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ISOMORPHIC FACTORISATIONS OF COMPLETE GRAPHS INTO FACTORS WITH A GIVEN DIAMETER

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1. Introduction

F. Harary, R. W. Robinson and N. C. Wormald have proved that the complete graph K_n is decomposable into m isomorphic factors if and only if m divides $n(n-1)/2$. See [5, Divisibility Theorem]. The papers [6], [8] and [10] deal with the same problem, but the factors are required to have a prescribed diameter d. Just this additional requirement is of interest to us. Using the results in [2], [5] and [7] we give the answer for $d = 2$ and m sufficiently large and for $3 \leqq d \leqq 2m - 1$, $m \geqq 3$, too.

We give some definitions, remarks and previous results. Let G be a graph and $V(G)$ ($E(G)$) its vertex (edge) set. The subgraph F of G is called a factor of G if $V(F) = V(G)$. The system $\{F_1, F_2, \ldots, F_m\}$ of factors of G forms a factorisation (a decomposition into factors) if

$$\cup E(F_i) = E(G) \text{ and } E(F_i) \cap E(F_j) = \emptyset \text{ for } i \neq j.$$

J. Bosák, A. Rosa and Š. Znám [4] initiated the studies of decompositions of complete graphs into factors with given diameters. Many papers deal with the problem of [4] or with various modifications of this one. The papers [1], [2], [3], [7], [9] and [12] are devoted to the case when all factors have the same diameter. Note that the isomorphism of factors is not required. It is convenient (cf. [4]) to denote by $F_m(d)$ the smallest integer n such that the complete graph K_n can be decomposed into m factors with diameter d; if such an integer does not exist, then we put $F_m(d) = \infty$. The significance of the function $F_m(d)$ resides in the validity of the following assertion (proved in [4]): K_n is decomposable into m factors with equal diameter d if and only if $n \geqq F_m(d)$.

An interesting case is $d = 2$. The results $F_2(2) = 5$ and $F_3(2) = 12$ or 13 were proved already in [4]. In [3], [9], [2] and [1] there were found lower and upper bounds for $F_m(2)$ if $m \geqq 4$. The best upper bound was stated by J. Bosák who in [2] proved that for every integer $m \geqq 2$ there holds $F_m(2) \leqq 6m$. Let us remark that the factors of his construction of the decomposition of K_{6m} are isomorphic. B. Bollobás in [1] proved that for $m \geqq 6$ we have

$$F_m(2) \geqq 6m - 9. \tag{1}$$

A significant reslut on $F_m(2)$ was achieved by Š. Znám (see [12]) who proved that $F_m(2) = 6m$ if m is sufficiently large ($m > 10^{17}$).

The second author of the present paper proved in [7] that $F_m(d) = 2m$ for $m \geqq 3$ and $3 \leqq d \leqq 2m - 1$. Proving this assertion a construction was used in which the factors are isomorphic for d odd. This fact was noticed by P. Tomasta who began to study the problem of decompositions of complete graphs and hypergraphs into isomorphic factors with a given diameter systematically (see [10] and [11]). Indepedently, the same problem was studied by the authors of [6]. Clearly, K_n can be decomposed into m isomorphic factors only if $n(n-1)/2$ is divisible by m. (In this case we shall say that n is admissible with respect to m.) As we noted above, in [5] it was proved that this necessary condition is also sufficient. Denote by $G_m(d)$ the smallest integer n such that the complete graph $K_n (n > 1)$ has an isomorphic factorisation into factors of diameter d; if such an integer does not exist, then put $G_m(d) = \infty$. Because an isomorphic factorisation of K_n does not exist for an integer $N > G_m(d)$, which is not admissible with respect to m, it is convenient to define a function $H_m(d)$ to be the smallest admissible integer n such that for all admissible $N \geqq n$ the complete graph K_N has an isomorphic factorisation into factors of diameter d. If such an integer does not exist, put $H_m(d) = \infty$. It is obvious that

