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FOURIER COEFFICIENTS OF CONTINUOUS LINEAR MAPPINGS ON HOMOGENEOUS BANACH SPACES

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Introduction. Let **T** be the quotient group $R/2\pi Z$ (R and Z denoting the additive group of reals, integers, respectively). Let $H(\mathbf{T})$ be a homogeneous Banach space on **T** ([6], p. 14) with the norm $\|\ \|_H$. Let X be a quasi-complete locally convex (Hausdorff) topological vector space and $u: H(\mathbf{T}) \to X$ a continuous linear mapping. The Fourier coefficients of the mapping u are, by definition, the elements of X of the form $\hat{u}(n) = u(e^{-int})$, $n \in Z$. Let (x_n) be a two-way sequence of elements of X. In this paper the necessary and sufficient conditions are given for (x_n) to be the Fourier coefficients of some continuous, weakly compact or compact linear mapping $u: H(\mathbf{T}) \to X$, in particular if $H(\mathbf{T}) = C(\mathbf{T})$, to be the Fourier—Stieltjes coefficients of a regular vector measure on **T** with values in X (cf. also [7], [10] and [11]). The results are a generalization of the results of ([6], p. 34 ff.) proved for a two-way sequence of complex numbers.

1. Recall that $H(\mathbf{T})$ is a linear subspace of the Banach space $L^1(\mathbf{T})$ (of all complex-valued Lebesgue integrable functions on **T**) having a norm $\|\ \|_H \geqq \|\ \|_1$ under which it is a Banach space having the properties:

(1) If $f \in H(\mathbf{T})$ and $v \in \mathbf{T}$, then $f_v \in H(\mathbf{T})$ and $\|f_v\|_H = \|f\|_H$.

$$(f_v(t) = f(t - v))$$

(2) For all $f \in H(\mathbf{T})$, v, $v_0 \in \mathbf{T}$, $\lim_{v \to v_0} \|f_v - f_{v_0}\|_H = 0$.

Examples of homogeneous Banach spaces on **T** are (cf. [6]): the space $C(\mathbf{T})$ of all continuous functions, the space $C^n(\mathbf{T})$ of all n-times continuously differentiable functions, the spaces $L^p(\mathbf{T})$, $1 \leqq p < \infty$.

A trigonometric polynomial on **T** is a function $a = a(t)$ defined on **T** by $a(t) = \sum_{-n}^{n} a_j e^{ijt}$. Denote by p (**T**) the set of all trigonometric polynomials on **T**. We shall need the following theorem ([6], Th. 2.12).

Theorem 1.1. *For every $f \in H(\mathbf{T})$ we have $\sigma_n(f) \to f$, $n \to \infty$, in the $H(\mathbf{T})$ norm.*

Recall that

$$\sigma_n(f,t)=\sum_{-n}^{n}\left(1-\frac{|j|}{n+1}\right)\hat{f}(j)\,e^{ijt},$$

where $\hat{f}(j)$ is the jth Fourier—Lebesgue coefficient of f defined by

$$\hat{f}(j)=\frac{1}{2\pi}\int f(t)\,e^{-ijt}\,dt.$$

Let the locally convex topology of the space X be defined by a family $\mathcal{Q}=(q)$ of continuous seminorms. For a continuous seminorm q and for a linear mapping u from $H(\mathbf{T})$ into X we denote

$$\|u\|_q=\sup\{q(u(f)):f\in H(\mathbf{T}),\ \|f\|_H\leqq 1\}.$$

Lemma 1.2. *Let $u:H(\mathbf{T})\to X$ be a continuous linear mapping. For every $a=\sum_{-n}^{n}a_j\,e^{ijt}$ we have $u(a)=\sum_{-n}^{n}a_j\hat{u}(-j)$ and $q(u(a))\leqq\|a\|_H\|u\|_q$ for every continuous seminorm q.*

Theorem 1.3. (Parseval's formula) *Let $f\in H(\mathbf{T})$ and u: $H(\mathbf{T})\to X$ be a continuous linear mapping. Then*

$$u(f)=\lim_{N\to\infty}\sum_{-N}^{N}\left(1-\frac{|j|}{N+1}\right)\hat{f}(j)\hat{u}(-j).$$

Proof. Since, by theorem 1.1, $f=\lim_{n\to\infty}\sigma_n(f)$ in the $H(\mathbf{T})$ norm, it follows from lemma 1.2 and the continuity of u that the assertion is true.

