on general score functions, can be found in the Section 3.

2. MAIN RESULTS

It is supposed throughout the paper that $X_{j1}, \ldots, X_{jn_j}$ is a random sample from the distribution with a continuous distribution function $F_j(x)$ and these $j = 1, \ldots, k$ random samples are independent. The topic is the statistical inference on the null hypothesis

$$H_0: \quad F_1 \equiv F_2 \equiv \ldots \equiv F_k\,. \tag{2.1}$$

Behaviour of the tests under the alternative will be described by means of the following assumption.

(A1) For $j = 1, \dots, k$ the sample $X_{j1}, \dots, X_{jn_j}$ is a random sample from the distribution of the random variable

$$\zeta_j = \sigma_j \varepsilon_j + \mu_j \tag{2.2}$$

where $\sigma_j > 0$, μ_j are real numbers and $\varepsilon_1, \dots, \varepsilon_k$ are independent identically distributed random variables with the continuous distribution function

$$F(x) = P(\varepsilon_j < x) \, . \tag{2.3}$$

Let

$$N = n_1 + \dots + n_k \tag{2.4}$$

denote the total sample size. The sample sizes are such that

$$\min(n_1, \dots, n_k) \longrightarrow \infty \, , \tag{2.5}$$

and for the relative sample sizes

$$\hat{p}_j = \frac{n_j}{N} \tag{2.6}$$

the relations

$$\lim \hat{p}_j = p_j > 0 \, , \quad j = 1, \dots, k \tag{2.7}$$

hold.

It is assumed in (A1) that $n_j = n_j^{(u)}$, where $u = 1, 2, \dots$ denotes the index of the experiment. Hence also $N = N^{(u)}$, $\hat{p}_j = \hat{p}_j^{(u)}$ and by the limit in (2.7) one understands the limit as u tends to infinity. The location and scale parameters μ_j, σ_j are in (2.2) described as being fixed, and this is how (A1) will be used in the assertion (II) of Theorem 2.1. However, in the third part of the mentioned theorem they are allowed to vary with u in the way, described in (2.21). For the sake of brevity, in most cases the index u of the experiment will be omitted.

Under validity of (A1) the null hypothesis (2.1) can be expressed as

$$H_0: \qquad \mu_1 = \mu_2 = \dots = \mu_k \, , \qquad \sigma_1 = \sigma_2 = \dots = \sigma_k \, . \tag{2.8}$$

Now assume that

$$X = (X_{11}, \dots, X_{1n_1}, \dots, X_{j1}, \dots, X_{jn_j}, \dots, X_{k1}, \dots, X_{kn_k}) \tag{2.9}$$

denotes the pooled random sample and

$$R^{(N)} = (R_{11}, \dots, R_{1n_1}, \dots, R_{j1}, \dots, R_{jn_j}, \dots, R_{k1}, \dots, R_{kn_k}) \tag{2.10}$$

denotes its ranks, i.e., $R_{j1}, \dots, R_{jn_j}$ are the ranks of the sample from the jth population.

The multisample Ansari–Bradley statistic will be defined by means of the score vector

$$b_N = \begin{cases} (1, 2, 3, \dots, m, m, \dots, 3, 2, 1) & N = 2m \, , \\ (1, 2, 3, \dots, m, \frac{N+1}{2}, m, \dots, 3, 2, 1) & N = 2m + 1 \, , \end{cases} \tag{2.11}$$

and the partial sums

$$S_j^{(b)} = \sum_{i=1}^{n_j} b_N(R_{ji}), \quad j = 1, \dots, k. \tag{2.12}$$

Put

$$v_N^2 = \begin{cases} \frac{N(N^2-4)}{48(N-1)} & N \text{ even}, \\ \frac{(N+1)(N^2+3)}{48N} & N \text{ odd}, \end{cases} \tag{2.13}$$

$$\tilde{\mu}_N = \begin{cases} \frac{(N+2)}{4} & N \text{ even}, \\ \frac{(N+1)^2}{4N} & N \text{ odd}. \end{cases} \tag{2.14}$$

The multisample version of the Ansari–Bradley statistic is defined by the formula

$$T_B = \frac{1}{v_N^2} \sum_{j=1}^{k} n_j \left(\frac{S_j^{(b)}}{n_j} - \tilde{\mu}_N \right)^2 = \frac{1}{v_N^2} \sum_{j=1}^{k} \frac{\left(S_j^{(b)} - n_j \tilde{\mu}_N\right)^2}{n_j}, \tag{2.15}$$

which is equivalent to the expression for the multisample Ansari–Bradley statistic, given on p. 792 of [23].

Let

$$w_N^2 = \frac{N(N+1)}{12} \tag{2.16}$$

and the partial sum

$$S_j = \sum_{i=1}^{n_j} R_{ji}. \tag{2.17}$$

Then

$$T_K = \frac{1}{w_N^2} \sum_{j=1}^{k} n_j \left(\frac{S_j}{n_j} - \frac{N+1}{2} \right)^2 = \frac{1}{w_N^2} \sum_{j=1}^{k} \frac{\left(S_j - n_j \frac{N+1}{2}\right)^2}{n_j} \tag{2.18}$$

is the well-known Kruskal–Wallis test statistic.

Theorem 2.1. Put

$$T = T_K + T_B. \tag{2.19}$$

(I) Suppose that for the continuous distribution functions mentioned at the beginning of the section the hypothesis (2.1) holds. If also (2.5) is fulfilled, then the statistic (2.19) is asymptotically χ^2-distributed with $2(k-1)$ degrees of freedom.

(II) Suppose that (A1) holds. Then the test of (2.8) based on (2.19) is consistent, i. e., if (2.8) is not fulfilled, then for the statistic (2.19)

$$\lim P(T > t) = 1 \tag{2.20}$$

for every positive real number t.

(III) Let us assume that for $j = 1, \dots, k$ the sample $X_{j1}, \dots, X_{jn_j}$ is a random sample from the distribution of the random variable (2.2) where μ_j, σ_j depend on the index u of the experiment in such a way that

$$\sigma_j = \sigma_j^{(u)} = \sigma + \frac{\sigma_j^*}{\sqrt{N}}, \quad \mu_j = \mu_j^{(u)} = \mu + \frac{\mu_j^*}{\sqrt{N}}, \quad \sigma > 0,\ \mu, \sigma_j^*, \mu_j^* \text{ are real numbers.} \tag{2.21}$$

