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ON THE CROSSING NUMBERS OF CARTESIAN PRODUCTS OF STARS AND PATHS OR CYCLES

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ABSTRACT. The main results of this paper are that the crossing number of the Cartesian product $S_4 \times P_n$ is $2(n-1)$ for $n \geqq 1$ and that of the Cartesian product $S_4 \times C_n$ is $2n$ for $n \geqq 6$. Besides, in addition are given the crossing numbers of $S_4 \times C_3$, $S_4 \times C_4$ and $S_4 \times C_5$.

Ringeisen and Beineke [6], [7] determined the crossing numbers of the Cartesian products $C_3 \times C_n$, $C_4 \times C_n$ and $K_4 \times C_n$. Jendroľ and Ščerbová [3] found an upper bound for $\nu(S_m \times P_n)$ and for $\nu(S_m \times C_n)$ and the crossing numbers of graphs $S_3 \times P_n$ and $S_3 \times C_n$. In this paper we improve the upper bound for $\nu(S_m \times C_n)$ and we find the crossing numbers of graphs $S_4 \times P_n$ and $S_4 \times C_n$.

Preliminaries

Let G be a simple graph with the vertex set V and the edge set E. The *crossing number* $\nu(G)$ of a graph G is the minimum number of "crossings" in any "good" drawing of G in the plane. By a *drawing of* G in the plane Π we mean a collection of points P in Π and open arcs A in $\Pi - P$ for which there are correspondences between the vertices of G and P and between the edges of G and A such that the vertices of an edge correspond to the end-points of the open arcs. The drawing is called *good* if for all arcs in A, no two with a common end-point meet, no two meet in more than one point, and no three have a common point. A *crossing* in a good drawing is a point of intersection of two arcs in A. For a detailed account of problems and results concerning this topic, the reader is referred to Erdös and Guy [1], Harary [2] or Koman [5].

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Let C_n be the *cycle*, S_m the *star* $K_{1,m}$ and P_n the *path* of length n. For a definition of the Cartesian product see [2]. Let the vertex of degree m of S_m be denoted by label 0 and the other vertices of S_m having degree 1 by labels 1, 2, ..., m. Let the vertices of the path P_n be labelled successively by 0, 1, ..., n so that the end vertices have labels 0 and n, respectively; the vertex i is adjacent to the vertices $i-1$ and $i+1$ for all i, $i = 1, 2, \ldots, n-1$. The vertices of the cycle C_n are analogously denoted by $0, 1, \ldots, n-1$. The Cartesian product $S_m \times P_n$ has $(m+1)(n+1)$ vertices (i,j) for $i = 0, 1, \ldots, m$ and $j = 0, 1, \ldots, n$. In $S_m \times P_n$ there are adjacent pairs of vertices $(0,j)$ and (i,j) for $i = 1, 2, \ldots, m$, $j = 0, 1, \ldots, n$; (i,j) and $(i,j+1)$ for $i = 0, 1, \ldots, m$ and $j = 0, 1, \ldots, n-1$. In $S_m \times C_n$ containing $n(m+1)$ vertices (i,j) for $i = 0, 1, \ldots, m$, $j = 0, 1, \ldots, n-1$, there are adjacent pairs of vertices $(0,j)$ and (i,j) for $i = 1, 2, \ldots, m$, $j = 0, 1, \ldots, n-1$ and the pairs (i,j), $(i,j+1)$ for $i = 0, 1, \ldots, m$ and $j = 0, 1, \ldots, n-1$. (The second coordinates are taken modulo n.)

The graph $\overline{S_m \times P_n}$ is obtained from the graph $S_m \times P_n$ by the removal of edges $(0,j)(0,j+1)$, $j = 0, 1, \ldots, n-1$ and the graph $\overline{S_m \times C_n}$ from the graph $S_m \times C_n$ by the removal of edges $(0,j)(0,j+1)$, $j = 0, 1, \ldots, n-1$. (The coordinates are taken modulo n.)

Results

Theorem 1. *If* $m \geqq 1$, *then* $\nu(S_m \times C_n) \leqq K\left[\frac{m}{2}\right]\left[\frac{m-1}{2}\right]$, *where* $K = n-2$ *for* $n = 3$ *or* 4, $K = 4$ *for* $n = 5$ *and* $K = n$ *for* $n \geqq 6$.

