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H_2-OPTIMAL REJECTION WITH PREVIEW: GEOMETRIC CONSTRAINTS AND DYNAMIC FEEDFORWARD SOLUTIONS VIA SPECTRAL FACTORIZATION

Elena Zattoni

In this work, a feedforward dynamic controller is devised in order to achieve H_2-optimal rejection of signals known with finite preview, in discrete-time systems. The feedforward approach requires plant stability and, more generally, robustness with respect to parameter uncertainties. On standard assumptions, those properties can be guaranteed by output dynamic feedback, while dynamic feedforward is specifically aimed at taking advantage of the available preview of the signals to be rejected, in compliance with a two-degree-of-freedom control structure. The geometric constraints which prevent achievement of perfect rejection are first discussed. Then, the procedure for the design of the feedforward dynamic compensator is presented. Since the approach proposed in this work is based on spectral factorization via Riccati equation of a real rational matrix function directly related to the original to-be-controlled system, the delays introduced to model the preview of the signals to be rejected do not affect the computational burden intrinsic in the solution of the appropriate algebraic Riccati equation. A numerical example helps to illustrate the geometric constraints and the procedure for the design of the feedforward dynamic unit.

Keywords: optimal design, geometric approach, linear systems, discrete-time systems

AMS Subject Classification: 62K05, 93B27, 93C05, 93C55

1. INTRODUCTION

Signal decoupling, i. e. the problem of making the output of a dynamic system completely insensitive to some external signals, has been deeply investigated, particularly within the geometric approach [1, 19]. As to decoupling of previewed signals, i. e. signals which are available to the controller a certain amount of time ahead of their impact on the system, necessary and sufficient conditions were given in [9, 18] and, in a revised form which exploits the properties of self-bounded controlled invariant subspaces and also considers the case of infinite preview, in [12, 14]. Although expressed with various formalism, the above mentioned conditions contemplate (i) the structural aspect of the problem, by checking the inclusion of suitably defined subspaces, and (ii) the stabilizability aspect, by inspecting the location in the complex plane of some critical invariant zeros of the system. However, if the geometric conditions for exact decoupling are not satisfied, minimizing, according to some criterion, the effect at the output of the previewed signals is a convenient option. In the context of this work, H_2-optimization is preferred since, by converse, it returns perfect rejection as the zero-cost solution whenever the geometric conditions are fulfilled.

H_2-optimal control (and rejection) with preview is a widely treated issue in the recent control literature. Many different procedures to design the controller as well as the practical interest of the problem have been discussed, e. g., in [3, 7, 10, 13, 15]. Nonetheless, the approach introduced in this manuscript remarkably differs from those developed in the abovementioned articles. In fact, this work first addresses the geometric constraints which prevent achievement of exact rejection, then focuses on optimal rejection, according to an H_2-norm criterion. The objective is accomplished by means of a feedforward dynamic compensator which synthesizes the control action by processing the previewed signals, with no information on the actual values of the system variables. Other problems which may be simultaneously present, like, e. g., plant stabilization, can be managed by output dynamic feedback under usual hypotheses, according to a two-degree-of-freedom control scheme. The design of the feedforward dynamic unit is carried out by means of spectral factorization: the spectral factor included in the transfer function matrix of the precompensator is evaluated by solving an appropriate discrete algebraic Riccati equation.

An evident advantage of the proposed approach is that the delays introduced to model the preview of the signals to be rejected do not affect the computational burden of the algorithm which solves the problem. In fact, the algebraic Riccati equation whose solution provides the spectral factor to include in the transfer function matrix of the feedforward dynamic compensator solely involves a state space realization of the original to-be-controlled system. The extended system, which, including the delays, has a dynamic order also depending on the dimension of the signals to be rejected and on the number of samples of preview, is left apart from this computation, which is therefore performed on matrices whose dimensions depend on the dynamic order of the original to-be-controlled system.

Although factorization techniques are quite commonly used in control system design (see, e. g., [5, 16]) and the connection between spectral factorization of real rational matrix functions and solution of appropriate algebraic Riccati equations is known (see, e. g., [2, 4, 11]), to the best of the author's knowledge, these concepts have never been considered in the specific framework of H_2-optimal rejection of previewed signals by dynamic feedforward. In fact, as to application of factorization techniques to the design of feedforward controllers, very few contributions addressing this aspect can be found in the literature (see, e. g., [20] and references therein). Furthermore, the problems discussed in those papers, and the means used to solve them as well, generally differ from those considered in this work. For instance, in [20] the target is to devise a tracking controller achieving, in the single-input single-output case, a certain frequency shape of the overall transfer function and the set of all possible controllers is parameterized by solving a Diophantine equation. Furthermore, the approach to H_2-optimal control developed in the present work, being completely framed in the state space, represents a valid alternative, with the

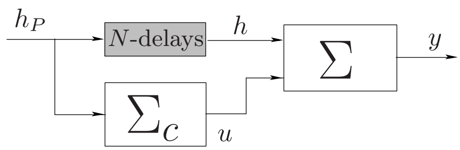

Fig. 1. Block diagram for previewed signal rejection: compact representation.

addition of preview, to the polynomial approach to the standard H_2-optimal control widely discussed in relevant works by Hunt, Šebek and Kučera [8], Grimble [6], and also by Šebek, Kwakernaak, Henrion and Pejchová [17].