Suppose that both (2.5) and (2.7) hold. If the distribution function (2.3) possesses with respect to the Lebesgue measure a bounded density f, which is continuous at every x (with the possible exception of the finite number of real numbers) and

$$\int_{-\infty}^{+\infty} |x| f(x)\,\mathrm{d}x < +\infty, \tag{2.22}$$

then the statistic (2.19) has asymptotically the chi-square distribution with $2(k-1)$ degrees of freedom and the noncentrality parameter

$$\delta_T = \delta_K + \delta_B, \quad \delta_K = 12\sum_{j=1}^{k} p_j(\nu_j^{(\varphi)})^2, \ \delta_B = 48\sum_{j=1}^{k} p_j(\nu_j^{(\psi)})^2, \tag{2.23}$$

where

$$\nu_j^{(\varphi)} = \int_{-\infty}^{+\infty} \left(\frac{\sigma_j^* - \overline{\sigma}}{\sigma} x + \frac{\mu_j^* - \overline{\mu}}{\sigma}\right) f^2(x)\,\mathrm{d}x, \tag{2.24}$$

$$\nu_j^{(\psi)} = \int_{-\infty}^{+\infty} \left(\frac{\sigma_j^* - \overline{\sigma}}{\sigma} x + \frac{\mu_j^* - \overline{\mu}}{\sigma}\right) \operatorname{sign}(0.5 - F(x)) f^2(x)\,\mathrm{d}x, \tag{2.25}$$

$$\overline{\sigma} = \sum_{j=1}^{k} p_j \sigma_j^*, \quad \overline{\mu} = \sum_{j=1}^{k} p_j \mu_j^*. \tag{2.26}$$

The statistic (2.19) is designed for the situation, when $F_j(x) = F((x - \mu_j)/\sigma_j)$ and F is a continuous distribution function. If the observed value of T is greater than the $1 - \alpha$ quantile of the chi-square distribution with $2(k-1)$ degrees of freedom, then the null hypothesis (2.1), corresponding in this case to (2.8), is rejected.

According to (II) of the previous theorem the test based on (2.19) is consistent at any fixed alternative $\mu_1, \sigma_1 > 0, \dots, \mu_k, \sigma_k > 0$ not fulfilling (2.8). For the Pitman alternatives (2.21) the noncentrality parameter (2.23) of T is the sum of components δ_K and δ_B, corresponding to the Kruskal–Wallis test and the Ansari–Bradley test, respectively. If in addition to the assumptions of Theorem 2.1 (III) for the density f the equality $f(x) = f(-x)$ holds for all x, then from the asymptotic local point of view according to (2.23) – (2.26) the Kruskal–Wallis statistic contributes to the overall power only through a response to the location and the Ansari–Bradley statistic only through a response to the scale parameter.

If the assumptions of the assertion (III) of the previous theorem hold in the normality setting, i. e., if the random variables $\varepsilon_1, \dots, \varepsilon_k$ are $N(0,1)$ distributed, then an application of (2.23) yields that in this case the noncentrality parameter

$$\delta_T = \frac{3}{\pi} \sum_{j=1}^{k} p_j \frac{(\mu_j^* - \overline{\mu})^2}{\sigma^2} + \frac{12}{\pi^2} \sum_{j=1}^{k} p_j \frac{(\sigma_j^* - \overline{\sigma})^2}{\sigma^2} . \tag{2.27}$$

The asymptotically optimal statistic for testing (2.8) based on the normality assumption is the likelihood ratio test statistic (its optimality in the sense of exact slopes follows from Theorem 3.1 of [21]). After some computation one obtains from the Corollary 1.2 and the formulas (1.21), (1.28) of [22] that in the normality setting under the local alternatives (2.21) the likelihood ratio statistic has asymptically chi-square distribution with $2(k-1)$ degrees of freedom and the non-centrality parameter

$$\lambda = \sum_{j=1}^{k} p_j \frac{(\mu_j^* - \overline{\mu})^2}{\sigma^2} + 2 \sum_{j=1}^{k} p_j \frac{(\sigma_j^* - \overline{\sigma})^2}{\sigma^2} . \tag{2.28}$$

Thus

$$0.6079 = \frac{6}{\pi^2} \le \frac{\delta_T}{\lambda} \le \frac{3}{\pi} = 0.9549 , \tag{2.29}$$

where the lower bound is attained if there is no change in the location and the upper bound is attained when there is no change in the scale parameter.

The following tables contain simulation results on the fit of the size of the test based on (2.19) with a chosen significance level or results concerning the comparison

Table 1. Simulation estimates of the tail probabilities under validity of (2.1) for k=3.

$n_1\ n_2\ n_3$	6, 6, 6		10, 10, 10		10, 10, 15	
α	0.05	0.1	0.05	0.1	0.05	0.1
$P\big(T_B > \chi^2_\alpha(k-1)\big)$	0.038	0.090	0.044	0.096	0.047	0.102
$P\big(T_K > \chi^2_\alpha(k-1)\big)$	0.041	0.098	0.050	0.103	0.046	0.093
$P\big(T > \chi^2_\alpha(2(k-1))\big)$	0.031	0.075	0.040	0.091	0.042	0.091

$n_1\ n_2\ n_3$	10, 15, 15		15, 15, 15		20, 20, 20	
α	0.05	0.1	0.05	0.1	0.05	0.1
$P\big(T_B > \chi^2_\alpha(k-1)\big)$	0.046	0.102	0.046	0.102	0.050	0.099
$P\big(T_K > \chi^2_\alpha(k-1)\big)$	0.048	0.101	0.046	0.097	0.052	0.104
$P\big(T > \chi^2_\alpha(2(k-1))\big)$	0.041	0.100	0.042	0.094	0.048	0.099

of this test with the Kruskal–Wallis and the Ansari–Bradley test. The simulation estimates in all tables are based on $N = 10000$ trials for each particular case. In Table 1 (as well as in the whole text) $\chi^2_\alpha(m)$ denotes the $1-\alpha$ quantile of the chi-square distribution with m degrees of freedom.

The simulation results from Table 1 suggest that for $k = 3$ the approximation of the exact critical constant of the statistic (2.19) with its asymptotic counterpart $\chi^2_\alpha(2(k-1))$ yields size of the test close to the nominal significance level when all sample sizes are at least 10, for smaller sample sizes the size of the test usually remains below the nominal value.

The Ansari–Bradley test, designed for testing the equality of the scale parameters, rejects the null hypothesis (2.1) if $T_B > \chi^2_\alpha(k-1)$, and the Kruskal–Wallis test designed for testing the equality of the location parameters rejects (2.1) if $T_K > \chi^2_\alpha(k-1)$. The behaviour of these tests in situations when the null hypothesis is violated only in one type of the parameter, is illustrated by simulation estimates of their power, when for $j = 1, 2, 3$ the jth random sample of size n_j is taken from the normal distribution with the mean μ_j and the variance σ_j^2.

Table 2. Simulation estimates of the power when the change occurs in the location parameter.

$\mu_1 = 0,\ \mu_2 = 0,\ \mu_3 = 0.5,\quad \sigma_1 = \sigma_2 = \sigma_3 = 1$								
$n_1\ n_2\ n_3$	10, 10, 10		10, 15, 15		15, 15, 15		15, 25, 35	
α	0.05	0.1	0.05	0.1	0.05	0.1	0.05	0.1
$P\left(T_B > \chi^2_\alpha(k-1)\right)$	0.042	0.091	0.042	0.091	0.046	0.096	0.045	0.092
$P\left(T_K > \chi^2_\alpha(k-1)\right)$	0.159	0.266	0.217	0.335	0.241	0.365	0.438	0.569
$P\left(T > \chi^2_\alpha(2(k-1))\right)$	0.105	0.198	0.147	0.257	0.166	0.280	0.325	0.464

Table 3. Simulation estimates of the power when the change occurs in the scale parameter.