Proof. The case for $n \geqq 6$ is proved in [3]. Let C^i, $i \in \{0, 1, \ldots, m\}$ denote the induced subgraph of $S_m \times C_n$ having the vertices (i,j) for $j = 0, 1, \ldots, n-1$. Let T^i, $i \in \{1, 2, \ldots, m\}$ be a subgraph of the graph $S_m \times C_n$ with the vertices $(0,j)$ and (i,j) for $j = 0, 1, \ldots, n-1$ and the edges $(0,j)(i,j)$ for $j = 0, 1, \ldots, n-1$ and $(i,j)(i,j+1)$ for $j = 0, 1, \ldots, n-1$. (The coordinates are taken modulo n.) In Figure 1(a) there is shown the drawing of $S_m \times C_n$ so that C^i is on the left from the line of vertices of C^0 for even i and on the right for odd i and the edges of C^0 have no crossing. The edges of two subgraphs T^i and T^{i-2k} for $i \in \{3, 4, \ldots, m\}$ and $k \in \left\{1, 2, \ldots, \left[\frac{i-1}{2}\right]\right\}$ cross each other in one point for $n = 3$ (Fig. 1(b)), in two points for $n = 4$ (Fig. 1(c)) and in four points for $n = 5$ (Fig. 1(d)). Edges of every T^i for $i = 3, 4, \ldots, m$ cross the edges of exactly $\left[\frac{i-1}{2}\right]$

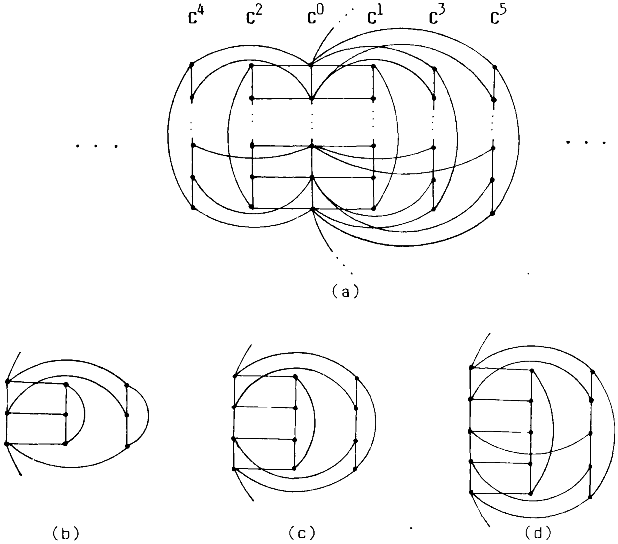


Fig. 1

subgraphs T^{i-2k} for $k = 1, 2, \ldots, \left[\frac{i-1}{2}\right]$. Since $\sum_{i=3}^{m}\left[\frac{i-1}{2}\right] = \left[\frac{m}{2}\right]\left[\frac{m-1}{2}\right]$, we obtain:

(i) $\nu(S_m \times C_3) \leqq \left[\frac{m}{2}\right]\left[\frac{m-1}{2}\right]$,

(ii) $\nu(S_m \times C_4) \leqq 2\left[\frac{m}{2}\right]\left[\frac{m-1}{2}\right]$,

(iii) $\nu(S_m \times C_5) \leqq 4\left[\frac{m}{2}\right]\left[\frac{m-1}{2}\right]$.

In the remainder of this paper we determine the precise values of the crossing numbers of the graphs $S_4 \times P_n$ and $S_4 \times C_n$. For this purpose let a_i, b_i, c_i, d_i and e_i denote the vertices $(0, i)$, $(1, i)$, $(2, i)$, $(3, i)$ and $(4, i)$, respectively, in the graphs $S_4 \times P_n$, $S_4 \times C_n$, $\overline{S_4 \times P_n}$ and $\overline{S_4 \times C_n}$. Let S^i denote the induced subgraph of $S_4 \times P_n$ ($S_4 \times C_n$, $\overline{S_4 \times P_n}$, $\overline{S_4 \times C_n}$) having vertices a_i, b_i, c_i, d_i and e_i. Let us remark that S^i is isomorphic to S_4. Let $H^{i,k}$ be a subgraph of $\overline{S_4 \times P_n}$ ($\overline{S_4 \times C_n}$) induced by the vertices of the stars S^i, S^{i+1}, ..., S^k for $0 \leqq i < k \leqq n$. The subgraph $H^{i,k} - S^i$ is obtained by the removal of all edges of the star S^i from the graph $H^{i,k}$. The cycle induced by the vertices $a_0, a_1, \ldots, a_{n-1}$ of the graph $S_4 \times C_n$ is called the *a-cycle*. In the same way are defined the *b-cycle*, the *c-cycle*, the *d-cycle* and the *e-cycle*.