Theorem 1.4. *Let (x_j) be a two-way sequence of elements of X. Then the following two conditions are equivalent:*

(a) *There is a continuous linear mapping $u:H(\mathbf{T})\to X$, with $\|u\|_q\leqq C_q<\infty$ for every continuous seminorm q, such that $\hat{u}(j)=x_j$ for all $j\in Z$.*

(b) *For all trigonometric polynomials $a=\sum_{-l}^{l}a_j\,e^{ijt}$ and all continuous seminorms q there holds $q\left(\sum_{-l}^{l}a_{-j}x_j\right)\leqq\|a\|_H C_q$.*

Proof. Clearly (a) implies (b). If we assume (b), then the linear mapping u defined on the space of all $a=\sum_{-l}^{l}a_j\,e^{ijt}\in p\,(\mathbf{T})$ by

$$u(a)=\sum_{-l}^{l}a_{-j}x_j$$

satisfies the inequality $q(u(a))\leqq C_q\|a\|_H$ for every $q\in\mathcal{Q}$, i.e. u is a continuous linear mapping on p (**T**) and hence using theorem 1.1, u admits a unique extension ([2], II. §3, Th. 6.2) $\bar{u}$ that is a continuous linear mapping on $H(\mathbf{T})$ with $\|\bar{u}\|_q \leqq C_q$ for all $q \in \mathcal{Q}$. Since $\bar{u}$ extends u, we obtain $\hat{u}(j)=x_j$.

We say that the function $F:\mathbf{T}\to X$ is integrable if, for every $x' \in X'$ (the space of all continuous linear forms on X), the function $t \to \langle F(t), x'\rangle$ is Lebesgue integrable, and if, for every $M \in \mathcal{B}(\mathbf{T})$ (Borel sets in **T**), there exists an element $x_M \in X$ such that

$$\langle x_M, x'\rangle = \int_M \langle F(t), x'\rangle \, dt, \quad x' \in X'.$$

If $M=\mathbf{T}$, we write $x_{\mathbf{T}}=\int F(t)\,dt$ (cf. [7], p. 6).

Let (x_j) be a two-way sequence of elements of X. Denote

$$\sigma_N(X, t)=\sum_{-N}^{N}\left(1-\frac{|j|}{N+1}\right) x_{-j}\, e^{-ijt}, \quad N=1, 2, \ldots$$

and by $S_N(X)$ the continuous linear mapping on $H(\mathbf{T})$ defined by

$$S_N(X)(f)=\frac{1}{2\pi}\int f(t)\sigma_N(X, t)\,dt, \quad f \in H(\mathbf{T}), \quad N=1, 2, \ldots$$

If $u \in L(H(\mathbf{T}), X)$ (the linear space of all continuous linear mappings of $H(\mathbf{T})$ into X) and if $x_j=\hat{u}(j)$, we shall write

$$\sigma_n(X, t)=\sigma_N(u, t) \quad \text{and} \quad S_N(X)=S_N(u).$$

We have

$$S_N(X)(f)=\frac{1}{2\pi}\int f(t)\sigma_N(X, t)\,dt=\sum_{-N}^{N}\left(1-\frac{|j|}{N+1}\right)\hat{f}(j)x_{-j},$$
$$f \in H(\mathbf{T}).$$

Theorem 1.5. *The members of a two-way sequence* (x_j) *in* X *are the Fourier coefficients of some* $u \in L(H(\mathbf{T}), X)$, *with* $\|u\|_q \leqq C_q<\infty$, *for all* $q \in \mathcal{Q}$, *if and only if* $\|S_N(X)\|_q \leqq C_q$, $N=1, 2, \ldots$