$\mu_1 = \mu_2 = \mu_3 = 0,\quad \sigma_1 = 1,\ \sigma_2 = 1,\ \sigma_3 = 1.5$								
$n_1\ n_2\ n_3$	10, 10, 10		10, 15, 15		15, 15, 15		15, 25, 35	
α	0.05	0.1	0.05	0.1	0.05	0.1	0.05	0.1
$P\left(T_B > \chi^2_\alpha(k-1)\right)$	0.131	0.225	0.177	0.281	0.199	0.309	0.345	0.468
$P\left(T_K > \chi^2_\alpha(k-1)\right)$	0.049	0.103	0.046	0.096	0.049	0.104	0.042	0.089
$P\left(T > \chi^2_\alpha(2(k-1))\right)$	0.093	0.183	0.125	0.219	0.143	0.246	0.245	0.367

The results in the Tables 2 and 3 show that while T may react to the parameter change weaker than the statistic designed especially for the underlying type of alternative, T reacts more strongly when compared with the statistic not designed for the given alternative (as the mentioned results show the latter may not at all react, because in Table 2 the power of the Ansari–Bradley test coincides with the nominal level of significance and the same situation is in Table 3 with the Kruskal–Wallis test). Therefore if one is not sure what type of the alternative (either location or scale change) will occur in practice, the test of (2.1) based on T is preferable to the Ansari–Bradley and to the Kruskal–Wallis test.

The alternative that the change will occur merely in the location or merely in the scale parameter can sometimes be perceived as not to be of the proper nature, because in some situations the increase of the response level (i. e., the increase of the location parameter) is accompanied with an increase of its variability. Behaviour of the previously mentioned tests in such a situation is illustrated by the power estimates given in the following table, where as in the previous cases n_j denotes size of the sample from the normal distribution with mean μ_j and the standard deviation σ_j.

Table 4. Simulation estimates of the power when the change occurs both in the location and in the scale parameter.

$\mu_1 = 0,\ \sigma_1 = 1,\ \ \mu_2 = 0.3,\ \sigma_2 = 1.5,\ \ \mu_3 = 0.8,\ \sigma_3 = 2$								
$n_1\ n_2\ n_3$	10, 10, 10		10, 15, 15		15, 15, 15		15, 25, 35	
α	0.05	0.1	0.05	0.1	0.05	0.1	0.05	0.1
$P\Big(T_B > \chi^2_\alpha(k-1)\Big)$	0.213	0.330	0.241	0.370	0.334	0.470	0.430	0.573
$P\Big(T_K > \chi^2_\alpha(k-1)\Big)$	0.137	0.232	0.151	0.253	0.199	0.307	0.263	0.392
$P\Big(T > \chi^2_\alpha(2(k-1))\Big)$	0.221	0.363	0.260	0.412	0.390	0.540	0.539	0.691

The simulation results presented in Table 4 suggest that when the change in the location parameter is comparable with the change in the scale, then it may happen that the test based on (2.19) will be more powerful than the Ansari–Bradley or the Kruskal–Wallis test.

Before proceeding with a multiple comparison procedure based on (2.19) we pay attention to procedures based on the components of this statistic.

It is observed on p. 131 of [4] that an analogue of the pairwise multiple comparison procedure using the Wilcoxon scores and constructed ibidem, can also be constructed in the joint ranking case. The following theorem is an extension of this assertion into a general framework. In this theorem the quantity $Q_k^{(\alpha)}$ fulfills the equality

$$P\left(\max_{1\le i,j\le k} |y_i - y_j| > Q_k^{(\alpha)} \,\middle|\, \mathcal{L}(y) = N_k(\mathbf{0}, \boldsymbol{I}_k)\right) = \alpha\,, \tag{2.30}$$

where y_i denotes the ith coordinate of y and $\boldsymbol{I}_k$ is the $k\times k$ identity matrix, i. e., $Q_k^{(\alpha)}$ denotes the $1-\alpha$ quantile of the maximum modulus of the $N_k(\mathbf{0}, \boldsymbol{I}_k)$ distribution.

Theorem 2.2. Let (2.1), (2.5) and (2.7) hold. Suppose that $\varphi : (0,1) \longrightarrow E^1$ is a function expressible as a finite sum of monotone square integrable functions such that for $\overline{\varphi} = \int_0^1 \varphi(u)\,\mathrm{d}u$

$$V_\varphi = \int_0^1 (\varphi(u) - \overline{\varphi})^2\ \mathrm{d}u$$

is a positive real number. Let the scores

$$a_N(j) = d_N\, \varphi\left(\frac{j}{(N+1)}\right), \quad j = 1, \dots, N\,; \tag{2.31}$$

where the real number $d_N \neq 0$. Put

$$\sigma_N^2 = \frac{1}{N-1}\sum_{j=1}^{N}(a_N(j) - \overline{a}_N)^2\,, \tag{2.32}$$

where $\overline{a}_N$ stands for the arithmetic mean of $a_N(1), \dots, a_N(N)$. For $j = 1, \dots, k$ let (cf. (2.10))

$$S_j^{(\varphi)} = \sum_{i=1}^{n_j} a_N(R_{ji}) \tag{2.33}$$

and for $j_1 < j_2$

$$D_{j_1j_2}^{(\varphi)} = \frac{\frac{S_{j_1}^{(\varphi)}}{n_{j_1}} - \frac{S_{j_2}^{(\varphi)}}{n_{j_2}}}{\sqrt{\frac{1}{n_{j_1}} + \frac{1}{n_{j_2}}}}\sqrt{\frac{2}{\sigma_N^2}}\,. \tag{2.34}$$

For the statistic $M_{n_1,\dots,n_k}^{(\varphi)} = \max\left\{\left|D_{j_1j_2}^{(\varphi)}\right|;\ 1 \le j_1 < j_2 \le k\right\}$ the convergence (cf. (2.30))

$$P\Big(M_{n_1,\dots,n_k}^{(\varphi)} > Q_k^{(\alpha)}\Big) \longrightarrow \gamma \le \alpha \tag{2.35}$$

holds, and if $p_1 = \dots p_k = \frac{1}{k}$, then $\gamma = \alpha$.

An application of the previous assertion yields the following theorem.

Theorem 2.3. Let (2.1), (2.5) and (2.7) hold.

(I) Suppose that

$$M_{n_1,\dots,n_k}^{(\varphi)} = \max\left\{\left|D_{j_1j_2}^{(\varphi)}\right|;\ 1 \le j_1 < j_2 \le k\right\}, \tag{2.36}$$

$$D_{j_1j_2}^{(\varphi)} = \sqrt{\frac{24}{N(N+1)}}\,\frac{\frac{S_{j_1}}{n_{j_1}} - \frac{S_{j_2}}{n_{j_2}}}{\sqrt{\frac{1}{n_{j_1}} + \frac{1}{n_{j_2}}}}\,, \tag{2.37}$$

where S_j is the partial sum (2.17). Then the convergence (2.35) holds and $\gamma = \alpha$ if $p_1 = \dots p_k = \frac{1}{k}$.