Notation. The symbols $\mathbb{R}$ and $\mathbb{C}$ stand for the set of real numbers and the set of complex numbers, respectively. The symbols $\mathbb{C}^{\circ}$, $\mathbb{C}^{\odot}$, and $\mathbb{C}^{\otimes}$ are respectively used for the unit circle, the open set inside the unit circle, and the open set outside the unit circle, in the complex plane. Sets, vector spaces and subspaces are denoted by script capital letters, like $\mathcal{X}$. The quotient space of a vector space $\mathcal{X}$ over a subspace $\mathcal{V} \subseteq \mathcal{X}$ is denoted by $\mathcal{X}/\mathcal{V}$. Matrices and linear maps are denoted by capital letters, like A. The restriction of a linear map A to an A-invariant subspace $\mathcal{J}$ is denoted by $A|_{\mathcal{J}}$. The image and the kernel of A are denoted by $\operatorname{im} A$ and $\ker A$, respectively. The symbols $\sigma(A)$, $\operatorname{tr}(A)$ and A' are respectively used for the spectrum, the trace, and the transpose of A. The symbol I is used to denote an identity matrix. For a real rational transfer function matrix $G(z)$, the notation $G(z)^{\sim}$ stands for $G(z^{-1})'$ and $G(z)^{-\sim}$ stands for $\left[G(z)^{-1}\right]^{\sim}$. For a real rational transfer function matrix $G(z)$ with all its poles in $\mathbb{C}^{\odot}$, the notation $\|G(z)\|_2$ stands for its H_2-norm.

2. PROBLEM FORMULATION

Let the discrete time-invariant linear system Σ be ruled by

$$x_{k+1} = A\,x_k + B\,u_k + H\,h_k, \tag{1}$$

$$y_k = C\,x_k + D\,u_k, \tag{2}$$

with $x \in \mathbb{R}^n$, $u \in \mathbb{R}^p$, $h \in \mathbb{R}^s$, $y \in \mathbb{R}^q$ respectively denoting the state, the control input, the external (to-be-rejected) input, the output. The external input signal h is assumed to be known (hence, to be available to a possible controller) ahead of its impact on the system, with preview $N \geq 0$. The set of all admissible external input sequences is defined as the set of all bounded sequences with values in $\mathbb{R}^s$, equal to zero with $k < N$. The set of all admissible control input sequences is defined as the set of all bounded sequences with values in $\mathbb{R}^p$, equal to zero with $k < 0$. The system Σ is assumed to be in the zero state with $k = 0$.

The artifice of inserting a cascade of N unit delays in the input h signal flow is adopted in order to obtain a new external input signal $h_{P,k} = h_{k+N}$, $k = 0, 1, \ldots$, which can directly be processed by a causal controller Σ_c, according to the scheme shown in Figure 1. The state equations of the original system Σ, expressed in terms

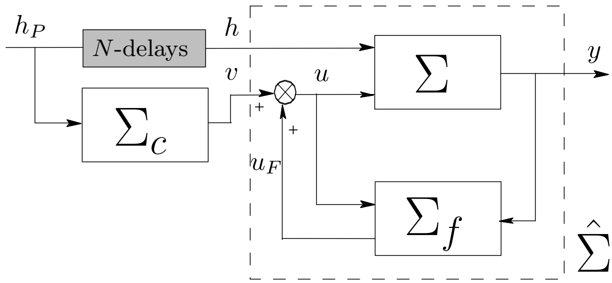

Fig. 2. Block diagram for previewed signal rejection with a prestabilized plant.

of h_P, are

$$\begin{aligned} x_{k+1} &= A\,x_k + B\,u_k + H\,h_{P,k-N}, \\ y_k &= C\,x_k + D\,u_k. \end{aligned}$$

The state equations of the controller Σ_c are

$$\begin{aligned} z_{k+1} &= A_c\,z_k + B_c\,h_{P,k}, \\ u_k &= C_c\,z_k + D_c\,h_{P,k}. \end{aligned}$$

2.1. Some remarks on the assumption of plant stability

Feedforward compensation requires that the system Σ be stable. On usual assumptions, stability can be guaranteed by output dynamic feedback. The result is a two-degree-of-freedom control structure as that shown in Figure 2. Let the dynamic feedback unit Σ_f be ruled by

$$w_{k+1} = (A+GC)\,w_k + (B+GD)\,u_k - G\,y_k, \tag{3}$$
$$u_{F,k} = F\,w_k, \tag{4}$$

with $w \in \mathbb{R}^n$ and $u_F \in \mathbb{R}^p$ respectively denoting the state and the output. Let the overall system $\hat{\Sigma}$ be obtained by connecting Σ, ruled by (1), (2), and Σ_f, ruled by (3), (4), with $v \in \mathbb{R}^p$ such that $u = u_F + v$. Let the state of $\hat{\Sigma}$ be defined as $\hat{x} = \begin{bmatrix} x^\top & w^\top \end{bmatrix}^\top$, so that the state equations of $\hat{\Sigma}$ can be written as

$$\begin{aligned} \hat{x}_{k+1} &= \hat{A}\,\hat{x}_k + \hat{B}\,v_k + \hat{H}\,h_k, \\ y_k &= \hat{C}\,\hat{x}_k + D\,v_k, \end{aligned}$$

with

$$\hat{A} = \begin{bmatrix} A & BF \\ -GC & A+BF+GC \end{bmatrix}, \quad \hat{B} = \begin{bmatrix} B \\ B \end{bmatrix}, \quad \hat{H} = \begin{bmatrix} H \\ O \end{bmatrix}, \quad \hat{C} = \begin{bmatrix} C & DF \end{bmatrix}.$$