Proof. The necessity. Let $x_j=\hat{u}(j)$ for some $u \in L(H(\mathbf{T}), X)$ with $\|u\|_q \leqq C_q$, $q \in \mathcal{Q}$. Then $S_N(X)=S_N(u)$, $N=1, 2, \ldots$ Recall that $\|\sigma_N(f)\|_H \leqq \|f\|_H$ for all $f \in H(\mathbf{T})$. Since, for $f \in H(\mathbf{T})$, $S_N(u)(f) = u(\sigma_N(f))$, we have

$$\|S_N(X)\|_q=\|S_N(u)\|_q=\sup\{q(S_N(u)(f)): f \in H(\mathbf{T}), \|f\|_H \leqq 1\}=$$
$$\sup\{q(u(\sigma_N(f))): f \in H(\mathbf{T}),$$
$$\|f\|_H \leqq 1\} \leqq$$
$$\leqq \sup\{q(u(f)): f \in H(\mathbf{T}), \|f\|_H \leqq 1\}=\|u\|_q \leqq C_q,$$

for all $q \in \mathcal{Q}$, $N=1, 2, \ldots$

The sufficiency. Take $a=\sum_{-l}^{l} a_j\, e^{-ijt}$. Then we have

$$\sum_{-l}^{l} x_{-j}a_j = \lim_{N\to\infty} \sum_{-N}^{N} \left(1-\frac{|j|}{N+1}\right) x_{-j}a_j = \lim_{N\to\infty} S_N(X)(a).$$

Thus

$$q\left(\sum_{-l}^{l} x_{-j}a_j\right) = \lim_{N\to\infty} q(S_N(X)(a)) \leqq \|a\|_H \limsup \|S_N(X)\|q \leqq \|a\|_H C_q.$$

According to theorem 1.4 there exists a $u \in L(H(\mathbf{T}), X)$ such that $x_j = \hat{u}(j)$ and $\|u\|_q \leqq C_q$ for all $q \in \mathcal{Q}$.

If $F: T \to X$ is an integrable function, the element of X of the form

$$\frac{1}{2\pi} \int e^{-ijt} F(t)\, dt$$

is called the Fourier—Lebesgue coefficient of F.

Theorem 1.6. *Let $F: (\mathbf{T}) \to X$ be an integrable function and put*

$$u(f) = \frac{1}{2\pi} \int f(t) F(t)\, dt, \quad f \in C(\mathbf{T}).$$

The members of a two-way sequence (x_j) in X are the Fourier—Lebesgue coefficients of F if and only if $\lim_{N\to\infty} S_N(X)(f) = u(f)$ for all $f \in C(\mathbf{T})$.

Proof. Let $x_j = \hat{F}(j)$, $j \in Z$. Clearly $f \to u(f)$ is a continuous linear mapping on $C(\mathbf{T})$ and thus $x_j = \hat{F}(f) = \hat{u}(j)$. By Parseval's formula we have

$$\lim_{N\to\infty} S_N(X)(f) = \lim_{N\to\infty} S_N(u)(f) = \lim_{N\to\infty} \sum_{-N}^{N} \left(1-\frac{|j|}{N+1}\right) \hat{f}(j)\hat{u}(-j) = u(f),$$

for all $f \in C(\mathbf{T})$. Conversely we have $x_{-j} = \lim_{N\to\infty} S_N(X)(e^{ijt}) = u(e^{ijt}) = \hat{u}(-j)$, i.e. $x_j = \hat{F}(j) = \hat{u}(j) = \frac{1}{2\pi} \int e^{-ijt} F(t)\, dt$.

For a similar result we quote ([7], Th. 2).

Let now X' be the conjugate of a separable Banach space X. Let $x'(\cdot): \mathbf{T} \to X'$ be a function such that $x'(\cdot)x$ is measurable for every $x \in X$ and $\operatorname{vrai\,sup}_{t\in\mathbf{T}} \|x'(t)\| = C < \infty$. Then the equality

$$(uf)(x) = \frac{1}{2\pi} \int x'(s) x f(t)\, dt, \quad f \in L^1(\mathbf{T}), \quad x \in X$$

defines a continuous linear mapping $u: L^1(\mathbf{T}) \to X'$ with the norm C ([4], VI. 8. 6). Hence we may define the jth Fourier—Lebesgue coefficient $\hat{x}'(j)$ of such a function $x'(\cdot)$ as the element of X such that