(II) Suppose that

$$M_{n_1,\dots,n_k}^{(\psi)} = \max\left\{\left|D_{j_1j_2}^{(\psi)}\right|;\ 1 \le j_1 < j_2 \le k\right\}, \tag{2.38}$$

$$D_{j_1j_2}^{(\psi)} = \sqrt{\frac{2}{v_N^2}}\,\frac{\frac{S_{j_1}^{(b)}}{n_{j_1}} - \frac{S_{j_2}^{(b)}}{n_{j_2}}}{\sqrt{\frac{1}{n_{j_1}} + \frac{1}{n_{j_2}}}}\,, \tag{2.39}$$

where v_N^2 is defined in (2.13) and $S_j^{(b)}$ is the partial sum (2.12). Then the convergence

$$P\Big(M_{n_1,\dots,n_k}^{(\psi)} > Q_k^{(\alpha)}\Big) \longrightarrow \gamma \le \alpha \tag{2.40}$$

holds and $\gamma = \alpha$ if $p_1 = \dots p_k = \frac{1}{k}$.

By means of the previous theorem we construct multiple comparisons procedures, used in conjunction with the concerned test rejecting the null hypothesis (2.1) if the test statistic exceeds the quantile of the chi-square distribution.

The Kruskal–Wallis statistic (2.18) is designed for the situation, when $F_j(x) = F(x-\mu_j)$. If for the observed value of T_K the inequality $T_K > \chi^2_\alpha(k-1)$ holds, then the null hypothesis (2.1) is rejected. Declare the j_1th and the j_2th populations to be different (i.e., the location parameters $\mu_{j_1} \neq \mu_{j_2}$), if for (2.37) the inequality

$$|D^{(\varphi)}_{j_1 j_2}| > Q^{(\alpha)}_k \tag{2.41}$$

holds. If $n_1 = \dots = n_k = n$, then the rule (2.41) becomes

$$\frac{1}{n}\sqrt{\frac{12}{k(kn+1)}}|S_{j_1} - S_{j_2}| > Q^{(\alpha)}_k\,,$$

which is the Nemenyi method for equal sample sizes, derived in [17]. It should be noted here that the rule (2.41) is an improvement of the rule (110) from p. 166 of [17] because of the reduction of the size of its critical constant.

The multisample version (2.15) of the Ansari–Bradley statistic is designed for the situation, when $F_j(x) = F(x/\sigma_j)$. If for the observed value of T_B the inequality $T_B > \chi^2_\alpha(k-1)$ holds, then the null hypothesis (2.1) is rejected. Declare the j_1th and the j_2th populations to be different (i.e., the scale parameters $\sigma_{j_1} \neq \sigma_{j_2}$), if for (2.39) the inequality

$$|D^{(\psi)}_{j_1 j_2}| > Q^{(\alpha)}_k \tag{2.42}$$

holds. If $n_1 = \dots = n_k = n$, then the rule (2.42) becomes

$$\frac{|S^{(b)}_{j_1} - S^{(b)}_{j_2}|}{\sqrt{n v^2_N}} > Q^{(\alpha)}_k\,. \tag{2.43}$$

Theorem 2.4. Let (2.1), (2.5) and (2.7) hold. Suppose that (cf. (2.36) – (2.39))

$$M_{n_1,\dots,n_k} = \max\{M^{(\varphi)}_{n_1,\dots,n_k}\,,\ M^{(\psi)}_{n_1,\dots,n_k}\}\,. \tag{2.44}$$

If $\alpha \in (0,1)$ and $\beta = 1 - \sqrt{1-\alpha}$, then the convergence (cf. (2.30))

$$P\Big(M_{n_1,\dots,n_k} > Q^{(\beta)}_k\Big) \longrightarrow \gamma \leq \alpha \tag{2.45}$$

holds and $\gamma = \alpha$ if $p_1 = \dots p_k = \frac{1}{k}$.

As has already been mentioned, the statistic (2.19) is designed for the situation, when $F_j(x) = F((x-\mu_j)/\sigma_j)$. If for the observed value of T the inequality $T >$

$\chi^2_\alpha(2(k-1))$ holds, then the null hypothesis (2.8) is rejected. Declare the j_1th and the j_2th populations to be different if for (2.37), (2.39) at least one of the inequalities

$$|D^{(\varphi)}_{j_1j_2}| > Q^{(\beta)}_k\,, \tag{2.46}$$

$$|D^{(\psi)}_{j_1j_2}| > Q^{(\beta)}_k\,, \tag{2.47}$$

holds; here

$$\beta = 1 - \sqrt{1-\alpha} \tag{2.48}$$

and as in the previous cases, the constant $Q^{(\beta)}_k$ is defined by means of (2.30). The validity of (2.46) is interpreted as the difference $\mu_{j_1} \neq \mu_{j_2}$ of the location parameters and the validity of (2.47) as the difference $\sigma_{j_1} \neq \sigma_{j_2}$ of the scale parameters. The tables of the constants fulfilling (2.30) are published in [8], but since for the usual significance levels α the quantity (2.48) has values not included in these tables, the use of the approximation

$$\beta = \frac{\alpha}{2}$$

can be recommended, because the use of the rule (2.46)–(2.47) with a chosen β corresponds to the significance level $\alpha = 2\beta - \beta^2$ of the test and the critical constants Q used in the multiple comparisons rules of this paper are of approximative asymptotic nature.

3. PROOFS

The assertion of Theorem 2.1 (I) on the limiting null distribution will be carried out by means of the following theorem.

Theorem 3.1. Let us assume that $\varphi : (0,1) \to E^1$, $\psi : (0,1) \to E^1$ and each of these functions is expressible as a finite sum of monotone square integrable functions. Put

$$\begin{gathered} \overline{\varphi} = \int_0^1 \varphi(u)\,\mathrm{d}u\,, \qquad \overline{\psi} = \int_0^1 \psi(u)\,\mathrm{d}u \\ V_\varphi = \int_0^1 (\varphi(u)-\overline{\varphi})^2\,\mathrm{d}u\,, \quad V_\psi = \int_0^1 (\psi(u)-\overline{\psi})^2\,\mathrm{d}u\,, \\ V_{\varphi,\psi} = \int_0^1 (\varphi(u)-\overline{\varphi})(\psi(u)-\overline{\psi})\,\mathrm{d}u\,, \end{gathered} \tag{3.1}$$

and suppose that the matrix

$$\boldsymbol{V} = \begin{pmatrix} V_\varphi & V_{\varphi,\psi} \\ V_{\varphi,\psi} & V_\psi \end{pmatrix} \tag{3.2}$$

is regular. For $j = 1, \dots, k$ let (cf. (2.10), (2.4))

$$\begin{gathered} S^{(\varphi)}_j = \sum_{i=1}^{n_j} \varphi\left(\frac{R_{ji}}{N+1}\right), \quad \sigma^{2,\varphi}_N = \frac{1}{N-1}\sum_{i=1}^{N}\left(\varphi\Big(\frac{i}{N+1}\Big) - \tilde{\varphi}\right)^2, \\ \tilde{\varphi} = \frac{1}{N}\sum_{i=1}^{N}\varphi\left(\frac{i}{N+1}\right), \end{gathered} \tag{3.3}$$

and the quantities $S_j^{(\psi)}$, $\sigma_N^{2,\psi}$, $\tilde{\psi}$ are defined similarly. Let both (2.1) and (2.5) hold.