Hence, by virtue of the well-known separation property, if (A,B) is stabilizable and (A,C) is detectable, matrices F and G exist, such that

$\sigma(\hat{A}) = \sigma(A+BF) \uplus \sigma(A+GC) \subset \mathbb{C}^{\odot}$. Consequently, the system Σ will henceforth be assumed asymptotically stable and attention will be focused on the design of the dynamic feedforward unit. More details on the structural properties of feedforward control schemes with prestabilized plants can be found in [21]. Furthermore, it is worth mentioning that the choice of the particular stabilizing feedback among the infinitely many stabilizing feedbacks may be exploited in connection with the specific control target (H_2-optimal rejection, perfect decoupling, etc). However, a deeper investigation of this aspect is considered to be beyond the scope of this work.

2.2. A geometric insight into signal rejection with preview

As mentioned in Section 1, recourse to optimization is motivated by those situations where, due to intrinsic properties of the systems, exact decoupling cannot be achieved: i. e., referring to the system Σ ruled by (1), (2), the output y cannot be made identically zero for any admissible external input signal h by means of a stable compensator. In order to recall the necessary and sufficient conditions for exact decoupling of previewed signals in their simplest form, the original system (1), (2) is replaced by the system

$$\bar{x}_{k+1} = \bar{A}\,\bar{x}_k + \bar{B}\,u_k + \bar{H}\,h_k, \tag{5}$$

$$\bar{y}_k = \bar{C}\,\bar{x}_k, \tag{6}$$

with

$$\bar{A} = \begin{bmatrix} A & O \\ C & O \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} B \\ D \end{bmatrix}, \quad \bar{H} = \begin{bmatrix} H \\ O \end{bmatrix}, \quad \bar{C} = \begin{bmatrix} O & I \end{bmatrix},$$

which is equivalent to (1), (2) with respect to decoupling, but does not exhibit the feedthrough matrix from the control input u to the output $\bar{y}$. In fact, the system (5), (6) is derived from (1), (2) by resorting to the well-known contrivance of inserting a unit delay in the output y signal flow (see e. g. [1]). Some geometric objects referring to (5), (6) are introduced. To avoid notation clutter, the symbols $\bar{\mathcal{B}}$ and $\bar{\mathcal{H}}$ respectively stand for $\operatorname{im}\bar{B}$ and $\operatorname{im}\bar{H}$, while $\bar{\mathcal{C}}$ stands for $\ker\bar{C}$. Hence, $\bar{\mathcal{V}}^* = \max\mathcal{V}(\bar{A},\bar{\mathcal{B}},\bar{\mathcal{C}})$ is the maximal $(\bar{A},\bar{\mathcal{B}})$-controlled invariant subspace contained in $\bar{\mathcal{C}}$, while $\bar{\mathcal{S}}^* = \min\mathcal{S}(\bar{A},\bar{\mathcal{C}},\bar{\mathcal{B}})$ is the minimal $(\bar{A},\bar{\mathcal{C}})$-conditioned invariant subspace containing $\bar{\mathcal{B}}$. Also the notion of internal stabilizability of a controlled invariant subspace plays a key role in the statement of the above-mentioned conditions. An $(\bar{A},\bar{\mathcal{B}})$-controlled invariant subspace $\bar{\mathcal{V}}$ is said to be internally stabilizable if $\sigma((\bar{A}+\bar{B}\bar{F})|_{\bar{\mathcal{V}}}) \subset \mathbb{C}^{\odot}$ for some $\bar{F}$ such that $(\bar{A}+\bar{B}\bar{F})\,\bar{\mathcal{V}} \subseteq \bar{\mathcal{V}}$. As was shown in [12, 14], exact decoupling of external input signals known with finite preview is achievable with stability if and only if (i) $\bar{\mathcal{H}} \subseteq \bar{\mathcal{V}}^* + \bar{\mathcal{S}}^*$ and (ii) $\bar{\mathcal{V}}_m = \bar{\mathcal{V}}^* \cap \min\mathcal{S}(\bar{A},\bar{\mathcal{C}},\bar{\mathcal{B}}+\bar{\mathcal{H}})$ is internally stabilizable. In that case, the preview N must be at least equal to the number of steps for the minimal conditioned invariant subspace algorithm to converge. In fact, $\bar{\mathcal{S}}^*$ is the last term of the sequence $\bar{\mathcal{S}}_0 = \bar{\mathcal{B}}$, $\bar{\mathcal{S}}_i = \bar{A}(\bar{\mathcal{S}}_{i-1} \cap \bar{\mathcal{C}}) + \bar{\mathcal{B}}$, $i = 1,\ldots,\rho$, where ρ is the least integer such that $\bar{\mathcal{S}}_{\rho+1} = \bar{\mathcal{S}}_\rho$. Furthermore, as was shown in [14], exact decoupling of external input signals known with infinite preview is achievable with stability if and only if (i) $\bar{\mathcal{H}} \subseteq \bar{\mathcal{V}}^* + \bar{\mathcal{S}}^*$ and (ii') $\bar{\mathcal{V}}_m$ has no unassignable internal eigenvalues on $\mathbb{C}^{\circ}$, i.e. $\sigma((\bar{A}+\bar{B}\bar{F})|_{\bar{\mathcal{V}}_m/\bar{\mathcal{R}}_{\bar{\mathcal{V}}^*}})\cap\mathbb{C}^{\circ}=\emptyset$, where $\bar{F}$ is such that $(\bar{A}+\bar{B}\bar{F})\,\bar{\mathcal{V}}_m\subseteq\bar{\mathcal{V}}_m$ and $\bar{\mathcal{R}}_{\bar{\mathcal{V}}^*}=\bar{\mathcal{V}}^*\cap\bar{\mathcal{S}}^*$.

