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A METHOD OF INVERSION OF THE LAPLACE TRANSFORM

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ABSTRACT. An approximate method of the Laplace transform inversion is given which is particularly appropriate for stress analysis problems in quasi-static linear viscoelasticity.

1. Introduction

The Laplace transform is useful in solving some ordinary and partial differential equations and integral equations and arises in many fields of engineering mathematics. However, the exact determination of the original function $f(t)$ from its Laplace transform

$$F(p) = \int_0^\infty e^{-pt} f(t)\,dt \tag{1}$$

is often a great difficulty. In many cases, numerical methods must be used.

In determining a function $f(t)$ from its Laplace transform $F(p)$ one applies either a partial fraction expansion or an integration along some contour in the complex p-plane; one thus obtains $f(t)$ in terms of the poles and residues of $F(p)$, or from the values of $F(p)$ on a contour of the p-plane. Both methods have obvious disadvantages for a numerical analysis.

In the following we propose to develop a method for determining $f(t)$ in terms of the values of $F(p)$ on a sequence of equidistant points

$$p_j = p_0 + jh\,, \qquad j = 0, 1, \dots, n \tag{2}$$

on the real p-axis, where p_0 is a real number in the region of existence of $F(p)$, and h is an arbitrary positive integer. That $F(p)$ is uniquely determined from

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its values at the above points, is know [4]. It should therefore be possible to express $f(t)$ directly in terms of $F(p_0 + jh)$.

An approximate method of the Laplace transform inversion is given, which is particularly appropriate for stress analysis problems in quasi-static linear viscoelasticity. A viscoelastic response can be easily calculated from an associated elastic solution, which is known numerically, using experimentally determined material properties, regardless of the complexity of the property dependence of the elastic solution and it holds that the Laplace transform of a viscoelastic stress has singularities only on the non-positive real p-axis, and that all poles are simple, except at the origin, where a double pole may occur. S c h a p e r y [5] shows that for the mentioned problems, $f(t)$ can be written in the form

$$f(t) = \sum_{i=1}^{s} \phi_i \, e^{-\gamma_i t} \tag{3}$$

and the time dependence of the exact inversion therefore suggests that a Dirichlet series

$$f_k(t) = \sum_{i=1}^{k} S_i \, e^{-m_i t} \tag{4}$$

can be used as a reasonable approximation to the solution $f(t)$, where the m_i are prescribed positive constants, and the S_i are unspecified coefficients.

Under the assumption that a finite sequence of the $(n+1)$ values of $F(p)$ at the equidistant points p_j, $j = 0, 1, \dots, n$, $n > 2k$ is given, the method presented here extends these results also for unspecified positive constants m_i. Our method determines the coefficients S_i, m_i, $i = 1, 2, \dots, k$ by solving linear overdetermined systems and a polynomial equation of the kth order.

2. Numerical inversion of the Laplace transform

From an experiment we get the values F_j, $j = 0, 1, \dots, n$ by equal spacing h, so that $p_{j+1} - p_j = h$, $j = 0, 1, \dots, n-1$ and there holds $F(p_j) = F_j$, $j = 0, 1, \dots, n$. Then with regard to the termwise Laplace transformation of (4) we get a system of $(n+1)$ equations of the form

$$F_j = \sum_{i=1}^{k} S_i \frac{1}{(p_0 + jh) + m_i}, \qquad j = 0, 1, \dots, n. \tag{5}$$

If we consider the first differences d_j from the values F_j, i.e.

$$d_j = F_{j+1} - F_j, \qquad j = 0, 1, \dots, n-1, \tag{6}$$

we can express d_j in the form

$$d_j = -h \sum_{i=1}^{k} \frac{S_i}{(p_0 + jh + m_i)(p_0 + (j+1)h + m_i)}, \tag{7}$$

or shortly

$$d_j = \sum_{i=1}^{k} d_{j,i}, \tag{8}$$

where the relation

$$d_{j+1,i} = \frac{p_0 + jh + m_i}{p_0 + (j+2)h + m_i} d_{j,i} \equiv u_{j,i}\, d_{j,i} \tag{9}$$

holds for $j = 0, 1, \ldots, n-2$ and $i = 1, 2, \ldots, k$.