(I) Suppose that the relative sample sizes (2.6) are such that $\hat{p}_j \to p_j$. Put

$$Z = (\xi_1, \dots, \xi_k, \eta_1, \dots, \eta_k)', \quad \xi_j = \frac{S_j^{(\varphi)} - n_j \tilde{\varphi}}{\sqrt{n_j \sigma_N^{2,\varphi}}}, \quad \eta_j = \frac{S_j^{(\psi)} - n_j \tilde{\psi}}{\sqrt{n_j \sigma_N^{2,\psi}}}. \tag{3.4}$$

Then

$$Z \longrightarrow N_{2k}(0, K_V \otimes A(p)) \tag{3.5}$$

in distribution. Here

$$K_V = \begin{pmatrix} 1 & \frac{V_{\varphi,\psi}}{\sqrt{V_\varphi V_\psi}} \\ \frac{V_{\varphi,\psi}}{\sqrt{V_\varphi V_\psi}} & 1 \end{pmatrix}, \quad A(p) = I_k - \sqrt{p}(\sqrt{p})', \quad \sqrt{p} = \begin{pmatrix} \sqrt{p_1} \\ \vdots \\ \sqrt{p_k} \end{pmatrix} \tag{3.6}$$

and $\otimes$ denotes the Kronecker product of matrices.

(II) The statistic

$$S^2 = \frac{V_\varphi V_\psi}{V_\varphi V_\psi - (V_{\varphi,\psi})^2} (Q_\varphi + Q_\psi - Q_{\varphi,\psi}), \tag{3.7}$$

where

$$Q_\varphi = \frac{1}{\sigma_N^{2,\varphi}} \sum_{j=1}^k n_j \Big(\frac{S_j^{(\varphi)}}{n_j} - \tilde{\varphi} \Big)^2, \quad Q_\psi = \frac{1}{\sigma_N^{2,\psi}} \sum_{j=1}^k n_j \Big(\frac{S_j^{(\psi)}}{n_j} - \tilde{\psi} \Big)^2, \tag{3.8}$$

$$Q_{\varphi,\psi} = \frac{2V_{\varphi,\psi}}{\sqrt{V_\varphi V_\psi \sigma_N^{2,\varphi} \sigma_N^{2,\psi}}} \sum_{j=1}^k n_j \Big(\frac{S_j^{(\varphi)}}{n_j} - \tilde{\varphi} \Big) \Big(\frac{S_j^{(\psi)}}{n_j} - \tilde{\psi} \Big), \tag{3.9}$$

converges in distribution to the chi-square distribution with $2(k-1)$ degrees of freedom.

Since the previous theorem can be proved similarly as the Theorem on p. 170 of [7] by means of Lemma a on p. 164 ibidem, the proof is omitted.

Proof of Theorem 2.1. (I) Suppose that the functions of the argument $x \in (0,1)$

$$\varphi(x) = x, \quad \psi(x) = \min\{x, 1-x\}. \tag{3.10}$$

Then the quantities (3.1) are

$$V_\varphi = \frac{1}{12}, \quad V_\psi = \frac{1}{48}, \quad V_{\varphi,\psi} = 0, \tag{3.11}$$

and φ, ψ fulfill the assumptions of the previous theorem. Since $j = (N+1)\varphi(j/(N+1))$ and the coordinates of the vector (2.11) fulfill the equality

$$b_N(j) = (N+1)\psi(\frac{j}{N+1}), \quad j = 1, \dots, N, \tag{3.12}$$

the validity of the assertion (I) of Theorem 2.1 can be verified by means of Theorem 3.1. □

The assertions (II) and (III) of Theorem 2.1 deal with the behaviour of the statistic under the alternative. Their proof will be carried out by means of the version of the Chernoff–Savage theorem stated in the next text. Since this version uses the following assumptions (A2) on existence of the derivatives of the score function which are slightly different from usually used conditions, we prefer to include it into the paper in order to make clear what a precise kind of assertion forms the base for the concerned proof.

(A2) $\psi : (0,1) \to E^1$ and there exist bounded functions $g_\psi^{(i)} : (0,1) \to E^1$, $i = 1,2$ and finitely many real numbers $a_0 = 0 < \ldots < a_s = 1$ such that for all $u \in (0,1) - \{a_0, \ldots, a_s\}$ the first two derivatives of ψ exist and

$$\psi'(u) = g_\psi^{(1)}(u)\,, \qquad \psi''(u) = g_\psi^{(2)}(u)\,,$$

$g_\psi^{(1)}$ is right-continuous and

$$\psi(t_2) - \psi(t_1) = \int_{t_1}^{t_2} g_\psi^{(1)}(t)\,\mathrm{d}t\,, \qquad g_\psi^{(1)}(u_2) - g_\psi^{(1)}(u_1) = \int_{u_1}^{u_2} g_\psi^{(2)}(t)\,\mathrm{d}t$$

for all $0 < t_1 < t_2 < 1$, the second equality holds whenever $u_1 < u_2$ belong to (a_i, a_{i+1}) and $i = 0, \ldots, s-1$.

Theorem 3.2. Suppose that (2.10) denotes the ranks of the pooled random sample (2.9), the relations (2.5), (2.7) hold and put

$$\hat{H}(x) = \sum_{j=1}^{k} \hat{p}_j F_j(x)\,, \qquad H(x) = \sum_{j=1}^{k} p_j F_j(x)\,. \tag{3.13}$$

(I) Assume that the function $\psi : (0,1) \to E^1$ fulfils (A2). Let (cf. (3.3))

$$T_j^{(\psi)} = \frac{S_j^{(\psi)}}{n_j}\,, \qquad \mu_j^{(\psi)} = \int_{-\infty}^{+\infty} \psi(\hat{H}(x))\,\mathrm{d}F_j(x)\,, \tag{3.14}$$

$$\boldsymbol{T}^{(\psi)} = (T_1^{(\psi)}, \ldots, T_k^{(\psi)})'\,, \qquad \boldsymbol{\mu}^{(\psi)} = (\mu_1^{(\psi)}, \ldots, \mu_k^{(\psi)})'\,. \tag{3.15}$$