Lemma 1. *If D is a good drawing of* $\overline{S_4 \times P_n}$, $n \geqq 2$, *in which every star* S^i, $i = 0, 1, \ldots, n$, *has at most one crossing, then D has at least* $2(n-1)$ *crossings.*

Lemma 2. *If D is a good drawing of* $\overline{S_4 \times C_n}$ $(S_4 \times C_n)$, $n \geqq 4$, *in which every star* S^i, $i = 0, 1, \ldots, n-1$, *has at most one crossing, then D has at least* $2n$ *crossings.*

Proof of Lemma 1. We show that in every drawing $D^{i, i+2}$ of $H^{i, i+2}$ induced by D, $i = 0, 1, \ldots, n-2$, there are at least two crossings and in every drawing $D^{i, i+k}$ of $H^{i, i+k}$ induced by D, $i = 0, 1, \ldots, n-k$, there are at least two more crossings than the minimum of crossings in the drawing $D^{i, i+k-1}$ induced by $D^{i, i+k}$.

If we say in the following that the drawing $D^{i, k}$ induces a map in the plane, we mean the crossings of the edges as new vertices.

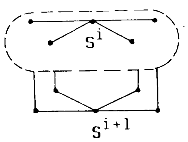
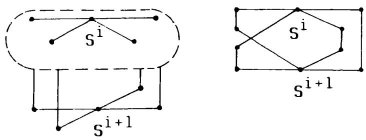


Fig. 2



(a) (b)

Fig. 3

Consider the drawing $D^{i, i+1}$ of $H^{i, i+1}$ induced by D. If in the drawing $D^{i, i+1}$ the edges of S^{i+1} have no crossing, then $D^{i, i+1}$ induces the map in the plane with at most two vertices b_{i+1}, c_{i+1}, d_{i+1} and e_{i+1} (the end vertices of S^{i+1}) on the boundary of every region (Fig. 2). If this $D^{i, i+1}$ is induced by $D^{i, i+2}$ of $H^{i, i+2}$, then the edges of $H^{i+1, i+2} - S^{i+1}$ in $D^{i, i+2}$ have at least two crossings. If in the drawing $D^{i, i+1}$ one edge of S^{i+1} is crossed, then $D^{i, i+1}$ induces the map with at most three end vertices of S^{i+1} on the boundary of every region (Fig. 3). In this case the edges of $H^{i+1, i+2} - S^{i+1}$ in $D^{i, i+2}$ have still at least one crossing.

Consider now the drawing $D^{i, i+k}$ of $H^{i, i+k}$ induced by D. If in the drawing $D^{i, i+k-1}$ induced by $D^{i, i+k}$ the edges of S^{i+k-1} cross no nonstar edge of $H^{i+k-2, i+k-1}$, then $D^{i, i+k-1}$ induces the map with at most two end vertices of S^{i+k-1} on the boundary of every region. In this case the edges of $H^{i+k-1, i+k} - S^{i+k-1}$ in $D^{i, i+k}$ have at least two crossings. If in the drawing $D^{i, i+k-1}$ the edge of S^{i+k-1} crosses any nonstar edge of $H^{i+k-2, i+k-1}$, then $D^{i, i+k-1}$ induces the map with at most three end vertices of S^{i+k-1} on the boundary of every region. In the drawing $D^{i, i+k-1}$ there is at least one more crossing than the minimum of crossings in the drawing $D^{i, i+k-1}$ from which it can be seen that in the drawing $D^{i, i+k}$ there are at least two more crossings than the minimum of crossings in the drawing $D^{i, i+k-1}$.