Hence, in the light of the necessary and sufficient conditions for previewed signal decoupling, it can more precisely be stated that resort to optimization is required when either the structural condition (i) is not fulfilled or the stabilizability conditions (ii) or (ii') (in the presence of finite or infinite preview, respectively) are not satisfied. Indeed, in the case where only the milder condition (ii') is satisfied and infinite preview is available, practical implementation of the geometric methods described in [14] necessarily implies truncation of signal processing for a certain, although arbitrarily high, value of the preview. This means that exact decoupling, albeit achievable from the theoretical point of view, is not practically feasible. Therefore, H_2-optimal rejection turns out to be a valid alternative also in that case, since it may help to reduce the so-called truncation error introduced by the geometric methods. By the way, it is worth noting that the necessary and sufficient conditions for exact decoupling are sharp, and very easy to check with the basic routines of the geometric approach first published in [1]. In other words, the geometric insight leads to the conclusion that perfect decoupling with finite preview is exactly achievable in practice when the structural condition (i) is satisfied along with the stabilizability condition (ii) – which means that the critical subset of the plant invariant zeros is inside the open unit disk of the complex plane – and, meanwhile, it provides the exact information on the preview time required to achieve the perfect solution. Furthermore, it yields the conclusion that perfect decoupling can arbitrarily be approached when the sole conditions (i) and (ii') are satisfied and infinite preview is available. Anyway, it provides the means to judge from the good knowledge of the case whether or not to resort to optimization techniques.

In this work, optimization in the H_2 sense is preferred to other criteria since it encompasses the perfect solution to the problem of signal rejection with preview as the zero-cost solution whenever this latter exists and the adequate preview is available.

The H_2-optimal rejection problem with preview can be stated as follows.

H_2-optimal rejection with preview: compact formulation. Refer to Figure 3. Let the system Σ, ruled by (1), (2), be asymptotically stable. Find a causal linear feedforward dynamic compensator Σ_c such that

(i) the overall control system Σ_o be asymptotically stable, and

(ii) the H_2 norm of Σ_o be minimal.

However, the following, detailed formulation of the problem, where the transfer function matrices of the different blocks are explicitly addressed, is functional to the developments in the sequel.

H_2-optimal rejection with preview: detailed formulation. Refer to Figure 3. Let the system Σ, ruled by (1), (2), be asymptotically stable. Let $G_1(z)$ and

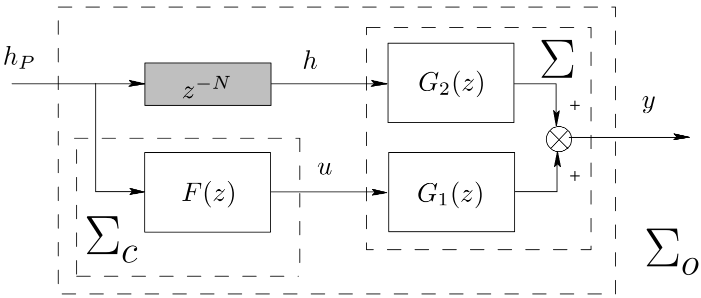

Fig. 3. Block diagram for previewed signal rejection: detailed representation.

$G_2(z)$ be the transfer function matrices from the respective inputs u and h to the output y, i. e.,

$$G_1(z) = C(zI-A)^{-1}B+D, \tag{7}$$
$$G_2(z) = C(zI-A)^{-1}H. \tag{8}$$

Find a causal linear feedforward dynamic compensator $F(z)$ such that
(i) $G_1(z)F(z)+z^{-N}G_2(z)$ be asymptotically stable, and
(ii) $\|G_1(z)F(z)+z^{-N}G_2(z)\|_2$ be minimal.

3. FEEDFORWARD DYNAMIC COMPENSATOR DESIGN

In this section, the feedforward dynamic compensator solving the H_2-optimal rejection problem with preview is derived.

