EXTREME DISTRIBUTION FUNCTIONS OF COPULAS

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In this paper we study some properties of the distribution function of the random variable \( C(X,Y) \) when the copula of the random pair \((X,Y)\) is \( M \) (respectively, \( W \)) – the copula for which each of \( X \) and \( Y \) is almost surely an increasing (respectively, decreasing) function of the other –, and \( C \) is any copula. We also study the distribution functions of \( M(X,Y) \) and \( W(X,Y) \) given that the joint distribution function of the random variables \( X \) and \( Y \) is any copula.

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1. INTRODUCTION

Let \( H_1 \) and \( H_2 \) be two bivariate distribution functions with common continuous one-dimensional margins \( F \) and \( G \) – the distribution functions considered are taken to be right-continuous. Let \((X,Y)\) be a random pair – all the random variables considered are defined on the same probability space \((\Omega, \mathcal{F}, P)\) – whose joint distribution function is \( H_2 \), and let \( \langle H_1|H_2\rangle(X,Y) \) denote the random variable \( H_1(X,Y) \). The \( H_2 \) distribution function of \( H_1 \), which we denote by \( (H_1|H_2) \), is given by

\[
(H_1|H_2)(t) = \Pr[\langle H_1|H_2\rangle(X,Y) \leq t] = \mu_{H_2}(\{(x,y) \in \mathbb{R}^2 | H_1(x,y) \leq t\}), \quad t \in [0,1],
\]

where \( \mu_{H_2} \) denotes the measure on \( \mathbb{R}^2 \) induced by \( H_2 \) \([7, 12]\). In this paper we study some properties of the distribution function of the random variable \( H_1(X,Y) \) when each variable of the random pair \((X,Y)\) is almost surely an increasing (respectively, decreasing) function of the other.

Since our methods involve the concept of a copula, we review this notion and some of its properties. A (bivariate) copula is the restriction to \([0,1]^2\) of a continuous (bivariate) distribution function whose margins are uniform on \([0,1]\). The importance of copulas stems largely from the observation that the joint distribution \( H \) of the random pair \((X,Y)\) with respective margins \( F \) and \( G \) can be expressed by \( H(x,y) = C(F(x),G(y)) \), for all \((x,y) \in [-\infty, \infty]^2\), where \( C \) is a copula that is
uniquely determined on Range $F \times \text{Range } G$ (Sklar’s Theorem) \cite{17, 18}. Let $\Pi$ denote the copula for independent random variables, i.e., $\Pi(u, v) = uv$ for all $(u, v) \in [0, 1]^2$. For a complete survey on copulas, see \cite{11}.

By Sklar’s Theorem, if $C_1$ and $C_2$ are two copulas and $(U, V)$ is a pair of uniform $[0, 1]$ random variables with copula $C_2$, and $(C_1|C_2)(U, V)$ denotes the random variable $C_1(U, V)$ – written $(C_1|C_2)$ when the meaning is clear –, then the $C_2$ distribution function of $C_1$ is given by

\[
(C_1|C_2)(t) = \Pr[(C_1|C_2)(U, V) \leq t] = \mu_{C_2}(\{(u, v) \in [0, 1]^2 | C_1(u, v) \leq t\}), \quad t \in [0, 1].
\]

Every copula $C$ of the random pair $(X, Y)$ satisfies the following inequalities:

\[
\max(u + v - 1, 0) = W(u, v) \leq C(u, v) \leq M(u, v) = \min(u, v), \quad \forall (u, v) \in [0, 1]^2.
\]

$M$ (respectively, $W$) is the copula for which each of $X$ and $Y$ is almost surely an increasing (respectively, decreasing) function of the other.

In the sequel, we shall use the following notation: For any pair of random variables $X$ and $Y$ with respective distribution functions $F$ and $G$, “$\leq_{st}$” denotes the stochastic inequality, i.e., $X \leq_{st} Y$ if, and only if, $F \geq G$; and $X \overset{d}{=} Y$ denotes the equality in distribution.

Distribution functions of copulas are employed – among other purposes – to construct orderings on the set of copulas (see \cite{12}). If $C$, $C_1$ and $C_2$ are copulas, two of those orderings are: (a) $C_1$ is $C$-larger than $C_2$ if $(C_1|C) \geq_{st} (C_2|C)$; and (b) $C_1$ is $C$-larger in measure than $C_2$ if $(C|C_1) \geq_{st} (C|C_2)$. As a consequence, two equivalences are given, namely: (c) $C_1$ is $C$-equivalent to $C_2$ (written $C_1 \equiv_C C_2$) if $(C_1|C) \overset{d}{=} (C_2|C)$; and (d) $C_1$ is $C$-equivalent in measure to $C_2$ if $(C|C_1) \overset{d}{=} (C|C_2)$.