Since $D^{0,2}$ induced by D has at least two crossings and in the drawing $D = D^{0,n}$ there are still $n-2$ stars, the drawing D has at least $2 + 2(n-2) = 2(n-1)$ crossings.

Proof of Lemma 2. The graph $\overline{S_4 \times C_n}$ contains the subraph $\overline{S_4 \times P_{n-1}}$. Let $F^{n-1,0}$ denote the edges $b_{n-1}b_0$, $c_{n-1}c_0$, $d_{n-1}d_0$ and $e_{n-1}e_0$ of $\overline{S_4 \times C_n}$. Consider now a good drawing D of $\overline{S_4 \times C_n}$. Let D^* be a good drawing of $\overline{S_4 \times P_{n-1}}$ induced by D. (D^* is ibtained from D by the removal of the edges $F^{n-1,0}$.)

The rest of the proof is based on the properties of the subgraphs $H^{0,1} - S^1$ and $H^{n-2,n-1} - S^{n-2}$ of the graph $\overline{S_4 \times P_{n-1}}$ which are the same as that of the subgraph $H^{i+k-1,i+k} - S^{i+k-1}$ described in the proof of Lemma 1. Therefore if in the map in the plane induced by D^* the stars S^0 and S^{n-1} have not the end vertices on the boundary of the same region, then the edges $F^{n-1,0}$ in D have at least four crossings. If there is a region with the end vertices of the stars S^0 and S^{n-1} on its boundary, then there are two end vertices for every of the stars S^0 and S^{n-1} on this boundary or two end vertices of S^0 (S^{n-1}) and three end vertices of S^{n-1} (S^0) or three end vertices for every of the stars S^0 and S^{n-1} on the boundary of this region. In the first case D^* has at least $2(n-2)$, in the second case at least $2(n-2)+1$ and in the third case at least $2(n-2)+2$ crossings. In all three cases the regions with the rest end vertices of S^0 on their boundaries cannot adjoin with the regions on whose boundaries there are the rest end vertices of S^{n-1}. From this it can be seen that D has at least $2n$ crossings.

If in the map induced by D^* there are more regions with the end vertices of the stars S^0 and S^{n-1} on their boundaries, then D^* has at least $2(n-2)+2$ crossings and the edges $F^{n-1,0}$ in D have at least two crossings. This completes the proof.

Theorem 2. $\nu(\overline{S_4 \times P_n}) = 2(n-1)$ *for* $n \geqq 1$.

Proof. Jendroľ and Ščerbová [3] proved that $\nu(S_4 \times P_n) \leqq 2(n-1)$ and therefore $\nu(\overline{S_4 \times P_n}) \leqq 2(n-1)$, too. By induction of n it is shown that $\nu(\overline{S_4 \times P_n}) \geqq 2(n-1)$. The case $n = 1$ is trivial. The inequality in the case $n = 2$ follows from the fact that $\overline{S_4 \times P_2}$ is homeomorphic to $K_{3,4}$ and $\nu(K_{3,4}) = 2$ (see [4]).

Assume that the result is valied for $n = k$, $k \geqq 1$. Let D be a good drawing of $\overline{S_4 \times P_{k+1}}$ with less than $2k$ crossings. By Lemma 1 in D there exists a star S^i with at least two crossings. By the removal of all edges of this star we obtain a graph homeomorphic to $\overline{S_4 \times P_k}$ with less than $2(k-1)$ crossings. This contradicts the induction hypothesis.

Theorem 3. $\nu(S_4 \times P_n) = 2(n-1)$ *for* $n \geqq 1$.

Proof. It is clear that $\nu(S_4 \times P_n) \geqq 2(n-1)$ because $\overline{S_4 \times P_n}$ is a subgraph of $S_4 \times P_n$. From [3] we konw that $\nu(S_4 \times P_n) \leqq 2(n-1)$.