$$\hat{x}'(j)x = \frac{1}{2\pi} \int x'(t) x\, e^{-ijt}\, dt, \quad x \in X.$$

If (x'_j) is a two-way sequence of elements of X', then if we put

$$\sigma_N(X', t)=\sum_{-N}^{N}\left(1-\frac{|j|}{N+1}\right)x'_{-j}\, e^{-ijt}, \quad N=1, 2, \ldots,$$

for each $x\in X$, the function $\sigma_N(X', \cdot)x$ is measurable and bounded on $\mathbf{T}$. Hence the equation

$$(S_N(X')(f))(x)=\frac{1}{2\pi}\int f(t)\sigma_N(X', t)x\, dt, \quad f\in L^1(\mathbf{T})$$

defines a continuous linear mapping $S_N(X')$ of $L^1(\mathbf{T})$ into X' whose norm is $\|S_N(X')\|=\sup_{t\in\mathbf{T}}\|\sigma_N(X', t)\|$ ([4], VI. 8. 6).

Theorem 1.7. *Let X' be the conjugate of a separable Banach space X. The members of a two-way sequence (x'_j) of elements of X' are the Fourier—Lebesgue coefficients of an essentially unique function $x'(\cdot):\mathbf{T}\to X'$ such that $x'(\cdot)x$ is measurable and essentially bounded for each $x\in X$, with $\operatorname{vrai\,sup}_{t\in\mathbf{T}}\|x'(t)\|=C$ if and only if*

$$\|S_N(X')\|\leqq C, \quad N=1, 2, \ldots .$$

Proof. If $x'_j=\hat{x}'(j)$ for some $x'(\cdot):\mathbf{T}\to X'$ with properties as in the theorem, then, for fixed $t\in\mathbf{T}$ and $x\in X$, we have

$$|\sigma_N(X', t)x|=\left|\sum_{-N}^{N}\left(1-\frac{|j|}{N+1}\right)x'_{-j}x\, e^{-ijt}\right|=$$

$$=\frac{1}{2\pi}\left|\int\left(\sum_{-N}^{N}\left(1-\frac{|j|}{N+1}\right)e^{ij(s-t)}\right)x'(s)x\, ds\right|=$$

$$=\frac{1}{2\pi}\left|\int K_N(s-t)x'(s)x\, ds\right|\leqq\|K_N\|_1\|x'(\cdot)x\|_1\leqq C\|x\|,$$

hence $\sup_{t\in\mathbf{T}}\|\sigma_N(X', t)\|\leqq C$, $N=1, 2, \ldots$, i.e. $\|S_N(X')\|\leqq C$, $N=1, 2, \ldots$

Conversely, let $\|S_N(X')\|\leqq C$, $N=1, 2, \ldots$ Then according to theorem 1.5 there exists a continuous linear mapping $u:L^1(\mathbf{T})\to X'$ such that $\hat{u}(j)=x'_j$ and $\|u\|\leqq C$. Hence there exists ([4], VI. 8.6) an essentially unique function $x'(\cdot):\mathbf{T}\to X'$ such that $x'(\cdot)x$ is measurable and essentially bounded for each $x\in X$ and

$$(u(f))(x)=\frac{1}{2\pi}\int x'(t)xf(t)\, dt, \quad f\in L^1(\mathbf{T}), \quad x\in X,$$

$$\|u\|=\operatorname{vrai\,sup}_{s\in\mathbf{T}}\|x'(s)\|\leqq C.$$

Further, $x'_j=\hat{x}'(j)$.

Let rca(**T**) denote the Banach space of all regular countably additive scalar measures μ defined on the σ-algebra $\mathscr{B}(\mathbf{T})$ of Borel sets in **T** with the total variation norm ([4], IV. 2. 17). Let X'' be the second dual (the strong bidual) of X ([5], 8. 7). Let $m \in \text{rca}(\mathbf{T}, X'')$, i.e. $m: \mathscr{B}(\mathbf{T}) \to X''$ is a set function such that for each $x' \in X'$ the scalar set function $m \circ x' = \langle x', m(\cdot) \rangle$ belongs to rca(**T**) and the mapping $x' \to m \circ x'$ of the space X' into rca(**T**) is continuous in $\sigma(X', X)$ and $\sigma(\mu, C(\mathbf{T}))$ topologies on X' and rca(**T**), respectively. The equation