It is known that if $F$ is a right-continuous distribution function such that $F(0^-) = 0$ and $F(t) \geq t$ for all $t \in [0, 1]$, then there exists a copula $C$ such that $(C|C)(t) = F(t)$ for all $t \in [0, 1]$ (see \cite{13, 16}). We now wonder whether this result can be generalized (in some sense) to other distribution functions of copulas. To be exact: if $C_0$ is a copula, and $F$ is a distribution function such that $(M|C_0)(t) \leq F(t) \leq (W|C_0)(t)$ for all $t \in [0, 1]$, does there exist a copula $C$ such that $(C|C_0)(t) = F(t)$ for all $t \in [0, 1]$? The answer is affirmative when $C_0 = M$. We will also provide some additional properties of the distributions $(C|M)$ and $(C|W)$ for any copula $C$.

2. THE $M$ DISTRIBUTION FUNCTION OF A COPULA

The diagonal section $\delta_C$ of a copula $C$ is the function given by $\delta_C(t) = C(t, t)$ for all $t \in [0, 1]$. A diagonal is a function $\delta \colon [0, 1] \to [0, 1]$ which satisfies the following properties:

(i) $\delta(1) = 1$,

(ii) $\delta(t) \leq t$ for all $t \in [0, 1]$,
(iii) \(0 \leq \delta(t') - \delta(t) \leq 2(t' - t)\) for all \(t, t' \in [0, 1]\) such that \(t \leq t'\) — i.e., \(\delta\) is increasing and 2-Lipschitz.

The diagonal section of any copula is a diagonal; and for any diagonal \(\delta\), there always exist copulas whose diagonal section is \(\delta\) \([5]\) (see also \([4, 14, 15]\)): for instance, the Bertino copula \(B_\delta\) \([6]\), which is given by

\[
B_\delta(u, v) = \min(u, v) - \min(s - \delta(s) \mid \min(u, v)) \\
\leq s \leq \max(u, v)), \ (u, v) \in [0, 1]^2.
\]

The diagonal section \(\delta_C\) of a copula \(C\) is the restriction to \([0, 1]\) of the distribution function of \(\max(U, V)\), whenever \((U, V)\) is a random pair distributed as \(C\). Let \(\delta_C^{-1}\) denote the \(\text{cadlag}\) inverse of \(\delta_C\), i.e., \(\delta_C^{-1}(t) = \sup\{u \in [0, 1] \mid \delta_C(u) \leq t\}\) for \(t\) in \([0, 1]\).

The following result gives a (partial) answer to the question posed at the end of Section 1.

**Theorem 1.** Let \(F\) be a right-continuous distribution function such that \(F(0^-) = 0, F(t) \geq t\) for all \(t \in [0, 1]\), and \(F'(t) \geq 1/2\) for almost every \(t\) in \([0, 1]\). Then there exists a copula \(C\) such that \((C|M)(t) = F(t)\) for all \(t \in [0, 1]\).

**Proof.** We know that \(\delta_C^{-1}\) is the restriction to the interval \([0, 1]\) of a distribution function with support on \([0, 1]\) and such that \((M|M)(t) \leq (C|M)(t) \leq (W|M)(t)\) for all \(t \in [0, 1]\). Since

\[
(C|M)(t) = \delta_C^{-1}(t), \ \forall \ t \in [0, 1]
\]

(see \([12]\)), and \(\delta_C\) is 2-Lipschitz, we have that \(\delta_C^{-1}\) must be a strictly increasing function (not necessarily continuous) whose derivative is greater or equal to 1/2 for almost every point in \([0, 1]\). Since the Bertino copula \(B_\delta\) associated with \(\delta\) satisfies \((B_\delta|M)(t) = \delta^{-1}(t) = F(t)\) for all \(t \in [0, 1]\) (see \([12]\)), this completes the proof. \(\Box\)

If \(C_1\) and \(C_2\) are two copulas, then we say that \(C_1 \equiv_M C_2\) if \((C_1|M)(t) = (C_2|M)(t)\) for all \(t \in [0, 1]\). The next example provides a class in this equivalence relation which contains more than one copula.