$$F_m(d) \leqq G_m(d) \leqq H_m(d). \tag{2}$$

In [6] it is conjectured that

$$G_m(d) = H_m(d) \tag{3}$$

for any $m \geqq 2$ and $d \geqq 2$. (The cases $m = 1$ and $d = 1$ are trivial.) Clearly, if we find the value $H_m(d)$ and prove the conjecture (3), then the problem of decomposition of K_n into m isomorphic factors of diameter d will be solved (cf. [6]). The truth of the conjecture (3) has been verified in some special cases. Namely, in [6] for $m = 2$ and any d, and for $m = 3$ if $d = 3, 4, 5, 6$. For $d = \infty$ there has been proved that if m is a power of an odd prime, then $G_m(\infty) = H_m(\infty) = m$. This result was improved in [8], where the following assertion was proved: Let $m > 1$ be an integer. Let $r > 1$ be the smallest integer which satisfies the congruence $n(n = 1) \equiv 0 \pmod{m}$ if m is odd or the congruence $n(n-1) \equiv 0 \pmod{2m}$ if m is even. Then $G_m(\infty) = H_m(\infty) = r$.

Note that the assertion solves the problem of the existence of an isomorphic factorisation of K_m into factors of diameter ∞ completely.

2. Results

Theorem 1. *Let $m \geqq 3$ be an integer. Then*

$$G_m(2) \leqq H_m(2) \leqq 6m\,,$$
$$G_m(2) = H_m(2) = 6m \text{ if } m \geqq 46\,.$$

Proof. Let $n \geqq 6m$ be any integer which is admissible with respect to m. To prove the first assertion, it is sufficient to prove that there exists an isomorphic factorisation of K_n into factors of diameter two. If $n = 6m$, we use Bosák's construction from [2]. As we note above, all factors of this construction are isomorphic. Thus, we can suppose $n > 6m$. We choose an arbitrary complete subgraph of K_n with $6m$ vertices and denote it by K_{6m}. The complete subgraph of K_n generated by the set $V(K_n) - V(K_{6m})$ will be denoted by K_{n-6m}. It is easy to see that also $n - 6m$ is admissible with respect to m. Hence, according to the Divisibility Theorem (see [5]) there exists an isomorphic factorisation of K_{n-6m} into m factors. We denote them by $G_1, G_2, \ldots, G_m$. Now, we use a simple extension of Bosák's construction (cf. [2, p. 60]). We define the sets:

$$A_1 = B_2 = \{1,3,4\}\,,\quad A_2 = B_1 = \{2,3,4\}\,,\quad A_3 = B_4 = \{3,5,6\}\,,$$
$$A_4 = B_3 = \{4,5,6\}\,,\quad A_5 = B_6 = \{5,1,2\}\,,\quad A_6 = B_5 = \{6,1,2\}\,.$$

The vertices of K_{6m} will be denoted by $a_{i,s}$, where $1 \leqq i \leqq m$, $1 \leqq s \leqq 6$ and the vertices of K_{n-6m} by $v_1, v_2, \ldots, v_{n-6m}$. We decompose K_n into factors $F_i (i = 1, 2, \ldots, m)$ as follows: Factor F_i contains the edges $a_{i,s}a_{i,t}$, where $1 \leqq s < t \leqq 6$, the edges $a_{i,s}a_{j,t}$, where $1 \leqq s \leqq 6$, $i < j \leqq m$, $t \in A_s$, the edges $a_{i,s}a_{j,t}$, where $1 \leqq s \leqq 6$. $1 \leqq j < i$, $t \in B_s$, the edges of G_i and the edges $a_{i,s}v_k$, where $1 \leqq s \leqq 6$, $1 \leqq k \leqq n - 6m$.

It is easy to check that the factors F_i form an isomorphic factorisation of K_n into factors of diameter two. The proof of the inequality $H_m(2) \leqq 6m$ is finished.

To prove the second assertion of the theorem we suppose that $G_m(2) = 6m - x$ for $m \geqq 46$, where x is a positive integer. Because $6m - x$ is admissible with respect to m, we have $(6m - x)(6m - x - 1)/2 = my$, where y is an integer. Therefore (as we can easily verify) $x^2 + x = 2mz$, where z is a positive integer. From this we have $x = (\sqrt{1 + 8mz} - 1)/2$. According to the inequality (1) which holds for $m \geqq 6$, we can write $6m - 9 \leqq F_m(2) \leqq G_m(2) = 6m - x$ i.e. $x \leqq 9$ which implies $m \leqq 45$, a contradiction. Thus $G_m(2) = H_m(2) = 6m$ for $m \geqq 46$.*)

*) R. Neděla has recently proved (oral communication) that $F_m(2) \geqq 6m - 6$ if $m \geqq 22$. Using this result it can be proved that the equality holds already for $m \geqq 22$.

