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SIMPLE 3–POLYTOPAL GRAPHS WITH EDGES OF ONLY TWO TYPES AND SHORTNESS COEFFICIENTS

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ABSTRACT. It is shown that the class of simple 3-polytopal graphs whose edges are incident with either two 7-gons or a 7-gon and a 4-gon, contains non-Hamiltonian members and even has shortness coefficient less then unity.

1. Introduction

In this paper we mean by a *graph* a finite connected undirected graph with no loops or multiple edges.

For any graph G let $v(G)$ denote the *number of vertices* and $h(G)$ the *length of a maximum cycle*. Thus G is *non-Hamiltonian* if and only if $h(G)$ is less than $v(G)$. The *shortness coefficient* $\varrho(\mathcal{G})$ of an infinite class $\mathcal{G}$ of graphs is defined by

$$\varrho(\mathcal{G}) = \liminf_{G\in\mathcal{G}} \frac{h(G)}{v(G)}, \qquad \text{see [6 or 7]}.$$

An edge of a trivalent planar graph is of *type* (p,q) if the faces containing it are a p-gon and a q-gon. The present paper deals with 3-connected trivalent planar graphs, i.e. simple 3-polytopal graphs, with only two types of edges. Evidently such graphs can exist only if its edges are of the type (p,p) or (p,q), $p \neq q$, $p, q \geq 3$.

Let $S(p,q)$ denote the class of simple 3-polytopal graphs in which all the edges are incident with two p-gons or a p-gon and a q-gon, $p \neq q$, $p,q \geq 3$.

So $S(p,q)$ is the class of simple 3-polytopal graphs the edges of which are of the type (p,p) or (p,q).

In the papers [5 and 7] it has been shown that the class $S(p,q)$ is infinite only for $6 \leq p \leq 10$ and $q = 3$, $6 \leq p \leq 7$ and $q = 4$, $p = 6$ and $q = 5$,

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or $p = 5$ and $q \geq 12$. According to G o o d e y , every member of $S(6,q)$ is Hamiltonian, for $q = 3$ [3] and $q = 4$ [2]. The same property has been shown by J e n d r o l' and M i h ó k for the class $S(5,12)$ [4]. In [7] O w e n s deals with the shortness coefficients of the classes $S(5,q)$. He proved that each class $S(5,q)$ has shortness coefficient less than one for all $q \geq 28$ and he also asked whether there are some non-Hamiltonian members in the classes $S(5,q)$ for $12 \leq q \leq 23$, or $q = 27$, and whether $\varrho(S(5,q)) < 1$ for $q = 24$, 25, 26.

This problem evoked an interest in this subject. In [8] O w e n s has shown that $\varrho(S(p,3)) < 1$ for $p = 8, 9$ and 10. The same inequality has been proved by the present author for $\varrho(S(5,q))$, $q = 26$, 27 [9] and for $\varrho(S(7,3))$ [10].

The following theorem supplements these results:

THEOREM.

(1) *There is a non-Hamiltonian member of* $S(7,4)$ *with* 1628 *vertices.*

(2) $\varrho(S(7,4)) \leq 1295/1296 < 1$.

2. Constructions and proof of the theorem

We begin to describe our constructions. Similarly as in [8] certain graphs which occur repeatedly as subgraphs will be denoted by capital letters and represented in diagrams by labelled circles. Numbers placed round such a circle show how many vertices the subgraph supplies to the adjoining faces of any graph in which it occurs. As the first example Fig. 1 shows the well-known Tutte "triangle" subgraph T [1, p. 165]. The "dangling" edges are not a part of the subgraph but show how it is to be joined into a graph. By a *path through a subgraph* we mean a path whose ends are not in the subgraph. By a *path of type* P_{ij} we mean a path through a subgraph that contains linking edges with the numerical labels i and j. The essential property of subgraph T is that every spanning path through it is of type P_{12} or P_{13}, not of type P_{23}. In other words, edge 1 is an a-edge.

Let A and B denote the subgraphs shown in Fig. 2. Small unlabelled circles in diagram A represent quadrangular faces. It is easily verified that every face within A (or B) is a quadrangle or a 7-gon and that $v(A) = 163$, $v(B) = 169$. Let U denote the subgraph formed from T by the two substitutions ($v \to B$ and $f \to F$) shown in Fig. 3, where v and f refer to labels in Fig. 1 and F is a subgraph defined in terms of two copies of B. The dangling edges of F are numbered to fix its orientation. Every interior face of U is either a quadrangle or a 7-gon and the outer boundary of U does not differ from that of T.

LEMMA 1. *No spanning path through* F *is of type* P_{46}.

P r o o f . Let Q be (if possible) a spanning path of type P_{46} through F.

Then it can be shown that all "heavy" edges of Fig. 3 must be in Q. Now we consider two cases.