Theorem 1. Let the system Σ, ruled by (1), (2), be asymptotically stable. Let $G_1(z)$ and $G_2(z)$ be defined as in (7), (8). Let $G_1(z)$ be of full column rank. Let $\Omega(z)$ be a real rational matrix function with the properties of being square, having all its poles and zeros in $\mathbb{C}^{\odot}$, and satisfying $\Omega(z)^{\sim}\Omega(z)=G_1(z)^{\sim}G_1(z)$. Then,

$$F(z) = -\,\Omega(z)^{-1}\left(z^{-N}\Omega(z)^{-\sim}G_1(z)^{\sim}G_2(z)\right)_-,$$

where $(\cdot)_-$ denotes the causal stable part of the argument, solves Problem 2.

Proof. First, note that

$$\begin{aligned}
&\|G_1(z)F(z)+z^{-N}G_2(z)\|_2^2\\
&\quad= \|\left(G_1(z)F(z)+z^{-N}G_2(z)\right)^{\sim}\|_2^2\\
&\quad= \frac{1}{2\pi}\int_{-\pi}^{\pi}\mathrm{tr}\left[\left(G_1(z)F(z)+z^{-N}G_2(z)\right)^{\sim}\left(G_1(z)F(z)+z^{-N}G_2(z)\right)\right]\Big|_{e^{j\omega}}\mathrm{d}\omega\\
&\quad= \|\Omega(z)F(z)+z^{-N}\Omega(z)^{-\sim}G_1(z)^{\sim}G_2(z)\|_2^2\\
&\qquad+\frac{1}{2\pi}\int_{-\pi}^{\pi}\mathrm{tr}\left[G_2(z)^{\sim}\left(I-G_1(z)\left(G_1(z)^{\sim}G_1(z)\right)^{-1}G_1(z)^{\sim}\right)G_2(z)\right]\Big|_{e^{j\omega}}\mathrm{d}\omega.
\end{aligned}$$

In fact,

$$\begin{aligned}
&\left(G_1(z)F(z)+z^{-N}G_2(z)\right)^{\sim}\left(G_1(z)F(z)+z^{-N}G_2(z)\right)\\
&\quad=\left(F(z)^{\sim}G_1(z)^{\sim}+z^{N}G_2(z)^{\sim}\right)\left(G_1(z)F(z)+z^{-N}G_2(z)\right)\\
&\quad=F(z)^{\sim}G_1(z)^{\sim}G_1(z)F(z)+z^{-N}F(z)^{\sim}G_1(z)^{\sim}G_2(z)\\
&\qquad+z^{N}G_2(z)^{\sim}G_1(z)F(z)+G_2(z)^{\sim}G_2(z)\\
&\quad=F(z)^{\sim}\Omega(z)^{\sim}\Omega(z)F(z)+z^{-N}F(z)^{\sim}G_1(z)^{\sim}G_2(z)\\
&\qquad+z^{N}G_2(z)^{\sim}G_1(z)F(z)+G_2(z)^{\sim}G_2(z)\\
&\qquad+G_2(z)^{\sim}G_1(z)\left(\Omega(z)^{\sim}\Omega(z)\right)^{-1}G_1(z)^{\sim}G_2(z)\\
&\qquad-G_2(z)^{\sim}G_1(z)\left(\Omega(z)^{\sim}\Omega(z)\right)^{-1}G_1(z)^{\sim}G_2(z)\\
&\quad=\left(F(z)^{\sim}\Omega(z)^{\sim}+z^{N}G_2(z)^{\sim}G_1(z)\,\Omega(z)^{-1}\right)\left(\Omega(z)\,F(z)+z^{-N}\Omega(z)^{-\sim}G_1(z)^{\sim}G_2(z)\right)\\
&\qquad+G_2(z)^{\sim}\left(I-G_1(z)\left(\Omega(z)^{\sim}\Omega(z)\right)^{-1}G_1(z)^{\sim}\right)G_2(z).
\end{aligned}$$

Since

$$G_2(z)^{\sim}\left(I-G_1(z)\left(G_1(z)^{\sim}G_1(z)\right)^{-1}G_1(z)^{\sim}\right)G_2(z)$$

does not depend on $F(z)$, it follows that

$$F(z)=-\,\Omega(z)^{-1}\left(z^{-N}\Omega(z)^{-\sim}G_1(z)^{\sim}G_2(z)\right)_{-}$$

minimizes the performance index with the constraint of stability. □

A real rational matrix function $\Omega(z)$ with the properties specified in Theorem 1 is analytic and invertible on $\mathbb{C}^{\circ}\cup\mathbb{C}^{\otimes}$. The following Theorem 2 provides a real rational matrix function $\Omega(z)$ with the properties specified in Theorem 1, on the assumptions that guarantee the existence and the uniqueness of the stabilizing solution of an appropriate discrete algebraic Riccati equation (see also [2, 4, 11]). Before stating Theorem 2, it is convenient to recall that a system represented by a quadruple (A,B,C,D) is left-invertible if $\mathcal{R}_{\mathcal{V}^*}=\{0\}$, where the definition of $\mathcal{R}_{\mathcal{V}^*}$ was introduced in Section 2 (see e. g. [1] for further details on left invertibility and [21] for a procedure to reduce a non-left-invertible system to a left-invertible system which is equivalent as far as signal rejection is concerned).