Lemma 3.

$$\nu(\overline{S_4\times C_3}) = 2,$$
$$\nu(\overline{S_4\times C_4}) = 4,$$
$$7 \leqq \nu(\overline{S_4\times C_5}) \leqq 8,$$
$$9 \leqq \nu(\overline{S_4\times C_6}) \leqq 12.$$

Proof. By Theorem 1 we have $\nu(S_4\times C_3) \leqq 2$, $\nu(S_4\times C_4) \leqq 4$, $\nu(S_4\times C_5) \leqq$ $\leqq 8$ and $\nu(S_4\times C_6) \leqq 12$ and by Theorem 2 we have $\nu(\overline{S_4\times P_2}) = 2$, $\nu(\overline{S_4\times P_3}) =$ $= 4$, $\nu(\overline{S_4\times P_4}) = 6$ and $\nu(\overline{S_4\times P_5}) = 8$. As $\overline{S_4\times P_n}$ is a subgraph of $\overline{S_4\times C_{n+1}}$ and $\overline{S_4\times C_n}$ is a subgraph of $S_4\times C_n$, it is easy to see that $\nu(\overline{S_4\times C_3}) = 2$, $\nu(\overline{S_4\times C_4}) =$ $= 4$, $6 < \nu(\overline{S_4\times C_5}) \leqq 8$ and $8 \leqq \nu(\overline{S_4\times C_6}) \leqq 12$.

We assume that D is a good drawing of $\overline{S_4\times C_5}$ with six crossings. The drawing D has the following properties:

Property (1a). None of the edges b_ib_{i+1}, c_ic_{i+1}, d_id_{i+1}, e_ie_{i+1} for $i = 0, 1, 2, 3, 4$ (The indices are taken modulo 5.) is crossed. (In the opposite case we remove these edges from $\overline{S_4\times C_5}$ and obtain a good drawing of $\overline{S_4\times P_4}$ with at most five crossings.)

Property (1b). For all j, $j = 0, 1, 2, 3, 4$, the star S^j has at most two crossings. (In the opposite case we can obtain a good drawing of the graph homeomorphic to $\overline{S_4\times C_4}$ with at most three crossings.)

In the drawing D no x-cycle has a crossing whenever $x\in\{b, c, d, e\}$ and no two edges of one star S^j cross each other. In the graph $\overline{S_4\times C_5}$ there are five stars S^j, $j = 0, 1, 2, 3, 4$ and from the properties (1a) and (1b) it follows that the drawing D of $\overline{S_4\times C_5}$ has at most five crossings. This contradicts the hypothesis $\nu(\overline{S_4\times C_5}) = 6$.

Let D be a good drawing of $\overline{S_4\times C_6}$ with eight crossings. Then the drawing D has the following properties:

Property (2a). None of the edges b_ib_{i+1}, c_ic_{i+1}, d_id_{i+1}, e_ie_{i+1} for $i = 0, 1, 2, 3, 4, 5$ (The indices are taken modulo 6.) is crossed.

Property (2b). For all j, $j = 0, 1, 2, 3, 4, 5$, the star S^j has at most one crossing. From the properties (2a) and (2b) it follows that this drawing D of $\overline{S_4\times C_6}$ has at most three crossings. This contradicts the hypothesis $\nu(\overline{S_4\times C_6}) = 8$.

Let T^x be a subgraph of the graph $S_4\times C_5$ ($S_4\times C_6$) with the vertices a_0, a_1, a_2, a_3, a_4, x_0, x_1, x_2, x_3, x_4 (a_0, a_1, a_2, a_3, a_4, a_5, x_0, x_1, x_2, x_3, x_4, x_5) for $x\in$ $\in\{b, c, d, e\}$ and with the edges of x-cycle and the edges a_0x_0, a_1x_1, a_2x_2, a_3x_3, a_4x_4 (a_0x_0, a_1x_1, $\ldots$, a_5x_5).

Theorem 4. $\nu(S_4\times C_3) = 2$, $\nu(S_4\times C_4) = 4$, $\nu(S_4\times C_5) = 8$.

Proof. By Theorem 1 and Lemma 3 we have first two equalities and it is easy to show that $7 \leqq \nu(S_4\times C_5) \leqq 8$, because $\nu(\overline{S_4\times C_5}) \geqq 7$.