**Example 1.** Let \(C\) be the copula given by \(C(u, v) = \max(0, u + v - 1, \min(u, v - 1/2)), \ (u, v) \in [0, 1]^2\). \(C\) is a \(\text{shuffle of Min}\) \([9]\), whose mass is spread uniformly on two line segments on \([0, 1]^2\): one joining the points \((0, 1/2)\) and \((1, 2), 1\), and the second one joining the points \((1/2, 1/2)\) and \((1, 0)\). Then it is easy to verify that \((C|M)(t) = (W|M)(t) = (1 + t)/2\) for all \(t \in [0, 1]\).

As a consequence of Theorem 1, we have the following
Corollary 2. Each equivalence class of the equivalence relation $\equiv_M$ on the set of copulas contains a unique Bertino copula.

Consider Spearman’s footrule coefficient [19], whose population version for a random pair $(X, Y)$ with copula $C$, is given by

$$\varphi_C = 1 - 3 \int_0^1 \int_0^1 |u - v| \, dC(u, v)$$

(see [11]). In terms of the $M$ distribution function of the copula $C$, this measure can be rewritten as

$$\varphi_C = 4 - 6 \int_0^1 (C|M)(t) \, dt$$

(see [12]). Given two copulas $C_1$ and $C_2$, $\langle C_1|M \rangle \leqst \langle C_2|M \rangle$ implies that $\varphi_{C_1} \leq \varphi_{C_2}$. However, the converse result is not true in general, as the following example shows.

Example 2. Let $C$ be the shuffle of Min given by $C(u, v) = \min(u, v, \max(1/3, u + v - 2/3))$, $(u, v) \in [0, 1]^2$. Its mass is spread uniformly on three line segments in $[0, 1]^2$: one joining the points $(0, 0)$ and $(1/3, 1/3)$, another one joining the points $(1/3, 2/3)$ and $(2/3, 1/3)$, and the third one joining the points $(2/3, 2/3)$ and $(1, 1)$. Then we have $(\Pi|M)(t) = \sqrt{t}$ for all $t$ in $[0, 1]$, and $(C|M)(t) = 2/3$ if $t \in [1/3, 2/3]$ and $(C|M)(t) = t$ otherwise. Thus, $\varphi_\Pi = 0 < 2/3 = \varphi_C$, but $(\Pi|M)(1/3) \simeq 0.577 < 0.67 \simeq (C|M)(1/3)$.

The “$M$-larger” ordering has several applications. For example, if $(U_i, V_i)$ are two uniform $[0, 1]$ random variables with copula $C_i$, $i = 1, 2$, then $C_1$ is $M$-larger than $C_2$ if, and only if, the order statistics of $U_i$ and $V_i$ are stochastically “inside” the interval determined by the order statistics of $U_2$ and $V_2$ [12]. The next result shows the relationship between the $M$-larger and the $M$-larger in measure orderings. To this end, we first note that, for any pair $(U, V)$ of random variables with associated copula $C$, the $C$ distribution function of $M$ is given by

$$(M|C)(t) = \Pr[\min(U, V) \leq t] = \Pr[U \leq t] + \Pr[U > t, V \leq t]$$

$$= t + \int_t^1 \Pr[V \leq t|U = u] \, du = t + \int_t^1 \frac{\partial C}{\partial u}(u, t) \, du$$

$$= t + t - C(t, t) = 2t - \delta_C(t)$$

for every $t$ in $[0, 1]$.

Proposition 3. Let $C_1$ and $C_2$ be two copulas. Then $\langle M|C_1 \rangle \leqst \langle M|C_2 \rangle$ if, and only if, $\langle C_1|M \rangle \leqst \langle C_2|M \rangle$.

Proof. Let $\delta_{C_1}$ and $\delta_{C_2}$ be the respective diagonal sections of $C_1$ and $C_2$. Then $C_1$ is $M$-larger in measure than $C_2$ if, and only if, $2t - \delta_{C_1}(t) \leq 2t - \delta_{C_2}(t)$ for all $t$
in \([0, 1]\), i.e., \(\delta_{C_2} \leq \delta_{C_1}\), which is equivalent to \(\delta_{C_1}^{(t-1)} \leq \delta_{C_2}^{(t-1)}\), that is, \((C_1|M)(t) \leq (C_2|M)(t)\) for all \(t\) in \([0, 1]\).