Theorem 2. Let the system Σ, ruled by (1), (2), be asymptotically stable. Let the system represented by the quadruple (A,B,C,D) have no invariant zeros on $\mathbb{C}^{\circ}$ and be left-invertible. Let $X\geq 0$ be the stabilizing solution of the discrete algebraic Riccati equation

$$X=-\,(A'XB+C'D)\,(B'XB+D'D)^{-1}(B'XA+D'C)+A'XA+C'C.$$

Let

$$\Omega(z)=(D'D+B'XB)^{\frac{1}{2}}\left(I+K(zI-A)^{-1}B\right),$$

with
$$K = (D'D + B'XB)^{-1} (B'XA + D'C).$$
Then, $\Omega(z)$ has the properties of being square, having all its poles and zeros in $\mathbb{C}^{\odot}$, and satisfying
$$\Omega(z)^{\sim}\Omega(z) = G_1(z)^{\sim}G_1(z).$$

Proof. First, it is shown that $\Omega(z)$ is square and has all its poles and zeros in $\mathbb{C}^{\odot}$. In fact, due to its definition, the following properties hold. (i) $\Omega(z)$ is a $p \times p$ real rational matrix function. (ii) The set of the poles of $\Omega(z)$ coincides with $\sigma(A)$ and $\sigma(A) \subset \mathbb{C}^{\odot}$ by assumption. (iii) The set of the zeros of $\Omega(z)$ coincides with $\sigma(A - BK)$. In fact, the Sherman–Morrison–Woodbury formula gives

$$\left(K\left(zI - A\right)^{-1}B + I\right)^{-1} = -K\left(zI - (A - BK)\right)^{-1}B + I.$$

Moreover, $\sigma(A - BK) \subset \mathbb{C}^{\odot}$ since the gain matrix K has been obtained from the stabilizing solution of the discrete algebraic Riccati equation.

Then, it is shown the relation between $\Omega(z)$ and $G_1(z)$. By definition of K, the discrete algebraic Riccati equation is written as

$$X - A'XA + K'(B'XB + D'D)K = C'C.$$

Hence, it can also be written as

$$(z^{-1}I - A')X(zI - A) + (z^{-1}I - A')XA + A'X(zI - A) + K'(B'XB + D'D)K = C'C.$$

Let $\Psi = (zI - A)^{-1}$, which implies $\Psi^{\sim} = (z^{-1}I - A')^{-1}$. Let $R = B'XB + D'D$. Then, by premultiplying by $\Psi^{\sim}$ and postmultiplying by Ψ, it follows that

$$X + XA\Psi + \Psi^{\sim}A'X + \Psi^{\sim}K'RK\Psi - \Psi^{\sim}C'C\Psi = 0,$$

and, by premultiplying by B' and postmultiplying by B,

$$B'XB + B'XA\Psi B + B'\Psi^{\sim}A'XB + B'\Psi^{\sim}K'RK\Psi B - B'\Psi^{\sim}C'C\Psi B = 0.$$

By definition of K and R, the discrete algebraic Riccati equation can be written as

$$\begin{aligned}&B'XB + B'XA\Psi B + B'\Psi^{\sim}A'XB\\&\qquad + B'\Psi^{\sim}(A'XB + C'D)R^{-1}(B'XA + D'C)\Psi B - B'\Psi^{\sim}C'C\Psi B = 0,\end{aligned}$$

or, equivalently, as

$$\begin{aligned}&R - D'D + B'XA\Psi B + B'\Psi^{\sim}A'XB\\&\qquad + B'\Psi^{\sim}(A'XB + C'D)R^{-1}(B'XA + D'C)\Psi B - B'\Psi^{\sim}C'C\Psi B = 0,\end{aligned}$$

and, finally, as

$$\begin{aligned}&R + B'\Psi^{\sim}(A'XB + C'D)R^{-1}(B'XA + D'C)\Psi B\\&\qquad = D'D - B'XA\Psi B - B'\Psi^{\sim}A'XB + B'\Psi^{\sim}C'C\Psi B.\end{aligned}$$

On the other hand, $G_1(z)^{\sim}G_1(z)$ can be written as

$$\begin{aligned} G_1(z)^{\sim}G_1(z) &= B'\Psi^{\sim}(A'XB+C'D)+(B'XA+D'C)\,\Psi B \\ &\quad +D'D-B'XA\Psi B-B'\Psi^{\sim}A'XB+B'\Psi^{\sim}C'C\,\Psi B\,. \end{aligned}$$

Finally, by definition of K and the final expression of the discrete algebraic Riccati equation, it follows that

$$G_1(z)^{\sim}G_1(z)=(I+B'\Psi^{\sim}K')\,R\,(I+K\,\Psi B)=\Omega(z)^{\sim}\Omega(z). \qquad \square$$

The procedure for the design of the feedforward dynamic compensator based on Theorems 1 and 2 and the remarks on the connections between the geometric approach and the H_2-optimal approach to rejection with preview are illustrated through a numerical example in Section 4.