We assume that D is a good drawing of $S_4\times C_5$ with at most seven crossings. The drawing D has the following properties:

Property (1). For all x, $x \in \{b, c, d, e\}$ the subgraph T^x of $S_4 \times C_5$ has at most three crossings. (In the opposite case $\nu(S_3 \times C_5) \leqq 3$, a contradiction see [3].)
Property (2). No edge of the a-cycle is crossed in D. (In the opposite case $\nu(\overline{S_4 \times C_5}) \leqq 6$.)
It is shown that every good drawing of the graph $S_4 \times C_5$ contradicts the property (1) or (2).
Let us have the a-cycle in the plane without crossings. This a-cycle divides the plane into two pentagonal regions ω_0 and ω_1. Regarding property (2) in the graph $S_4 \times C_5$ at least two of the x-cycles for $x \in \{b, c, d, e\}$ must lie in one of the regions ω_0 or ω_1. Then, without loss of generality, we may assume that in D the b-cycle and the c-cycle lie in the region ω_0. First the b-cycle is assumed. The edges of the b-cycle and the edges $a_i b_i$ for $i = 0, 1, 2, 3, 4$ divide the region ω_0 into new regions ω_i' (The possible crossings of the edges of T^b are considered as new vertices and no edge of the a-cycle can be crossed.) with at most two vertices of the a-cycle on the boundary of every region ω_i'. Now it can be seen in D that the c-cycle and the edges $a_i c_i$ for $i = 0, 1, 2, 3, 4$ cannot lie in the region ω_0 with at most three crossings at the edges of T^c because of property (1).

Theorem 5. *$\nu(S_4 \times C_n) = 2n$ for all $n \geqq 6$.*

Proof. The proof of this Theorem for $n = 6$ is similar to that of the identity $\nu(S_4 \times C_5) = 8$. By Theorem 1 and Lemma 3 we have $9 \leqq \nu(S_4 \times C_6) \leqq 12$, because $\nu(\overline{S_4 \times C_6}) \geqq 9$. We assume that D is a good drawing of $S_4 \times C_6$ with at most eleven crossings. The drawing D has the following properties:
Property (1). For all x, $x \in \{b, c, d, e\}$, the subgraph T^x of $S_4 \times C_6$ has at most five crossings because of $\nu(S_3 \times C_6) = 6$ (see [3]).
Property (2). The edges of the a-cycle have at most two crossings because of $\nu(\overline{S_4 \times C_6}) \geqq 9$, see Lemma 3.
We show that an assumption of the existence of a good drawing D of $S_4 \times C_6$ with less than twelve crossings contradicts the properties (1) or (2).

Case 1. Let in D the edges of the a-cycle cross each other. This a-cycle in the drawing D divides the plane into three or four regions. If there are at most five vertices of the a-cycle on the boundary of every region determined by the a-cycle in D, then the edges of the a-cycle must be crossed by edges of the subgraph T^x for every $x \in \{b, c, d, e\}$. This contradicts the property (2). Let ω_0 denote the region with six vertices of the a-cycle on its boundary. Every other region has at most four vertices a_i for $i = 0, 1, 2, 3, 4, 5$ on its boundary. (The crossings are considered again as new vertices.) The graph $S_4 \times C_6$ has four subgraphs T^x for $x \in \{b, c, d, e\}$. Regarding property (2) in this case at least three of the subgraphs T^x must lie in the region ω_0. Let us have the subgraph T^b in the region ω_0. The edges of T^b divide the region ω_0 into new regions ω_i' with at most two vertices a_i for $i = 0, 1, \ldots, 6$ on the boundary of every region ω_i'. Now it can be seen in D that no T^x for $x \in \{c, d, e\}$ can lie in the region ω_0 with at most five crossings at the edges of T^x because of property (1).

Case 2. Let in D no edges of the a-cycle cross each other. This a-cycle divides the plane into two hexagonal regions ω_0 and ω_1. In the drawing D the edges of the a-cycle can cross the edges of at most two subgraphs T^x for $x \in \in \{b, c, d, e\}$ (property (2)). Without loss of generality we assume that the edges of T^b and T^c cross no edge of the a-cycle. From the first case of this proof it can be seen that there is at most one of the graphs T^b and T^c being in one of the hexagonal regions. The subdrawing of D induced by vertices of T^b and T^c induces, in this case, the map in the plane so that at most two vertices a_i for $i = 0, 1, \ldots, 6$ are on the boundary of every region ω_i'. Now it can be seen in D that there are more than five crossings at the edges of every T^x for $x \in \{d, e\}$ and it contradicts the property (1).

For $n \geqq 7$ the proof proceeds by induction on n in the same way as in Theorem 2 using Theorem 1 and Lemma 2.

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