$$x'(u(f)) = \frac{1}{2\pi} \int f \, \mathrm{d} m \circ x', \quad f \in C(\mathbf{T}), \quad x' \in X',$$

defines a continuous linear mapping u on $C(\mathbf{T})$ into X ([4], VI. 7.2 and [12], §3, 3. Th.) for which $\|u\|_q = \|m\|_q(\mathbf{T})$, where (the q-semivariation of m on $E \in \mathscr{B}(\mathbf{T})$)

$$\|m\|_q(E) = \sup q \left(\sum_i^n c_i m(E_i) \right),$$

the supremum being taken over all finite families of scalars, $\|c_i\| \leqq 1$, and over all finite disjoint families E_i, $i = 1, \ldots, n$, $E_i \in \mathscr{B}(\mathbf{T})$ such that $\bigcup_{i=1}^n E_i = E$. We take $q \in \mathscr{Q}$ extended to X''.

Let (x_j) be a two-way sequence of elements of X. We say that x_j is the jth Fourier—Stieltjes coefficient of $m \in \text{rca}(\mathbf{T}, X'')$ and we write $x_j = \hat{m}(j)$, if

$$x' x_j = \frac{1}{2\pi} \int e^{-ijt} \, \mathrm{d} m(s) x'$$

for all $x' \in X'$.

Theorem 1.8. *The members of a two-way sequence (x_j) of elements of X are the Fourier—Stieltjes coefficients of some $m \in \text{rca}(\mathbf{T}, X'')$, with $\|m\|_q(\mathbf{T}) \leqq C_q$, $q \in \mathscr{Q}$ if and only if $\|S_N(X)\|_q \leqq C_q$, $q \in Q$, $N = 1, 2, \ldots$*

Proof. If there exists a set function $m \in \text{rca}(\mathbf{T}, X'')$ such that $x_j = \hat{m}(j)$, then the equation $x' u(f) = \frac{1}{2\pi} \int f \, \mathrm{d} m \circ x'$, $f \in C(\mathbf{T})$, $x' \in X'$, defines a continuous linear mapping $u: C(\mathbf{T}) \to X$ with $\|u\|_q = \|m\|_q(\mathbf{T}) \leqq C_q$, $q \in \mathscr{Q}$ ([4], VI. 7.2, [12], §3. 3. Th.). Thus x_j are the Fourier coefficients of u, hence according to theorem 1.5 we have $\|S_N(X)\|_q \leqq C_q$, $q \in \mathscr{Q}$, $N = 1, 2, \ldots$

Conversely, if $\|S_N(X)\|_q \leqq C_q$, $q \in \mathscr{Q}$, $N = 1, 2, \ldots$, then according to theorem 1.5 there exists a continuous linear mapping $u: C(\mathbf{T}) \to X$ such that $\hat{u}(j) = x_j$, $\|u\|_q \leqq \leqq C_q$, $q \in \mathscr{Q}$. Hence there exists ([4], VI. 7.2 and [12], §3. 3 Th.) a set function $m \in \text{rca}(\mathbf{T}, X'')$ with $\|m\|_q(\mathbf{T}) = \|u\|_q \leqq C_q$, $q \in \mathscr{Q}$. So $x_j = \hat{u}(j) = \hat{m}(j)$.

2. Fourier coefficients of weakly compact mappings.

Let V be a normed vector space. Recall that a linear mapping $u: V \to X$ is said to be weakly compact (compact) if, for a suitable neighborhood U of zero in V, $u(U)$ is a weakly relatively compact (a relatively compact) subset of X; equivalently, u transforms the bounded subsets of V into the weakly relatively compact (relatively compact) subsets of X.