4. AN ILLUSTRATIVE EXAMPLE

Let the system Σ be ruled by (1), (2) with

$$A=\begin{bmatrix}-0.1 & 1\\ 0 & -0.2\end{bmatrix},\quad B=\begin{bmatrix}1\\1\end{bmatrix},\quad H=\begin{bmatrix}0\\1\end{bmatrix},\quad C=\begin{bmatrix}1 & 0\end{bmatrix},\quad D=0.$$

The spectrum of the system matrix is $\sigma(A)=\{-0.1,-0.2\}$. The subspaces $\mathcal{V}^*=\max\mathcal{V}(A,\mathcal{B},\mathcal{C})$ and $\mathcal{S}^*=\min\mathcal{S}(A,\mathcal{C},\mathcal{B})$ respectively are

$$\mathcal{V}^*=\mathrm{im}\begin{bmatrix}0\\1\end{bmatrix},\quad \mathcal{S}^*=\mathrm{im}\begin{bmatrix}0.7071\\0.7071\end{bmatrix}.$$

Hence, the structural condition (i) of Section 2.2. for exact decoupling is satisfied. In fact, $\mathcal{H}\subseteq\mathcal{V}^*+\mathcal{S}^*=\mathbb{R}^2$. However, $\mathcal{V}_m=\mathcal{V}^*\cap\min\mathcal{S}(A,\mathcal{C},\mathcal{B}+\mathcal{H})=\mathrm{im}\ [0\ 1]'=\mathcal{V}^*$ and the invariant zero of the triple (A,B,C), which coincides with the unassignable internal eigenvalue of $\mathcal{V}_m$, is $z=1.2$. This implies that the stabilizability condition (ii) of Section 2.2. for exact decoupling is not satisfied, while only the milder condition (ii') is fulfilled. Consequently, H_2-optimal rejection is a convenient option.

Table 1. Coefficients of the Partial Fraction Expansion of $Q(z)$ for $N=0,1,\ldots,6$.

—	$N=0$	1	2	3	4	5	6
$\frac{1}{z+1.2}$	0.48	−0.4	0.3333	−0.2778	0.2315	−0.1929	0.1608
$\frac{1}{z+0.2}$	1.52	−7.6	38	−190	950	−4750	23750
$\frac{1}{z+0.1}$	−0.8	8	−80	$8\cdot 10^2$	$-8\cdot 10^3$	$8\cdot 10^4$	$-8\cdot 10^5$
$\frac{1}{z}$	0	0	41.6667	−609.7222	7049.7685	−75249.8071	776249.8393
$\frac{1}{z^2}$	0	0	0	41.6667	−609.7222	7049.7685	−75249.8071
$\frac{1}{z^3}$	0	0	0	0	41.6667	−609.7222	7049.7685
$\frac{1}{z^4}$	0	0	0	0	0	41.6667	−609.7222
$\frac{1}{z^5}$	0	0	0	0	0	0	41.6667

The stabilizing solution $X \geq 0$ of the discrete algebraic Riccati equation, the corresponding gain matrix K, and the spectrum of $A - BK$ respectively are

$$X = \begin{bmatrix} 1.0036 & -0.0436 \\ -0.0436 & 0.5236 \end{bmatrix},$$

$$K = \begin{bmatrix} -0.0667 & 0.6 \end{bmatrix},$$

$$\sigma(A - BK) = \{-0.8333, 0\}.$$

The real rational function $\Omega(z)$ is

$$\Omega(z) = 1.2 \frac{z(z+0.8333)}{(z+0.1)(z+0.2)}.$$

Hence, the sets of its poles and zeros match the sets $\sigma(A)$ and $\sigma(A - BK)$, respectively. Consequently,

$$\Omega(z)^{-1} = 0.8333 \frac{(z+0.1)(z+0.2)}{z(z+0.8333)},$$

and

$$\Omega(z)^{-\sim} = 0.02 \frac{(z+10)(z+5)}{z+1.2}.$$

The transfer functions $G_1(z)$ and $G_2(z)$ respectively are

$$G_1(z) = \frac{z+1.2}{(z+0.1)(z+0.2)},$$

and

$$G_2(z) = \frac{1}{(z+0.1)(z+0.2)}.$$

Hence,

$$G_1(z)^{\sim} = \frac{60z(z+0.8333)}{(z+10)(z+5)}.$$

Finally, the transfer function $F(z)$ of the feedforward dynamic compensator is

$$F(z) = -0.8333 \frac{(z+0.1)(z+0.2)}{z(z+0.8333)} \left(1.2\, z^{-N} \frac{z(z+0.8333)}{(z+0.1)(z+0.2)(z+1.2)} \right)_- .$$

The coefficients of the expansion in partial fractions of the rational function

$$Q(z) = 1.2\, z^{-N} \frac{z(z+0.8333)}{(z+0.1)(z+0.2)(z+1.2)}$$

are reported for different values of the preview N, from 0 to 6, in Table 1.

The transfer functions $F(z)$ of the dynamic feedforward compensator and the corresponding values of the H_2-norm of the overall control system, i.e.

$$\|G(z)\|_2 = \|G_1(z)F(z) + z^{-N} G_2(z)\|_2,$$

are reported for the same set of values of N in Table 2.