As a consequence of Proposition 3, the \(M\)-equivalence in measure coincides with the \(M\)-equivalence. We now show that the equality \(\langle M|C \rangle \overset{d}{=} \langle C|M \rangle\) only holds when \(C = M\).

**Proposition 4.** Let \(C\) be a copula. Then \(\langle M|C \rangle \overset{d}{=} \langle C|M \rangle\) if, and only if, \(C = M\).

**Proof.** Suppose \(\langle M|C \rangle \overset{d}{=} \langle C|M \rangle\), i.e., \(2t - \delta_C(t) = \delta_C^{(t-1)}(t)\) for all \(t\) in \([0, 1]\). Thus, \(\delta_C^{(t-1)}(t) = \sup\{u \in [0, 1] | \delta_C(u) \leq t\} \leq 2t\) for all \(t\) in \([0, 1]\), which implies that \(\delta_C(t) \geq t/2\) for all \(t\) in \([0, 1]\). Hence, \(2t - \delta_C^{(t-1)}(t) \geq t/2\) for all \(t\) in \([0, 1]\), i.e., \(\delta_C^{(t-1)}(t) \leq 3t/2\) for all \(t\) in \([0, 1]\), which implies that \(\delta_C(t) \geq 2t/3\) for all \(t\) in \([0, 1]\). After \(n\) iterations, we have that \(\delta_C(t) \geq nt/(n + 1)\) for all \(t\) in \([0, 1]\). Therefore, if \(n\) tends to infinity, we have that \(\delta_C(t) \geq t\), and hence, \(\delta_C(t) = t\) for all \(t\) in \([0, 1]\). Thus, we obtain that \(C = M\); otherwise, if there exists a point \((u, v)\) in \([0, 1]^2\) such that \(C(u, v) = M(u, v)\) with \(u \leq v\) (the case \(u \geq v\) is similar), then \(C(u, u) \leq C(u, v) = M(u, v) = u\), that is, there exists \(u\) in \([0, 1]\) such that \(\delta_C(u) < u\), which is absurd. The converse is trivial, completing the proof. \(\square\)

Let \(C_1\) and \(C_2\) be two copulas. We say that \(C_1\) is df-larger than \(C_2\) if \(\langle C_1|C_1 \rangle \geq_{st} \langle C_2|C_2 \rangle\) [2, 12, 13]. The following example shows that the df-larger and the M-larger orderings are not comparable.

**Example 3.**

(a) Consider the copulas \(\Pi\) and the shuffle of Min given by \(C(u, v) = \min(u, v, \max(0, u - 0.3, v - 0.612, u + v - 0.912))\), \((u, v) \in [0, 1]^2\), whose mass is spread on three line segments in \([0, 1]^2\): one joining the points \((0.0612\) and \((0.3, 0.912)\), the second one joining the points \((0.3, 0)\) and \((0.912, 0.612)\), and the third one joining the points \((0.912, 0.912)\) and \((1, 1)\). For every \(t\) in \([0, 1]\), we have \(\langle \Pi|\Pi \rangle(t) = t - t \ln t\), \(\langle \Pi|M \rangle(t) = \sqrt{t}\), \(\langle C|C \rangle(t) = \max(t, \min(2t, t + 0.3, 0.912))\), and \(\langle C|M \rangle(t) = \max(t, \min(t + 0.3, t + 0.912)/2)\) for all \(t\) in \([0, 1]\). Then, it is easy to check that \(\langle \Pi|\Pi \rangle \leq_{st} \langle C|C \rangle\); however, we have \(\langle \Pi|M \rangle(0) = 0 < 0.3 = \langle C|M \rangle(0)\) and \(\langle \Pi|M \rangle(0.912) \simeq 0.955 > 0.912 = \langle C|M \rangle(0.912)\).

(b) Consider now the copulas \(\Pi\) and \(A = (M + W)/2\) – recall that the convex linear combination of two copulas is again a copula. The mass distribution of \(A\) is spread uniformly on two line segments in \([0, 1]^2\): one connecting the points \((0, 0)\) to \((1, 1)\), and the second one connecting \((0, 1)\) to \((1, 0)\). Then, for every \(t\) in \([0, 1]\), we have \(\langle A|A \rangle(t) = \min(3t, 2 + t)/3\) and \(\langle A|M \rangle(t) = \min(2t, 2 + t + 1)/3\). Thus, it is easy to verify that \(\langle \Pi|M \rangle \leq_{st} \langle A|M \rangle\); but \(\langle \Pi|\Pi \rangle(0.25) \simeq 0.5966 < 0.75 = \langle A|A \rangle(0.25)\) and \(\langle \Pi|\Pi \rangle(0.75) \simeq 0.9658 > 0.9167 = \langle A|A \rangle(0.75)\).
3. THE W DISTRIBUTION FUNCTION OF A COPULA