Then the convergence in distribution

$$\sqrt{N}(\boldsymbol{T}^{(\psi)} - \boldsymbol{\mu}^{(\psi)}) \longrightarrow N_k(\boldsymbol{0},\, \Sigma) \tag{3.16}$$

holds. Here the diagonal elements of the asymptotic covariance matrix

$$\Sigma_{ii} = 2 \sum_{\substack{j=1 \\ j \neq i}}^{k} \sum_{\substack{t=1 \\ t \neq i}}^{k} \frac{p_j p_t}{p_i} I_{i,j,t} + 2 \sum_{\substack{j=1 \\ j \neq i}}^{k} p_j I_{j,i,i}\,,$$

and the off-diagonal elements

$$\Sigma_{ir} = \sum_{j=1}^{k} p_j (I_{j,i,r} + I_{j,r,i} - I_{i,j,r} - I_{i,r,j} - I_{r,i,j} - I_{r,j,i}),$$

where

$$I_{i,j,t} = \iint_{x<y} F_i(x)(1 - F_i(y)) g_{\psi}^{(1)}(H(x)) g_{\psi}^{(1)}(H(y))\, \mathrm{d}F_j(x) \mathrm{d}F_t(y).$$

(II) Suppose that for $j = 1, \dots, k$ the distribution function F_j depends on the index u of the experiment in such way that

$$F_j(x) = F\left(\frac{x - \mu_j^{(u)}}{\sigma_j^{(u)}}\right), \tag{3.17}$$

where F is the continuous distribution function (2.3),

$$\lim_{u\to\infty} \sigma_j^{(u)} = \sigma, \quad \lim_{u\to\infty} \mu_j^{(u)} = \mu \tag{3.18}$$

and $\sigma > 0$, μ are real numbers. Let the score functions φ, ψ fulfill (A2) and, similarly as in (3.14), (3.15),

$$\boldsymbol{T}^{(\varphi)} = (T_1^{(\varphi)}, \dots, T_k^{(\varphi)})', \quad \boldsymbol{\mu}^{(\varphi)} = (\mu_1^{(\varphi)}, \dots, \mu_k^{(\varphi)})'. \tag{3.19}$$

Then for

$$\boldsymbol{T} = \begin{pmatrix} \boldsymbol{T}^{(\varphi)} \\ \boldsymbol{T}^{(\psi)} \end{pmatrix}, \quad \boldsymbol{\mu} = \begin{pmatrix} \boldsymbol{\mu}^{(\varphi)} \\ \boldsymbol{\mu}^{(\psi)} \end{pmatrix}, \tag{3.20}$$

the weak convergence of distributions

$$\mathcal{L}\Big(\sqrt{N}(\boldsymbol{T} - \boldsymbol{\mu})\Big) \longrightarrow N_{2k}(\mathbf{0}, \Xi) \tag{3.21}$$

holds. Here (cf. (3.1))

$$\Xi = \begin{pmatrix} V_{\varphi}\boldsymbol{\Gamma} & V_{\varphi,\psi}\boldsymbol{\Gamma} \\ V_{\varphi,\psi}\boldsymbol{\Gamma} & V_{\psi}\boldsymbol{\Gamma} \end{pmatrix}, \quad \boldsymbol{\Gamma} = \mathrm{diag}\Big(\frac{1}{p_1}, \dots, \frac{1}{p_k}\Big) - \mathbf{1}\,\mathbf{1}', \quad \mathbf{1} = (1, \dots, 1)' \in E^k. \tag{3.22}$$

We remark that the previous theorem can be proved in the same way as Theorem 1 and its Corollary 2 in [2], Theorem 3.6.5 on p. 104 of [19] (the formulas for the asymptotic covariance matrix can be found also in [6]), the second part of the previous theorem can be proved similarly as Theorem 5.6.1 on p. 204 of [19].

Proof of Theorem 2.1. (II). Suppose that (2.8) does not hold and put (cf. (2.2))

$$q_{i,j} = \begin{cases} P(\zeta_i > \zeta_j) & i \neq j, \\ \frac{1}{2} & i = j. \end{cases} \tag{3.23}$$

If there exists i such that for the limits (2.7) the inequality $\sum_j p_j q_{i,j} \neq \frac{1}{2}$ holds, then according to Lemma 5.3 of [12] the test based on T_K is consistent and since $T \geq T_K$, the relation (2.20) holds. Assume therefore that

$$\sum_{j=1}^{k} p_j q_{i,j} = \frac{1}{2}, \quad i = 1, \ldots, k. \tag{3.24}$$

Since $T \geq T_B$, it is sufficient to prove that the test based on T_B is consistent. However,

$$T_B = (v_N^2/N^2)^{-1} \sum_{j=1}^{k} \tilde{t}_j^2, \quad \tilde{t}_j = \frac{1}{\sqrt{n_j N^2}} \left(S_j^{(b)} - n_j \tilde{\mu}_N \right),$$

which means that it is enough to show that for some j the test based on $|\tilde{t}_j|$ is consistent. But

$$\operatorname{Var}(\tilde{t}_j | H_0) \longrightarrow \frac{1 - p_j}{48}$$

and therefore it is sufficient to prove that for some j and

$$t_j = t_j^{(u)} = \frac{1}{n_j N} S_j^{(b)} \tag{3.25}$$

the equality

$$\lim_{u \to \infty} P \left(\frac{|t_j - E(t_j | H_0)|}{\sqrt{\operatorname{Var}(t_j | H_0)}} > \gamma \right) = 1 \tag{3.26}$$

holds for each positive real number γ. Assume for a while that j is fixed and put (cf. (3.13))

$$\mu_N = \frac{1}{2} - \int_{-\infty}^{+\infty} \left| \hat{H}(x) - \frac{1}{2} \right| \mathrm{d}F_j(x), \quad \mu_0 = E(t_j | H_0).$$

The limit $D_j = \lim_{u \to \infty} (\mu_N - \mu_0)$ exists and

$$D_j = \frac{1}{4} - \int_{-\infty}^{+\infty} \left| H(x) - \frac{1}{2} \right| \mathrm{d}F_j(x). \tag{3.27}$$

It follows from the formula (3.16) of the Chernoff–Savage theorem that (3.26) will hold if $D_j \neq 0$. We shall find an index j with this property.

Choose a number x_0 such that

$$H(x_0) = \frac{1}{2}. \tag{3.28}$$

Since the random variables (2.2) are independent, for $j \neq i$

$$F_i(x) = P(\zeta_i < x) = P(\zeta_i < \zeta_j | \zeta_j = x)$$

and therefore

$$\int_{-\infty}^{x_0} F_i(x)\,\mathrm{d}F_j(x) = P(\zeta_i < \zeta_j\,,\,\zeta_j < x_0)\,.$$

Making use of these properties of the conditional distribution, (3.24) and the fact that $F_j(x)$ is uniformly distributed, after some computation one obtains that

$$D_j = \frac{1}{4} + p_jF_j(x_0)^2 - F_j(x_0) + 2\int_{-\infty}^{x_0}\left(\sum_{i\neq j} p_iF_i(x)\right)\mathrm{d}F_j(x)\,.$$

Hence

$$D_j = \frac{1}{4} - F_j(x_0) + 2\int_{-\infty}^{x_0} H(x)\,\mathrm{d}F_j(x)\,. \tag{3.29}$$

Apply to the integral in this equality the integration by parts. Then (3.29) and (3.28) yield

$$D_j = \frac{1}{4} - 2\int_{-\infty}^{x_0} F_j(x)\,\mathrm{d}H(x)\,. \tag{3.30}$$