Let D^n_{k+1} be the set of $(n-k)$ subsets, each consisting of $(k+1)$ elements of the set $\{d_0, d_1, \ldots, d_{n-1}\}$, i.e. for a fixed j there holds

$$\{d_j, d_{j+1}, \ldots, d_{j+k}\} \in D^n_{k+1},$$

where

$$\begin{aligned} d_j &= \sum_{i=1}^{k} d_{j,i}, \\ d_{j+1} &= \sum_{i=1}^{k} d_{j+1,i} = \sum_{i=1}^{k} u_{j,i}\, d_{j,i}, \\ &\vdots \\ d_{j+k} &= \sum_{i=1}^{k} d_{j+k,i} = \cdots = \sum_{i=1}^{k} \left(\prod_{s=0}^{k-1} u_{j+s,i} \right) d_{j,i}. \end{aligned} \tag{10}$$

From (9) an interesting result for $u_{j+s,i}$ can be obtained, namely

$$u_{j+1,i} = \frac{u_{j,i} + 1}{3 - u_{j,i}}, \qquad u_{j+2,i} = \frac{1}{2 - u_{j,i}}, \ldots, \tag{11}$$

and for simplicity of notation we shall denote

$$u_{j+s,i} = \varphi_s(u_{j,i}), \qquad s = 0, 1, \ldots, k-1, \tag{12}$$

where $u_{j,i} = \varphi_0(u_{j,i})$.

And so, instead of (10) we have

$$\begin{aligned} d_j &= \sum_{i=1}^{k} d_{j,i}, \\ d_{j+r} &= \sum_{i=1}^{k} \Big(\prod_{s=0}^{r-1} \varphi_s(u_{j,i}) \Big) d_{j,i}, \qquad r = 1,2,\dots,k. \end{aligned} \tag{13}$$

Instead of solving the nonlinear systems (13) for S_i and m_i (see (7), (8)), we propose to solve the following (see (17)) linear system corresponding to (13).

Let M^k_{k-r} be the set of all subsets, each consisting of $(k-r)$ elements, of the set $\{1,2,\dots,k\}$. Take

$$\begin{aligned} q_{k-r} &= (-1)^{k-r} \sum_{\{s_1,s_2,\dots,s_{k-r}\}\in M^k_{k-r}} \Big(\prod_{t=1}^{k-r} \psi_{s_t}(u_{j,i}) \Big), \\ q_0 &= 1, \qquad r = 0,1,\dots,k-1 \end{aligned} \tag{14}$$

and $\psi_s(x)$ is given.

For example, if $k = 3$, $j = 1$, then there holds

$$q_1 = \frac{(u_{3,3}u_{2,3}u_{1,3} - u_{3,1}u_{2,1}u_{1,1})(u_{1,2} - u_{1,1}) - (u_{3,2}u_{2,2}u_{1,2} - u_{3,1}u_{2,1}u_{1,1})(u_{1,3} - u_{1,1})}{(u_{2,3}u_{1,3} - u_{2,1}u_{1,1})(u_{1,2} - u_{1,1}) - (u_{2,2}u_{1,2} - u_{2,1}u_{1,1})(u_{1,3} - u_{1,1})}. \tag{15}$$

Then, by multiplication of the whole system (13) by expressions q_{k-r}, $r = 0,1,\dots,k$, we get the system corresponding to (13)

$$\begin{aligned} q_k d_j &= q_k \sum_{i=1}^{k} d_{j,i}, \\ q_{k-r} d_{j+r} &= q_{k-r} \sum_{i=1}^{k} \Big(\prod_{s=0}^{r-1} \varphi_s(u_{j,i}) \Big) d_{j,i}, \qquad r = 1,2,\dots,k. \end{aligned} \tag{16}$$

The equation which originated from (16) does not contain by its sum $d_{j,i}$ and is of the form

$$\sum_{r=0}^{k} q_{k-r} d_{j+r} = 0. \tag{17}$$

This holds for each j, $j = 0, 1, \ldots, n-k-1$. Hence, (17) represents the linear system of $(n-k)$ equations containing k unknown parameters $q_1, q_2, \ldots, q_k$. And so equations (17) form the overdetermined system. Such overdetermined system can be solved in the L_1 sense, that is, the solution can be the q_i, $i = 1, 2, \ldots, k$, that minimize

$$\sum_{j=0}^{n-k-1} \left| \sum_{r=0}^{k} q_{k-r} d_{j+r} \right| . \tag{18}$$

The background material on the L_1 minimization can be found in Bloomfield and Steiger [2] and Barrodale and Roberts [1]. Any linear L_1 problem can be rephrased as a linear-programming problem [1]. The L_1 strategy for solving system (17) is presented by myself [3]. We propose to solve the system (17) by minimizing the quadratic deviation

$$E = \sum_{j=0}^{n-k-1} \left(\sum_{r=0}^{k} q_{k-r} d_{j+r} \right)^2 . \tag{19}$$