Remark 1. It is easy to check (examining the equality $x^2 + x = 2mz$, where $1 \leqq x \leqq 9$) that the equality $H_m(2) = 6m$ holds also for $3 \leqq m \leqq 45$ if $m \neq 3, 4, 5, 6, 7, 9, 10, 12, 14, 15, 18, 21, 28, 36, 45$. In particular $H_8(2) = 48$. It is known that $H_3(2) \leqq 13 < 18 = 6 \cdot 3$ (see [6]). From this the following problem arises.

Problem 1. *Which is the smallest integer m for which $H_m(2) = 6m$?*
We can see that such m is equal to one of the numbers 4, 5, 6, 7, 8.

Theorem 2. *Let m, d be integers such that $m \geqq 3$ and $3 \leqq d \leqq 2m - 1$. We have*

(*i*) $2m = F_m(d) \leqq G_m(d) = H_m(d) \leqq 2m + 1$.

Moreover,

(*ii*) $F_m(d) = G_m(d) = H_m(d) = 2m$ *if at least one of m, d is odd.*
(*iii*) $F_m(4) = G_m(4) = H_m(4) = 2m$,
(*iv*) $F_4(6) = G_4(6) = H_4(6) = 8$.

Proof. (i) The equality $F_m(d) = 2m$ (proved in [7]) together with the condition (2) imply $2m = F_m(d) \leqq G_m(d) \leqq H_m(d)$. Let m, d be integers under the conditions of the theorem and $n \geqq 2m + 1$ be an admissible integer with respect to m. To show $H_m(d) \leqq 2m + 1$ it is sufficient to decompose K_n into m isomorphic factors of diameter d. We choose an arbitrary complete subgraph of K_n having $2m$ vertices and denote it by K_{2m}. The vertices of K_{2m} will be denoted by $v_1, v_2, \ldots, v_{2m}$. For $j > 2m$ we define $v_j = v_s$ with $s \equiv j \pmod{2m}$, where $1 \leqq s \leqq 2m$. The complete subgraph generated by the set of the remaining vertices will be denoted by K_{n-2m} and its vertices by $u_1, u_2, \ldots, u_{n-2m}$. Clearly, $n - 2m$ is also admissible with respect to m. Thus, according to the Divisibility Theorem (see [5]), there exists an isomorphic factorisation of K_{n-2m} into m factors (we denote them $G_1, G_2, \ldots, G_m$). To decompose K_n we use a certain extension of the construction from [7]. Let us consider two cases:

(I) The diameter d is odd, i.e. $d = 2k - 1$. We decompose K_n into isomorphic factors $F_i (i = 1, 2, \ldots, m)$ as follows:

$$E(F_i) = E(G_i) \cup A_i \cup B_i,$$

where the set A_i is formed by the edges $v_i u_j$ and $v_{i+m} u_j$, where $1 \leqq j \leqq n - 2m$. For the set B_i we have two posibilities:

(a) If k is odd, we consider the path

(4) $$v_i v_{i+1} v_{i-1} v_{i+2} v_{i-2} v_{i+3} \ldots v_{i-(k-3)/2} v_{i+(k-1)/2}$$

and the path

(5) $$v_{i+m} v_{i+m+1} v_{i+m-1} v_{i+m+2} v_{i+m-2} v_{i+m+3} \ldots v_{i+m-(k-3)/2} v_{i+m+(k-1)/2} .$$

The set B_i consists of the edges of the paths (4) and (5), of the edge $v_i v_{i+m}$, of the edges $v_{i+(k-1)/2}v_s$, where $s = i-(k-1)/2,\ i-(k+1)/2,\ i-(k+3)/2, \ldots$ $\ldots, i-(2m-k-1)/2$, and of the edges $v_{i+m+(k-1)/2}v_t$, where $t = i+(k+1)/2$, $i+(k+3)/2,\ i+(k+5)/2, \ldots, i+(2m-k-1)/2$.