According to the assumptions

$$F_i(x) = F(a_ix + b_i)\,, \qquad i = 1,\dots,k\,. \tag{3.31}$$

Suppose that the equality

$$a_1 = \ldots = a_k \tag{3.32}$$

would hold and for the sake of simplicity of notation assume that $b_1 \le b_2 \le \ldots \le b_k$. Since (2.8) does not hold, obviously $b_{i_0} < b_{i_0+1}$ for some i_0. But by means of (3.32)

$$q_{k,i} = P(\varepsilon_i - \varepsilon_k \le b_i - b_k)$$

and since ε_i, ε_k are i.i.d. with a continuous distribution function, $q_{k,i} \le \frac{1}{2}$ and for i_0 this inequality is strict, which yields a contradiction with (3.24). Hence (3.32) does not hold and as the multiplication of the random variables X_{ij}'s by the same positive constant does not change the values of ranks, one may assume that

$$1 = a_1 \ge \ldots \ge a_k > 0$$

and $a_i > a_{i+1}$ for some i. Put $i_0 = \max\{t; a_t = a_1\}$ and

$$x^* = \min\left\{\frac{b_t - b_1}{a_1 - a_t};\, t = i_0 + 1,\dots,k\right\}.$$

Since the ranks of $X_{11} - \mu,\dots,X_{kn_k} - \mu$ do not depend on the constant μ and this transformation leaves the values of a_i's unchanged, assume without the loss of generality that $x^* = 0$. Hence there exists $j > 1$ such that

$$a_1 = 1 > a_j > 0\,, \qquad b_1 = b_j\,. \tag{3.33}$$

(α) Let there exists x_0 satisfying (3.28) such that

$$x_0 \leq 0\,. \tag{3.34}$$

Put $G(x) = F_1(x)$. Then (3.33) implies that $F_j(x) = G(a_j x)$ and by (3.34) for all $x < x_0$

$$F_j(x) = G(a_j x) \geq G(x) = F_1(x)\,. \tag{3.35}$$

If (3.35) holds with the equality sign for each $x < x_0$, then for $x < x_0$

$$G(x) = G\left(\frac{x}{a_j}\right) = \ldots = G\left(\frac{x}{a_j^m}\right)\,.$$

Thus $G(x) = 0$, which together with (3.30) means that $D_1 \neq 0$. It is therefore sufficient to assume that for some z

$$G(a_j z) > G(z)\,, \qquad x_0 > a_j z > z\,.$$

For this z put

$$x_L = \inf\{x;\, G(x) > G(z)\}\,, \qquad x_U = \sup\{x;\, G(x) < G(a_j z)\}\,.$$

Then $z \leq x_L < x_U \leq a_j z$, for all numbers $x \in (x_L, x_U)$ the inequality in (3.35) is strict and this interval has positive measure (with respect to the Lebesgue–Stieltjes measure induced by G), which together with (3.35) and (3.30) means that $D_1 \neq D_j$, and therefore at least one of these numbers is different from zero.

(β) Suppose that the number $\inf\{x;\, H(x) \geq \frac{1}{2}\}$ is positive. Since the transformation $\tilde{X}_{ij} = -X_{ij}$ preserves the value of the statistic (2.12), considering instead of (2.9) the random variables $\tilde{X}_{ij}$ one obtains the situation from (α), and the proof is completed. □

We remark that the following assertion is similar to Theorem 5.6.4 on p. 205 of [19].

Lemma 3.1. Suppose that (A1) is fulfilled with (2.21) and the distribution function (2.3) possesses a density which has the properties, postulated in the assumptions of the assertion (III) of Theorem 2.1. Assume further that the score functions φ, ψ fulfill (A2), the matrix (3.2) is regular and define by means of (3.14)

$$\begin{aligned}
\hat{T}^{(\varphi)} &= \left(\sqrt{\hat{p}_1}(T_1^{(\varphi)} - E(T_1^{(\varphi)}|H_0)), \ldots, \sqrt{\hat{p}_k}(T_k^{(\varphi)} - E(T_k^{(\varphi)}|H_0))\right)', \\
\hat{T}^{(\psi)} &= \left(\sqrt{\hat{p}_1}(T_1^{(\psi)} - E(T_1^{(\psi)}|H_0)), \ldots, \sqrt{\hat{p}_k}(T_k^{(\psi)} - E(T_k^{(\psi)}|H_0))\right)', \\
\hat{T} &= \left(\hat{T}^{(\varphi)\prime}, \hat{T}^{(\psi)\prime}\right)',
\end{aligned}$$

Then (cf. (2.4), (3.2), (3.6))

$$N\hat{T}'\left(V^{-1} \otimes A(\hat{p})\right)\hat{T} \longrightarrow \chi^2_{2(k-1)}(\delta) \tag{3.36}$$

in distribution, and the noncentrality parameter of this chi-square distribution with $2(k-1)$ degrees of freedom is

$$\delta = \boldsymbol{\nu}'(\boldsymbol{V}^{-1} \otimes \boldsymbol{\kappa})\boldsymbol{\nu}. \tag{3.37}$$

Here

$$\begin{aligned}
\boldsymbol{\kappa} &= \operatorname{diag}(p_1, \dots, p_k) - \boldsymbol{p}(\boldsymbol{p})', \quad \boldsymbol{\nu} = (\nu_1^{(\varphi)}, \dots, \nu_k^{(\varphi)}, \nu_1^{(\psi)}, \dots, \nu_k^{(\psi)})', \\
\nu_j^{(\varphi)} &= \int_{-\infty}^{+\infty} \left(\frac{\sigma_j^* - \overline{\sigma}}{\sigma} x + \frac{\mu_j^* - \overline{\mu}}{\sigma} \right) g_\varphi(F(x)) f^2(x)\, \mathrm{d}x, \\
\nu_j^{(\psi)} &= \int_{-\infty}^{+\infty} \left(\frac{\sigma_j^* - \overline{\sigma}}{\sigma} x + \frac{\mu_j^* - \overline{\mu}}{\sigma} \right) g_\psi(F(x)) f^2(x)\, \mathrm{d}x,
\end{aligned}$$

$\overline{\mu}$, $\overline{\sigma}$ are defined in (2.26) and $g_\varphi(z) = \varphi'(z)$, $g_\psi(z) = \psi'(z)$ at the points where these derivatives exist.