Case 1: Edge 8 is in Q. Then the edges 9 and 12 are not in Q and the edges 11, 10 and 15 must be in Q. The intersection of Q with the quadrangle g is a path of type $P_{10\ 11}$, which is impossible, because path Q cannot contain a cycle.

Case 2: Edge 8 is not in Q. Then the edges 9 and 12 are in Q and the edge 10 is not in Q. Thus the edges 13 and 14 are in Q. So the intersection of Q with the quadrangle g is a path of type $P_{12\ 13}$, which is impossible, too. Since in each case we get a contradiction, no such path Q exists and the lemma follows.

The following lemma shows the property of U which makes it useful to us.

LEMMA 2. *For every spanning path through U there exists a spanning path through T which is of the same type.*

P r o o f. Since there is a spanning path through the vertex v that contains any two of its three incident edges, only the substitution $f \to F$ need be considered. The nonempty intersection of F with a path through U is of the type

$$P_{46}, P_{45}, P_{47}, P_{45} \cup P_{67} \quad \text{or} \quad P_{47} \cup P_{56}$$

only, allowing for symmetry. The nonempty intersection of f with a path through T has the same property. It is easy to find in f a spanning path (or pair of paths) of each type except P_{46}. By Lemma 1, no such spanning path exists in F, either. This completes the proof of the lemma.

Now let W be defined in terms of U as in Fig. 4. The three interior faces of W that do not lie in U are 7-gons.

LEMMA 3. *W has an a-edge.*

P r o o f. We first show that the subgraph U has an a-edge. Let Q be (if possible) a spanning path through U which does not contain edge 1 (see Fig. 1). Then Q is of type P_{23}. Thus, by Lemma 2, there exists a spanning path through T which is of type P_{23}, but it leads to a contradiction with the existence of an a-edge in T. So every spanning path through W contains the a-edge of U and the six vertices of $W - U$. It is easy to check that such a path necessarily includes the linking edge labelled 1 in Fig. 4.

Let J_1 be as shown in Fig. 4. Evidently $J_1 \in S(7,4)$ and J_1 is non-Hamiltonian, because it contains three copies of the subgraph W, the a-edges of which are concurrent. So every cycle in J_1 omits at least one of them and therefore omits at least one vertex of the corresponding copy of the subgraph W. This completes the proof of Theorem (1).

The graph J_1 contains nine copies of the subgraph A. We denote by X the subgraph of J_1 that remains when one copy of A is deleted. By inspection, $v(J_1) = 1628$ and $v(X) = 1465$. Since X and B each contribute three vertices to the three adjoining faces of any graph in which either occurs, $S(7,4)$ is closed under the replacement of the copies of B by copies of X. It is easy to verify that no path through X spans X.

We now use the fact that X contains eight copies of B to construct an infinite sequence $\langle J_n \rangle$ of non-Hamiltonian members of $S(7,4)$, starting with J_1. For $n \geq 1$, let J_{n+1} be the graph obtained from J_n when one copy of B in one (any one) of its subgraphs of type X is replaced by the new copy of X. So $h(J_n) \leq v(J_n) - n$ and, since $v(J_n) = v(J_1) + (n-1)\big(v(X) - v(B)\big) = 332 + 1296n$, we obtain $\varrho\big(S(7,4)\big) \leq 1295/1296 < 1$ and this completes the proof of Theorem.

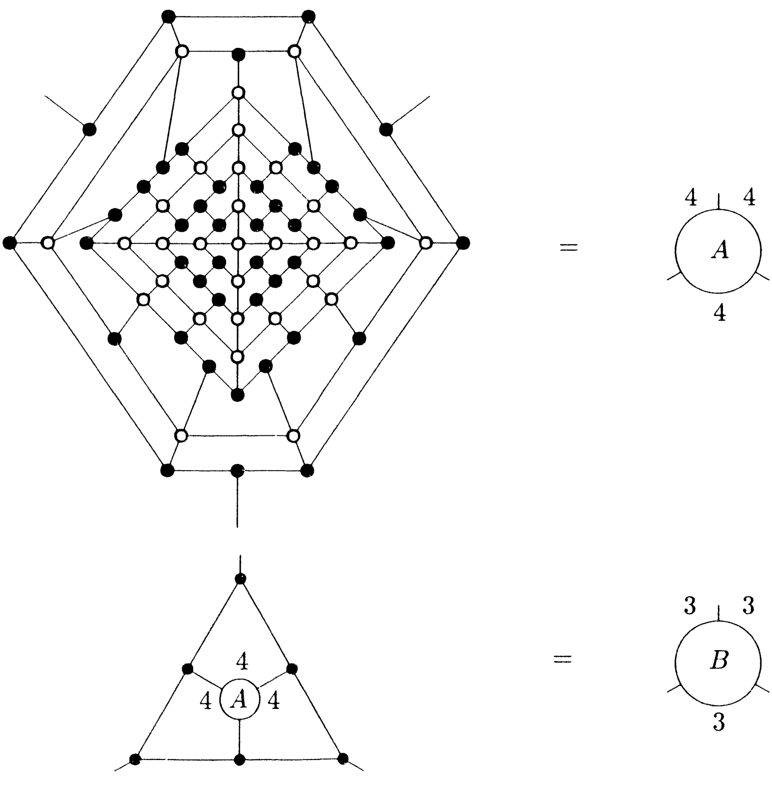

Figure 2. The subgraphs A and B.