The values of the H_2-norm of the transfer function $G(z)$ of the overall control system are plotted versus the values of N in Figure 4.

Table 2. Transfer Function $F(z)$ and H_2-norm of the Corresponding $G(z)$ for $N = 0, 1, \dots, 6$.

N	$F(z)$	$\|G(z)\|_2$
0	$-0.6\,\dfrac{z - 0.0111}{z(z + 0.8333)}$	0.7236
1	$-0.3333\,\dfrac{z + 2.1}{z(z + 0.8333)}$	0.6030
2	$0.2777\,\dfrac{z^2 - 0.9z - 2.5}{z^2(z + 0.8333)}$	0.5025
3	$-0.2315\,\dfrac{z^3 - 0.8999z^2 + 1.1001z + 2.9996}{z^3(z + 0.8333)}$	0.4188
4	$0.1929\,\dfrac{z^4 - 0.8999z^3 + 1.0998z^2 - 1.3201z - 3.5996}{z^4(z + 0.8333)}$	0.3489
5	$-0.1607\,\dfrac{z^5 - 0.8999z^4 + 1.1006z^3 - 1.3199z^2 + 1.5842z + 4.3199}{z^5(z + 0.8333)}$	0.2910
6	$0.1339\,\dfrac{z^6 - 0.9004z^5 + 1.1002z^4 - 1.3211z^3 + 1.5843z^2 - 1.9017z - 5.1854}{z^6(z + 0.8333)}$	0.2422

By prosecuting the construction of Table 1 and Table 2, it can be seen that for higher values of N the H_2-norm of the overall control system tends to zero: for instance, $\|G(z)\|_2 = 0.1169$, $\|G(z)\|_2 = 0.0189$, $\|G(z)\|_2 = 7.9516 \cdot 10^{-5}$, $\|G(z)\|_2 = 1.9961 \cdot 10^{-8}$, with $N = 10, 20, 50, 100$, respectively. Hence, as N approaches infinity, the H_2-optimal solution approaches the exact solution.

This trend complies with the theoretical considerations of Section 2. In fact, the system satisfies the structural condition (i) and the stabilizability condition (ii'). Hence, the exact solution (or, the zero-cost solution) is conceptually obtainable with infinite preview.

5. CONCLUSION

If the geometric conditions for total neutralization of signals known with preview are not satisfied, optimization is a convenient alternative. In this work, H_2-optimization was privileged since, by converse, it returns the exact solution whenever this latter is admissible. Preview was efficiently exploited by means of a dynamic feedforward scheme, while stabilization was left to output dynamic feedback, according to a two-degree-of-freedom control structure. The dynamic feedforward compensator was derived through a direct and tight procedure based on the spectral factorization of a real rational matrix function which is directly connected to the original, to-be-controlled system through the solution of an appropriate discrete algebraic Riccati

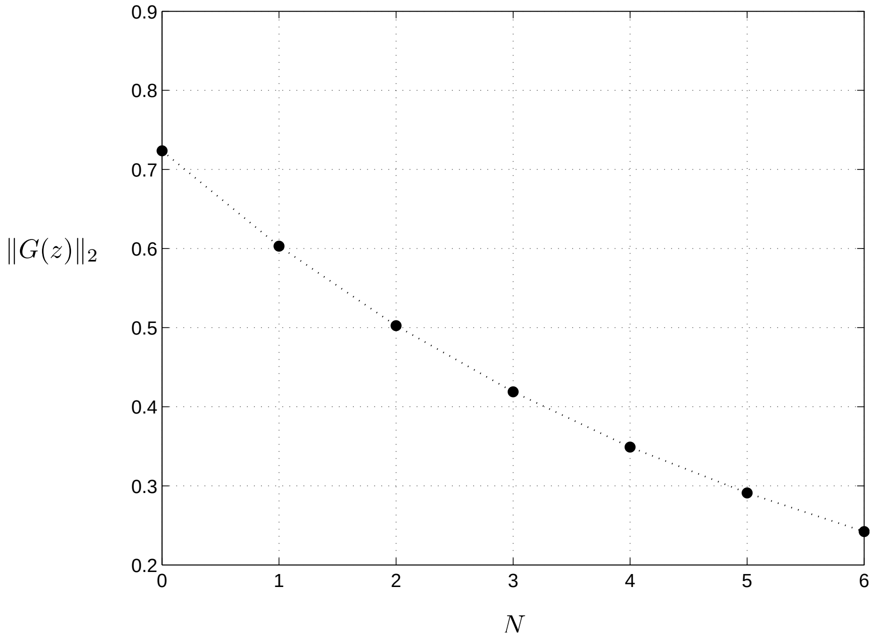

Fig. 4. Plot of $\|G(z)\|_2$ for $N = 0, 1, \ldots, 6$.

equations. Hence, the computational burden implicit in the proposed methodology only depends on the dynamic order of the to-be-controlled system and is not affected by the length of the preview time interval. A numerical example first illustrated the geometric constraints which prevent perfect rejection to be achieved, then described the application of the design procedure.

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