Theorem 2.1. *Let (x_j) be a two-way sequence of elements of X. Then the following two conditions are equivalent:*

(a) *There is a weakly compact (compact) linear mapping $u: H(\mathbf{T}) \to X$ with $\|u\|_q \leqq C_q$, $q \in \mathcal{Q}$ such that $\hat{u}(j) = x_j$ for all $j \in Z$.*

(b) *For all trigonometric polynomials $a = \sum_{-l}^{l} a_j e^{ijt}$ and all $q \in \mathcal{Q}$ there holds*

$$q\left(\sum_{-l}^{l} a_{-j} x_j\right) \leqq \|a\|_H C_q$$

and the set

$$A = \left\{ \frac{1}{\|a\|_H} \sum_{-l}^{l} a_{-j} x_j : \text{for all } a \in p\ (\mathbf{T}) \right\}$$

is contained in a weakly compact (compact) subset of X.

Proof. If (a) holds, then the mapping u is necessarily continuous with $\|u\|_q \leqq C_q$, $q \in \mathcal{Q}$, and since the set A is the range of u on the set of all trigonometric polynomials of H-norm one, A is contained in a weakly compact (compact) subset W of X.

Conversely, let the set A be contained in a weakly compact (compact) subset W of X. The closed absolutely convex hull $\overline{\text{aco}}(W)$ of the set W is a closed convex bounded and so complete subset of X because X is quasi-complete and hence $\overline{\text{aco}}(W)$ is a weakly compact (compact) subset of X ([8], p. 244 and 328). Therefore the closed absolutely convex hull $\overline{\text{aco}}(A)$ of the set A is a weakly compact (compact) subset of X. Since $p\ (\mathbf{T})$ is dense in $H(\mathbf{T})$, the continuous linear mapping $u: H(\mathbf{T}) \to X$, existing according to theorem 1.4, maps every bounded set in $H(\mathbf{T})$ into a relatively weakly compact (relatively compact) subset of X. Since $H(\mathbf{T})$ is a Banach space, we obtain that u is a weakly compact (compact) linear mapping such that $\hat{u}(j) = x_j$ and $\|u\|_q \leqq C_q$, $q \in \mathcal{Q}$.

Theorem 2.2. *The members of a two-way sequence (x_j) in X are the Fourier coefficients of some weakly compact (compact) mapping $u \in L(H(\mathbf{T}), X)$ if and only if there exists a weakly compact (compact) subset W of X such that $S_N(X)(f) \in W$ for all $f \in H(\mathbf{T})$, $\|f\|_H \leqq 1$ and for $N = 1, 2, \ldots$*

Proof. The necessity. If $x_j = \hat{u}(j)$ for some weakly compact (compact) $u \in L(H(\mathbf{T}), X)$, then there exists a weakly compact (compact) subset W of X such that

$$S_N(X)(f) = \sum_{-N}^{N} \left(1 - \frac{|j|}{N+1}\right) \hat{f}(j)\hat{u}(-j) =$$

$$= u\left(\sum_{-N}^{N} \left(1 - \frac{|j|}{N+1}\right) \hat{f}(j)\, e^{ijt}\right) = u(\sigma_N(f)) \in W, \quad N = 1, 2, \ldots$$

for all $f \in H(\mathbf{T})$, $\|f\|_H \leqq 1$, because $\|\sigma_N(f)\|_H \leqq 1$.

The sufficiency. For every trigonometrical polynomial $a = \sum_{-l}^{l} a_j\, e^{ijt}$ we have

$$\frac{1}{\|a\|_H} \sum_{-l}^{l} a_{-j} x_j = \frac{1}{\|a\|_H} \sum_{-l}^{l} a_j x_{-j} = \lim_{N\to\infty} \frac{1}{\|a\|_H} \sum_{-N}^{N} \left(1 - \frac{|j|}{N+1}\right) a_j x_{-j} =$$

$$= \lim_{N\to\infty} S_N(X)\left(\frac{a}{\|a\|_H}\right) \in W$$

and for some positive C_q, $q \in \mathcal{Q}$, $q\left(\sum_{-l}^{l} a_{-j} x_j\right) \leqq \|a\|_H C_q$. It follows from theorem 2.1 that there exists a weakly compact (compact) linear mapping $u: H(\mathbf{T}) \to X$ such that $x_j = \hat{u}(j)$.

Close to the preceding theorem is the following.