(b) If k is even, we consider the path

(6) $$v_i v_{i+1} v_{i-1} v_{i+2} v_{i-2} \ldots v_{i+(k-2)/2} v_{i-(k-2)/2}$$

and the path

(7) $$v_{i+m} v_{i+m+1} v_{i+m-1} v_{i+m+2} v_{i+m-2} v_{i+m+3} \ldots v_{i+m+(k-2)/2} v_{i+m-(k-2)/2}.$$

The set B_i consists of the edges of the paths (6) and (7), of the edge $v_i v_{i+m}$, of the edges $v_{i-(k-2)/2}v_s$, where $s = i+k/2,\ i+(k+2)/2,\ i+(k+4)/2, \ldots, i+$ $+(2m-k)/2$, and of the edges $v_{i+m-(k-2)/2}v_t$, where $t = i+m+k/2$, $i+m+(k+2)/2,\ i+m+(k+4)/2, \ldots, i+m+(2m-k)/2$.

(II) The diameter $d\,(4 \leqq d \leqq 2m-2)$ is even. In order to define the factor F_i^* of a decompositon of K_n into m isomorphic factors of diameter $2k-2\,(k \geqq 3)$ we take the factor F_i of an isomorphic factorisation of K_n into factors of diameter $2k-1$ defined in (I) and replace the set A_i by the set A_i^* which is formed by the edges $v_{i+1}u_j$ und $v_{i+m+1}u_j$, where $1 \leqq j \leqq n-2m$.

In all these cases we can check that the factors of the systems $\{F_1, F_2, \ldots, F_m\}$ or $\{F_1^*, F_2^*, \ldots, F_m^*\}$, respectively, form an isomorphic factorisation of K_n and that all the factors have the required diameter. For instance in the case (I) (a) the distance $d = 2k-1$ in the factor F_i is realized by an arbitrary path of the greatest length (having $2k$ vertices) of the tree generated by the set B_i. See Fig. 1, where the factor F_i (without edges of G_i) is drawn.

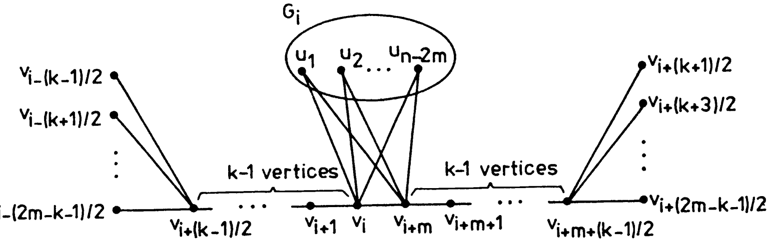

Fig. 1.

(ii) To prove (ii) we must show that K_{2m} is decomposable into m isomorphic factors if at least one of m, d is odd. If d is odd, then we use the construction of [7], i.e. the factors $F_i (i = 1, 2, \ldots, m)$ are defined by $E(F_i) = B_i$, where B_i is the set defined in (I). It remains to decompose K_{2m} into m isomorphic factors of diameter d if $m \leqq 3$ is odd and d is even. First we suppose $d \geqq 6$. We decompose K_{2m} into m isomorphic factors $F_1^*, F_2^*, \ldots, F_m^*$ of diameter d as follows. In the case $d = 2k - 2$, where k is odd, we take the set B_i from (I) (a). If $i = 1, 3, \ldots, m$, we add to it the edge $v_{i+(k-1)/2}v_{i+(k+1)/2}$ and remove from it the edge v_iv_{i+1}. If $i =$ - $= 2, 4, \ldots, m - 1$, we add to it the edge $v_{i-(2m-k-1)/2}v_{i+m+(k-1)/2}$ and remove from it the edge $v_{i+m}v_{i+m+1}$. In the case $d = 2k - 2$, where k is even, we take the set B_i from (I) (b). If $i = 1, 3, \ldots, m$, we add to it the edge $v_{i-(k-2)/2}v_{i+m+(2m-k)/2}$ and remove from it the edge $v_{i+m}v_{i+m+1}$. If $i = 2, 4, \ldots, m - 1$, we add to it the edge $v_{i+(2m-k)/2}v_{i+m-(k-2)/2}$ and remove from it the edge v_iv_{i+1}. In both cases we get from the set B_i a set B_i'. Put $E(F_i') = B_i'$ for $i = 1, 2, \ldots, m$. It is easy to verify that the system F_i' forms an isomorphic factorisation of K_{2m} into m factors of diameter d. It remains to decompose K_{2m} into m isomorphic factors of diameter four. We shall do it in

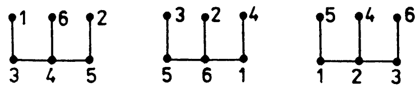

Fig. 2.