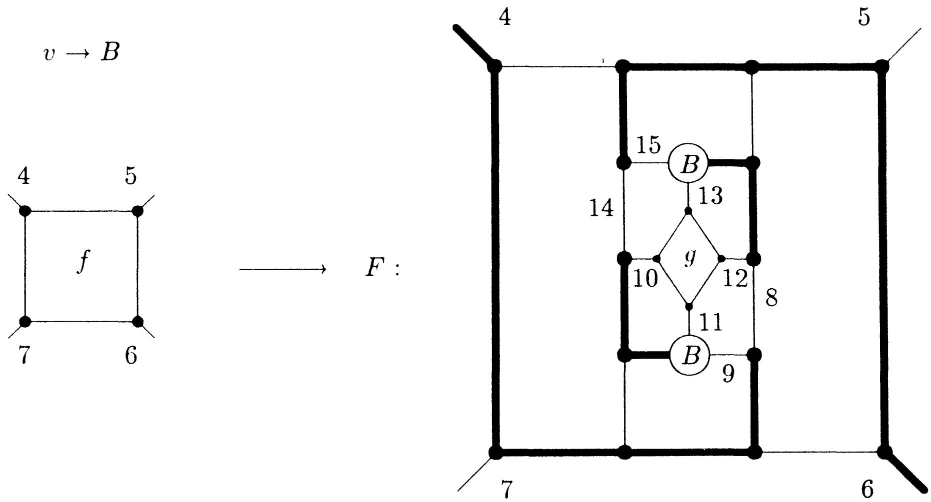

Figure 3. Two substitutions.

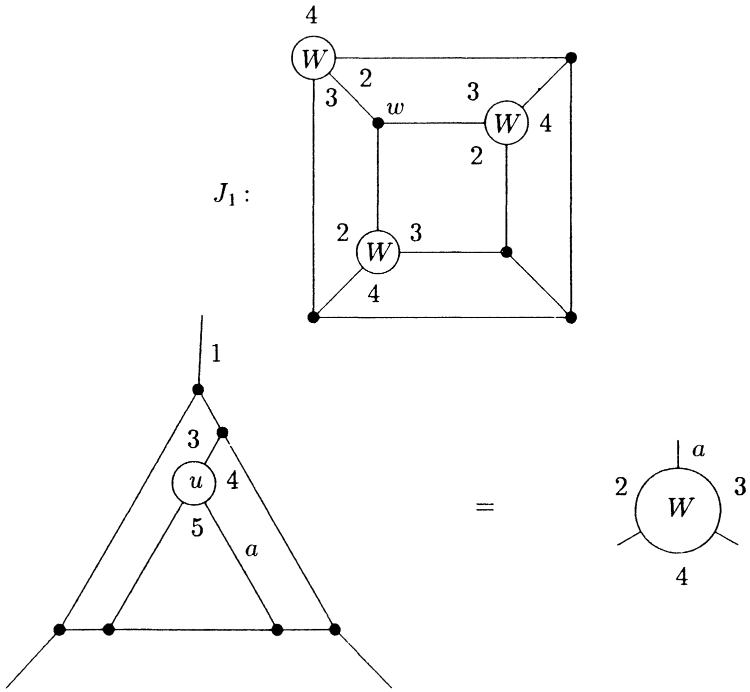

Figure 4.

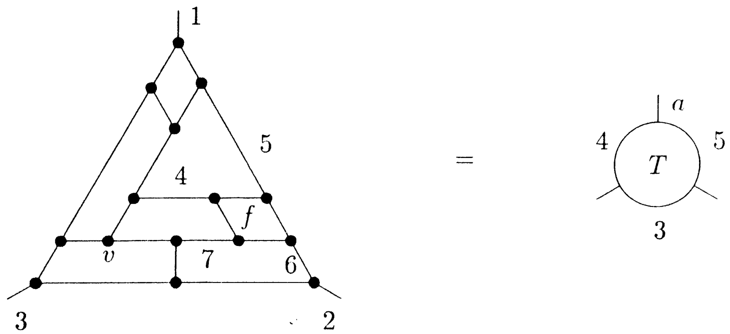

Figure 1. The Tuttetriangle.

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