Theorem 2.3. *The members of a two-way sequence (x_j) in X are the Fourier coefficients of a some weakly compact (compact) linear mapping $u \in L(H(\mathbf{T}), X)$, with $\|u\|_q \leqq C_q$ for all $q \in \mathcal{Q}$, if and only if $\|S_N(X)\|_q \leqq C_q < \infty$, $N = 1, 2, \ldots$ and there exists a weakly compact (compact) subset W of X such that $S_N(X)(f) \in W$, $N = 1, 2, \ldots$ for all $f \in H(\mathbf{T})$, $\|f\|_H \leqq 1$.*

Proof. Similarly as in theorem 2.2.

Theorem 2.4. *Let X be a Banach space and (x_j) a two-way sequence of elements of X. The elements x_j are the Fourier—Lebesgue coefficients of some measurable weakly compact valued (compact valued) function $g: \mathbf{T} \to X$, i.e. $g(\mathbf{T})$ is a weakly relatively compact (relatively compact) subset of X, if and only if there exists a weakly compact (compact) subset W of X such that $S_N(X)(f) \in W$ for $N = 1, 2, \ldots$ and all $f \in L^1(\mathbf{T})$, $\|f\|_1 \leqq 1$.*

Proof. The necessity. If $x_j = \hat{g}(j) = \frac{1}{2\pi} \int e^{-ijt} g(t)\, dt$ with g weakly compact valued (compact valued), then the relation

$$u(f) = \frac{1}{2\pi} \int fg\, dt, \quad f \in L^1(\mathbf{T})$$

defines a weakly compact (compact) linear mapping $u: L^1(\mathbf{T}) \to X$ ([5], 9.4.7 and 9.4.8). Then $x_j = \hat{u}(j)$ and according to theorem 2.3 there exists a weakly compact (compact) subset W of X such that $S_N(X)(f) = S_N(u)(f) \in W$, $N = 1, 2, \ldots$ and all $f \in L^1(\mathbf{T})$, $\|f\|_1 \leqq 1$.

The sufficiency. If

$$S_N(X)(f) \in W, \quad N = 1, 2, \ldots \quad \text{and all} \quad f \in L^1(\mathbf{T}), \quad \|f\|_1 \leqq 1,$$

for some weakly compact (compact) subset W of X, then according to theorem 2.3 there exists a weakly compact (compact) linear mapping $u: L^1(\mathbf{T}) \to X$ such that $\hat{u}(j) = x_j$. Hence ([5], 9.4.7 and 9.4.8; [4], VI. 8.10 and VI. 8.11) there exists a measurable weakly compact valued (compact valued) function $g: \mathbf{T} \to X$ such that

$$u(f) = \frac{1}{2\pi} \int fg \, \mathrm{d}t, \quad f \in L^1(\mathbf{T}).$$

So $x_j = \hat{g}(j)$.

Theorem 2.5. *Given a two-way sequence (x_j) of elements of X, there exists a regular vector measure $m: \mathcal{B}(\mathbf{T}) \to X$ with $\|m\|_q(\mathbf{T}) \leqq C_q$, $q \in \mathcal{Q}$ such that x_j are the Fourier—Stieltjes coefficients of m if and only if $\|S_N(X)\|_q \leqq C_q$, $q \in \mathcal{Q}$, $N = 1, 2, \ldots$ and there exists a weakly compact subset W of X such that*

$$S_N(X)(f) \in W, \quad N = 1, 2, \ldots, \quad f \in C(\mathbf{T}), \quad \|f\|_\infty \leqq 1.$$

Proof. If there exists a regular vector measure $m: \mathcal{B}(\mathbf{T}) \to X$ with $\|m\|_q \leqq C_q$, $q \in \mathcal{Q}$, such that $x_j = \hat{m}(j) = \frac{1}{2\pi} \int e^{-ijt} \, \mathrm{d}m(t)$, then the equation $u(f) = \frac{1}{2\pi} \int f \, \mathrm{d}m$, $f \in C(\mathbf{T})$, defines a weakly compact linear mapping on $C(\mathbf{T})$ into X ([7], Proposition 1, [9], Theorem 3.1) with $\|u\|_q = \|m\|_q(\mathbf{T}) \leqq C_q$, $q \in \mathcal{Q}$, ([3], Theorem 12). Thus x_j are the Fourier coefficients of u, hence according to theorem 2.3 there exists a weakly compact subset W of X such that $S_N(X)(f) \in W$, $N = 1, 2, \ldots$, $f \in C(\mathbf{T})$, $\|f\|_\infty \leqq 1$, and we have $\|S_N\|_q \leqq C_q$, $q \in \mathcal{Q}$, $N = 1, 2, \ldots$