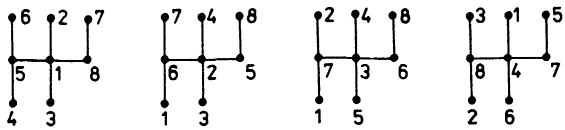

Fig. 3.

(iii). An isomorphic factorisation of K_6 (K_8) into 3 (4) factors of diameter four can be seen in Fig. 2 (Fig. 3). Hence, we may suppose $m \geqq 5$. As above, we denote the vertices of K_{2m} by $v_1, v_2, \ldots, v_{2m}$. We decompose K_{2m} as follows. The factor F_1 contains the edge v_1v_{m+1}, the edges v_1v_s, where $2 \leqq s \leqq m - 1$, the edge v_mv_{2m-1}, the edges $v_{m+1}v_t$, where $m + 2 \leqq t \leqq 2m$. The factor F_2 contains the edge v_2v_{m+2}, the edges v_2v_s, where $3 \leqq s \leqq m + 1$, the edge v_1v_m and the edges $v_{m+2}v_t$, where $m + 3 \leqq t \leqq 2m$. The factor $F_i (i = 3, 4, \ldots, m - 1)$ containts the edge v_iv_{i+m}, the edges v_iv_r, where $i + 1 \leqq r \leqq i + m - 1$, the edge $v_{i+m-1}v_1$, the edges $v_{i+m}v_s$, where $i + m + 1 \leqq s \leqq 2m$ and the edges $v_{i+m}v_t$, where $2 \leqq t \leqq i - 1$. The factor F_m contains the edge v_mv_{2m}, the edges v_mv_s, where $m + 1 \leqq s \leqq 2m - 2$, the edge $v_{2m-1}v_1$ and the edges $v_{2m}v_t$, where $1 \leqq t \leqq m - 1$. One can verify that F_i form a desired factorisation.

(iv) To prove $F_4(6) = 8$ we decompose K_8 into 4 isomorphic factors of diameter 6, which is done in Fig. 4.

Remark 2. From Theorem 2 it follows that the problem of decomposition of K_n into m isomorphic factors of diameter d is completely solved for $m \geqq 3$ and $3 \leqq d \leqq 2m - 1$ with the only exception when both m, d ($\geqq 6$) are even. In this case the value of $H_m(d)$ is equal to 2m or $2m + 1$. To obtain the exact value of $H_m(d)$ it is necessary to solve the following

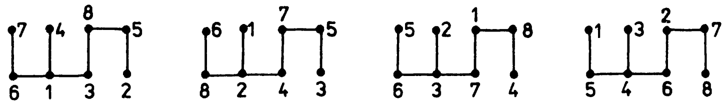

Fig. 4.

Problem 2. *Let m, d be even integers such that* $m \geqq 6$ *and* $6 \leqq d \leqq 2m - 2$. *Is it possible to decompose* K_{2m} *into m isomorphic factors of diameter d*?

Remark 3. It would by very interesting to find exact values or at least lower and upper bounds of $H_m(d)$ if $m \geqq 3$ and $d \geqq 2m$. The only known value is $H_3(6) = 9$ (see [4] and [6]). If m is a power of an odd prime, then $H_m(d) \leqq$-$\leqq md - 2m$ for $d \geqq 5$. These upper bounds were found in [10]. From results of [7] it follows that $H_m(2m) \geqq 2m + 3$.

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ИЗОМОРФНАЯ ФАКТОРИЗАЦИЯ ПОЛНЫХ ГРАФОВ НА ФАКТОРЫ С ДАННЫМ ДИАМЕТРОМ

Pavol Híc — Daniel Palumbíny

Резюме

В статье рассматривается вопрос разложения полного графа на m изоморфных факторов с данным диаметром d. Задача полностью решена для случаев: 1. $d = 2$ если $m \geqq 46$, 2. $m \geqq 3$, если $3 \leqq d \leqq 2m -$ и, по крайней мере, одно из чисел m, d нечетное.