The opposite diagonal section $\omega_C$ of a copula $C$ is the function given by $\omega_C(t) = C(t, 1 - t)$ for all $t$ in $[0, 1]$. An opposite diagonal is a function $\omega: [0, 1] \to [0, 1]$ which satisfies the following properties:

(i) $\omega(1) = 0$,

(ii) $\omega(t) \leq \min(t, 1 - t)$ for all $t$ in $[0, 1]$,

(iii) $\omega(t') - \omega(t) \leq t' - t$ for all $t, t'$ in $[0, 1]$ such that $t \leq t'$ – i.e., $\omega$ is 1-Lipschitz.

The opposite diagonal section of any copula is an opposite diagonal; and for any opposite diagonal $\omega$, there exist copulas whose opposite diagonal section is $\omega$: for instance, the copula $J_\omega$ given by

$$J_\omega(u, v) = \max\left(0, u + v - 1, \frac{u + v - 1 + \omega(u) + \omega(1 - v)}{2}\right)$$

for all $(u, v)$ in $[0, 1]^2$ (see [3]).

The following result provides a probabilistic interpretation of the opposite diagonal section of a copula (in the sequel, we will denote the distribution function of a random variable $X$ either by $\text{df}(X)$ or a letter such as $F$).

**Proposition 5.** Let $(U, V)$ be a pair of random variables with associated copula $C$. Then

$$\omega_C(t) = \frac{1}{2} \cdot (\Pr[\min(U, 1 - V) \leq t < \max(U, 1 - V)]).$$

**Proof.** The copula $C'$ associated with the random pair $(U, 1 - V)$ is given by $C'(u, v) = u - C(u, 1 - v)$ for every $(u, v)$ in $[0, 1]^2$ (see [11]). Then we have that

$$\Pr[\min(U, 1 - V) \leq t] = \Pr[u \leq t] + \Pr[1 - V \leq t] - \Pr[u \leq t, 1 - V \leq t] = t + t - C'(t, t) = t + \omega(t),$$

and

$$\Pr[\max(U, 1 - V) \leq t] = \Pr[\max(U, 1 - V) \leq t] = \Pr[u \leq t, 1 - V \leq t] = C'(t, t) = t - C(t, 1 - t) = t - \omega(t),$$

whence the result easily follows. \(\square\)

Let $(U, V)$ be a random pair with copula $C$. The $W$ distribution function of $C$ is given by

$$(C|W)(t) = \Pr[C(U, V) \leq t] = \Pr[C(U, 1 - U) \leq t] = \Pr[\omega_C(U) \leq t] = \lambda(\{u \in [0, 1] | \omega_C(u) \leq t\}),$$
where $\lambda$ denotes the Lebesgue measure in $\mathbb{R}$.

Distribution functions of copulas are also employed in constructing new measures of association. Thus, for instance, given a copula $C$, it seems reasonable to obtain a measure $\chi_C$ – in the same sense that Spearman’s footrule coefficient $\varphi_C$ – based on the $W$ distribution function of $C$, and given by the linear expression

$$\chi_C = a \int_0^1 (C|W)(t) \, dt + b$$

where $a$ and $b$ are two real numbers. If we consider $\chi_W = -1$ and $\chi_\Pi = 0$ for this measure – for the Spearman’s footrule coefficient we have $\varphi_M = 1$, $\varphi_\Pi = 0$, and $\varphi_W = -1/2$ –, since $(\Pi|W)(t) = 1 - \sqrt{\max(0,1-4t)}$ and $(M|W)(t) = \min(2t,1)$ for all $t$ in $[0,1]$, then we obtain

$$\chi_C = 5 - 6 \int_0^1 (C|W)(t) \, dt.$$

The coefficient $\chi_C$ can be also written as

$$\chi_C = 6 \int_0^1 C(t,1-t) \, dt - 1 = 3 \int_0^1 \int_0^1 |1-u-v| \, dC(u,v) - 1.$$  

This coefficient – which first appeared in this last form in [1] – satisfies $\chi_M = 1/2$. Observe also that the population version $\gamma_C$ of the known Gini’s rank correlation coefficient [8, 10, 11] of a copula $C$ can be written as $\gamma_C = 2(\varphi_C + \chi_C)/3$.