Conversely, if $\|S_N\|_q \leqq C_q$, $q \in \mathcal{Q}$, $N = 1, 2, \ldots$ and there exists such a weakly compact subset W of X that $S_N(X)(f) \in W$, $N = 1, 2, \ldots$, $\|f\|_\infty \leqq 1$, then according to theorem 2.3 there exists a weakly compact linear mapping $u: C(\mathbf{T}) \to X$ such that $\hat{u}(j) = x_j$ with $\|u\|_q \leqq C_q$, $q \in \mathcal{Q}$. Hence there exists ([7], Proposition 1, [3], Theorem 12) a regular vector measure $m: \mathcal{B}(\mathbf{T}) \to X$ such that

$$u(f) = \frac{1}{2\pi} \int f \, \mathrm{d}m, \quad f \in C(\mathbf{T}), \quad \|m\|_q(\mathbf{T}) = \|u\|_q \leqq C_q, \quad q \in \mathcal{Q}.$$

So $x_j = \hat{u}(j) = \hat{m}(j)$.

Note. We have obtained the last theorem as a consequence of theorem 2.3. For another approach cf. ([10], Theorem 2, [11], Theorem 2). A similar theorem in case of any locally compact abelian group is proved in ([7], Theorem 1).

Corollary. *Let X be a semi-reflexive locally convex space. The elements of the two-way sequence (x_j) in X are the Fourier—Stieltjes coefficients of some regular vector measure $m:\mathcal{B}(\mathbf{T})\to X$, $\|m\|_q(\mathbf{T})\leqq C_q$, $q\in\mathcal{Q}$ if and only if*

$$\|S_N(X)\|_q\leqq C_q,\quad N=1,2,\ldots,\quad q\in\mathcal{Q}.$$

Proof. A locally convex space X is semi-reflexive if and only if every bounded subset of X is weakly relatively compact ([5], 8.4.2, [8], §23, 3(1)). Every semi-reflexive space is quasi-complete ([8], §23, 3(2), [13], IV. 5.5). Now it suffices to use theorem 2.5.

The corollary is applicable, for example, to all quasi-complete nuclear spaces ([13], IV. 5. 5).

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КОЭФФИЦИЕНТЫ ФУРЬЕ НЕПРЕРЫВНЫХ ЛИНЕЙНЫХ ОТОБРАЖЕНИЙ НА ОДНОРОДНЫХ ПРОСТРАНСТВАХ БАНАХА

Милослав Духонь

Резюме

Пусть **T** — одномерный тор. Пусть $H(\mathbf{T})$ — однородное пространство Банаха, т.е. подпространство пространства Банаха $L^1(\mathbf{T})$ всех комплексных интегрируемых по Лебегу функций определенных на T, имеющее норму $\| \, \|_H \geqq \| \, \|_1$, со свойствами инвариантности при сдвиге и непрерывности сдвига.

Пусть X – квазиполное локально выпуклое топологическое векторное пространство и $u: H(\mathbf{T}) \to X$ — непрерывное линейное отображение. Коэффициентами Фурье отображения u называются элементы X вида $\hat{u}(j) = u\,(e^{-ijt})$, j — целое число. В работе доказываются результаты следующего типа.

Пусть (x_j) — бесконечная в обе стороны последовательность элементов пространства X. Элементы x_j являются коэффициентами Фурье некоторого непрерывного (слабо компактного или компактного) линейного отображения $u: H(\mathbf{T}) \to X$, $\|u\|_q \leqq C_q$, $q \in \mathcal{Q}$, тогда и только тогда, когда

$$\|S_N(X)\|_q \leqq C_q, \quad q \in \mathcal{Q}, \quad N = 1, 2, \ldots$$

(и существует слабо компактное или компактное подмножество W в X такое, что

$$S_N(X)(f) \in W, \quad N = 1, 2, \ldots, \quad f \in H(\mathbf{T}), \quad \|f\|_H \leqq 1).$$