Proof. Let $F_j(x) = F(a_j x + b_j)$ be the distribution function of (2.2). Since

$$\int h(x)\, \mathrm{d}F(a_j x + b_j) = \int h\Big(\frac{x - b_j}{a_j}\Big)\, \mathrm{d}F(x)$$

after some computation one obtains from (3.14) that

$$\begin{aligned}
\mu_j^{(\varphi)} &= \int_{-\infty}^{+\infty} \varphi(y_N(x))\, \mathrm{d}F(x), \quad y_N(x) = \sum_{i=1}^{k} \hat{p}_i F(x_{i,N}), \\
x_{i,N} &= \frac{\sqrt{N}\sigma + \sigma_j^*}{\sqrt{N}\sigma + \sigma_i^*} x + \frac{\mu_j^* - \mu_i^*}{\sqrt{N}\sigma + \sigma_i^*}.
\end{aligned} \tag{3.38}$$

Put $y(x) = F(x)$. Then

$$\sqrt{N}\Big(\varphi(y_N(x)) - \varphi(y(x))\Big) = \frac{\varphi(y_N) - \varphi(y)}{y_N - y} \sum_{i=1}^{k} \hat{p}_i \left(\frac{F(x_{i,N}) - F(x)}{x_{i,N} - x} \right) \sqrt{N}(x_{i,N} - x)$$

and applying to this function the Lebesgue theorem one obtains from (3.38) that (cf. (3.1))

$$\sqrt{N}(\mu_j^{(\varphi)} - \overline{\varphi}) = \nu_j^{(\varphi)} + o(1). \tag{3.39}$$

It is easy to see from (A2) that φ satisfies the Lipschitz condition and (cf. (3.3))

$$\sqrt{N}|\tilde{\varphi} - \overline{\varphi}| = o(1). \tag{3.40}$$

Taking into account (3.39), (3.40) and (3.21) one easily obtains obtains that

$$\sqrt{N}\hat{\boldsymbol{T}} \longrightarrow N_{2k}(\boldsymbol{D}\boldsymbol{\nu}, \boldsymbol{W}),$$

where $\boldsymbol{D} = \operatorname{diag}(\sqrt{p_1}, \dots, \sqrt{p_k}, \sqrt{p_1}, \dots, \sqrt{p_k})$, $\boldsymbol{W} = \boldsymbol{V} \otimes \boldsymbol{A}(\boldsymbol{p})$. Since $\boldsymbol{V}^{-1} \otimes \boldsymbol{A}(\boldsymbol{p})$ is the Moore–Penrose inverse of $\boldsymbol{W}$, the rest of the proof follows from Theorem 9.2.3 on p. 173 of [20]. □

Proof of Theorem 2.1. (III) The functions (3.10) fulfill the assumptions of the previous Lemma with (3.11) and since with the notation from (2.19) and the previous lemma

$$T = N\hat{T}'\Big(V^{-1} \otimes A(\hat{p})\Big)\hat{T} + o_P(1),$$

the rest of the proof can be carried out by means of (3.36). □

The proof of Theorem 2.2 is an extension of the proof on pp. 130–131 of [4] into a setting based on the joint ranking and requiring the score function to be a function fulfilling imposed regularity conditions.

Proof of Theorem 2.2. Taking into account (2.34) it is evident that one may assume that for the scores in (2.33) the equality $a_N(j) = \varphi(j/(N+1))$ holds. Then making use of Lemma a on p. 164 of [7] and Theorem 2.1 of [11] it is easy to see that the random vectors

$$S^{(\varphi)} = \Big(D_{12}^{(\varphi)}, D_{13}^{(\varphi)}, \dots, D_{1k}^{(\varphi)}, D_{23}^{(\varphi)}, \dots, D_{2k}^{(\varphi)}, \dots, D_{k-1k}^{(\varphi)}\Big)'$$

are asymptotically normal with mean **0** and asymptotic covariances $cov(D_{j_1j_2}^{(\varphi)}, D_{j_3j_4}^{(\varphi)})$ given by the formula (here $j_1 < j_2$, $j_3 < j_4$)

$$\begin{array}{ll} 2 & j_1 = j_3\,,\ j_2 = j_4\,, \\ \dfrac{2}{p_j\sqrt{\left(\frac{1}{p_j}+\frac{1}{p_{j_2}}\right)\left(\frac{1}{p_j}+\frac{1}{p_{j_4}}\right)}} & j_1 = j_3 = j\,,\ j_2 \neq j_4\,, \\ \dfrac{2}{p_i\sqrt{\left(\frac{1}{p_{j_1}}+\frac{1}{p_i}\right)\left(\frac{1}{p_{j_3}}+\frac{1}{p_i}\right)}} & j_1 \neq j_3\,,\ j_2 = j_4 = i\,, \\ \dfrac{-2}{p_j\sqrt{\left(\frac{1}{p_{j_1}}+\frac{1}{p_j}\right)\left(\frac{1}{p_j}+\frac{1}{p_{j_4}}\right)}} & j_2 = j_3 = j\,, \\ 0 & j_1, j_2, j_3, j_4 \text{ mutually different.} \end{array} \tag{3.41}$$

However, if $\mathbf{Y} = (Y_1, \dots, Y_k)'$ is a random vector which is normally distributed with mean **0** and the covariance matrix $\mathrm{diag}(\frac{1}{p_1}, \dots, \frac{1}{p_k})$, then putting

$$U_{ij} = \sqrt{2}\frac{Y_i - Y_j}{\sqrt{\frac{1}{p_i}+\frac{1}{p_j}}}\,,$$

$$U = (U_{12}, U_{13}, \dots, U_{1k}, U_{23}, \dots, U_{2k}, \dots, U_{k-1k})$$

one finds out that the normally distributed random vector U has the covariance structure (3.41). Hence

$$P\Big(M_{n_1,\dots,n_k}^{(\varphi)} > Q_k^{(\alpha)}\Big) \to \gamma = P\Big(\max_{i<j}|U_{ij}| > Q_k^{(\alpha)}\Big) \le \alpha,$$

where the last inequality follows from the Hayter theorem in [9], and the inequality holds with the equality sign if $p_1 = \ldots = p_k = 1/k$. □

Proof of Theorem 2.3 and 2.4. Suppose that φ, ψ are the functions (3.10). Then $b_N(j) = (N+1)\psi(j/(N+1))$ and (cf. (2.14), (2.13))

$$\bar{b}_N = \tilde{\mu}_N\,, \qquad \frac{1}{N-1}\sum_{j=1}^{N}(b_N(j) - \bar{b}_N)^2 = v_N^2\,.$$

Hence one may assume that for the partial sums $S_j^{(\varphi)}$, $S_j^{(\psi)}$ appearing in the formulas (2.33), (2.34) for $M^{(\varphi)}_{n_1,\ldots,n_k}$ and $M^{(\psi)}_{n_1,\ldots,n_k}$ the validity of the equalities

$$S_j = S_j^{(\varphi)}\,, \qquad S_j^{(b)} = S_j^{(\psi)}$$

holds. Thus the assumptions of Theorem 2.2 are obviously fulfilled which implies that the Theorem 2.3 is true.

Making use of Theorem 3.1 (I), (3.11) and Theorem 2.3 one obtains that

$$\begin{aligned}\lim P\Big(M_{n_1,\ldots,n_k} \le Q_k^{(\beta)}\Big) &= \lim P\Big(M^{(\varphi)}_{n_1,\ldots,n_k} \le Q_k^{(\beta)}\Big)\lim P\Big(M^{(\psi)}_{n_1,\ldots,n_k} \le Q_k^{(\beta)}\Big)\\ &\ge (1-\beta)^2\end{aligned}$$

and this relation holds with the equality sign, if $p_1 = \ldots = p_k = 1/k$. □

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