Unlike the relationship between the $M$-larger and the $M$-larger in measure orderings, there is no analogue to Proposition 3 for the $W$-larger and the $W$-larger in measure orderings, as the next example shows. The example also provides a class in the equivalence relation $\equiv_W$ – recall that if $C_1$ and $C_2$ are two copulas, then $C_1 \equiv_W C_2$ if $(C_1|W)(t) = (C_2|W)(t)$ for all $t$ in $[0,1]$ – which contains more than one copula. First note that, if $(U,V)$ is a random pair with copula $C$, then the $C$ distribution function of $W$ is given by

$$(W|C)(t) = \Pr[U + V - 1 \leq t] = \Pr[U \leq t] + \Pr[U > t, V \leq 1 + t - U]$$

$$= t + \int_t^1 \Pr[V \leq 1 + t - u \mid U = u] \, du$$

$$= t + \int_t^1 \frac{\partial C}{\partial u}(u,1 + t - u) \, du$$

for every $t$ in $[0,1]$.

**Example 4.** Let $C$ be the shuffle of Min given by $C(u,v) = \min(u,v, \max(1/2, u + v - 1))$, $(u,v) \in [0,1]^2$. Its mass is spread uniformly on two line segments in $[0,1]^2$: one joining the points (0,0) and (1/2,1/2), and the second one joining the points (1/2,1) and (1,1/2). Then it is easy to verify that $(C|W)(t) = (M|W)(t) = \min(2t,1)$ for all $t$ in $[0,1]$. But, on the other hand, we have $(W|C)(t) = 1/2$ if $t \in [0,1/2)$ and $(W|C)(t) = 1$ if $t \in [1/2,1]$, and $(W|M)(t) = (1 + t)/2$. 


To see the “utility” of the \(C\) distribution function of \(W\), where \(C\) is the copula of the random pair \((U, V)\), we provide the following result, which describes the relationship between this distribution function and the distribution function of the random variable \(U + V\). In what follows, we will use some notation. Let \(f\) be a real function defined on \([a, b]\) (or on a dense subset of \([a, b]\), including \(a\) and \(b\)) having only removable or jump discontinuities. Then \(\ell^+ f\) and \(\ell^- f\) are the functions defined on \([a, b]\) via \(\ell^+ f(x) = f(x^+)\) and \(\ell^- f(x) = f(x^-)\), where \(f(x^+)\) (respectively, \(f(x^-)\)) denotes the limit – if it exists – by the right (respectively, left) of \(f\) in \(x\). Let \(\hat{C}\) denote the survival copula of \(C\), i.e., \(\hat{C}(u, v) = u + v - 1 + C(1 - u, 1 - v)\) for every \((u, v) \in [0, 1]^2\) (see [11]).

**Proposition 6.** Let \((U, V)\) be a pair of random variables with associated copula \(C\). Then we have

\[
\text{df}(U + V)(t) = \begin{cases} 
\ell^+ (1 - (W|\hat{C})(1-t)), & \text{if } t \in [0, 1] \\
(W|C)(t-1), & \text{if } t \in [1, 2]. 
\end{cases}
\]

**Proof.** Let \(t \in [0, 1]\). Then we have

\[
\text{df}(U + V)(t) = \mu_C((u, v) \in [0, 1]^2 \mid u + v \leq t) \\
= \mu_C((u, v) \in [0, 1]^2 \mid (1-u) + (1-v) - 1 \geq 1 - t) \\
= \mu_C((1-u', 1-v') \in [0, 1]^2 \mid u' + v' - 1 \geq 1 - t)) \\
= \mu_C((u', v') \in [0, 1]^2 \mid W(u', v') \geq 1 - t) \\
= 1 - \mu_C((u', v') \in [0, 1]^2 \mid W(u', v') < 1 - t) \\
= 1 - \ell^-((W|\hat{C})(1-t)) \\
= \ell^+ (1 - (W|\hat{C})(1-t)),
\]

where we have done the transformations \(u' = 1 - u\), \(v' = 1 - v\). On the other hand, for every \(t \in [1, 2]\), we have

\[
(W|C)(t) = \mu_C((u, v) \in [0, 1]^2 \mid u + v - 1 \leq t) \\
= \mu_C((u, v) \in [0, 1]^2 \mid u + v \leq t + 1)) \\
= \text{df}(U + V)(t + 1),
\]

which completes the proof. \(\square\)

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