Consequently, the formula (14) (and (15) also) are only of theoretical interest. In actual computations, the values $q_1, q_2, \ldots, q_k$ are calculated from the system

$$\frac{\partial E}{\partial q_j} = 0, \qquad j = 1, 2, \ldots, k, \qquad q_0 = 1 . \tag{20}$$

Let $q = (q_0, q_1, \ldots, q_k)$ be the vector that spans the null space of the system (17). Then it is easy to show that from (17) after substituting (16) for each i, $i = 1, 2, \ldots, k$, we have

$$q_k + \sum_{r=1}^{k} q_{k-r} \left(\prod_{s=0}^{r-1} \varphi_s(u_{j,\cdot}) \right) = 0 . \tag{21}$$

The problem of finding the $u_{j,\cdot}$ that fulfills the equation (21) can be rephrased as the problem of finding the roots of the equation

$$\sum_{r=0}^{k} w_r \cdot u_{j,\cdot}^{k-r} = 0 , \tag{22}$$

where $w_r = w(q)$ and $u_{j,i} \in (0,1)$, $i = 1, 2, \ldots, k$. Now, m_i, $i = 1, 2, \ldots, k$ results from relation (9).

In order to calculate S_i, $i = 1,2,\dots,k$, we use the values of m_i, $i = 1,2,\dots,k$, which are computed using (22) and (9). The equations (5) form the overdetermined system

$$F_j - \sum_{i=1}^{k} S_i \frac{1}{p_j + m_i} = 0\,, \qquad j = 0,1,\dots,n \tag{23}$$

of $(n+1)$ equations for k unknowns S_i, $i = 1,2,\dots,k$. The strategy for solving the system (23) is to minimize the quadratic deviation. So, form a system analogous with (20) we have S_i, $i = 1,2,\dots,k$.

3. Numerical examples

We consider here the problem of the numerical inversion of the Laplace transform.

Example 1. We would now like to discus a mentioned technique based on finding the approximation $f_k(t)$, $k = 3$, of the exact solution $f(t)$.

Consider the values F_j, $j = 0,1,\dots,15$, be equal spacing $h = 5$ given in the Table.

Table:

t_j	1	6	11	16	21	26	31
F_j	3.125-2	1.488-3	4.735-4	2.298-4	1.353-4	8.903-5	6.300-5

t_j	36	41	46	51	56	61	66
F_j	4.692-5	3.630-5	2.891-5	2.357-5	1.958-5	1.653-5	1.413-5

t_j	71	76
F_j	1.223-5	1.068-5

(24)

In our case, the equation (13) has the form

$$\begin{aligned} d_j &= \sum_{i=1}^{3} d_{j,i}\,, & d_{j+2} &= \sum_{i=1}^{3} \frac{u_{j,i}+1}{3-u_{j,i}}\, u_{j,i}\, d_{j,i}\,, \\ d_{j+1} &= \sum_{i=1}^{3} u_{j,i}\, d_{j,i}\,, & d_{j+3} &= \sum_{i=1}^{3} \frac{u_{j,i}}{2-u_{j,i}} \cdot \frac{u_{j,i}+1}{3-u_{j,i}}\, d_{j,i}\,. \end{aligned} \tag{25}$$

Using the result of (20), we can obtain from (22) the next formula, i.e. $u_{j,i}$, $i = 1,2,3$ are roots of the equation

$$(q_2 - q_1)u^3 + (1 + q_3 - 5q_2 + q_1)u^2 + (1 - 5q_3 + 6q_2 + 2q_1)u + 6q_3 = 0\,, \tag{26}$$

where q_i, $i = 1, 2, 3$ are calculated by minimizing the quadratic deviation (19). According to (26) we have

$$1.61543u^3 - 2.11210u^2 + 0.803905u - 0.0828195 = 0. \tag{27}$$

By solving this equation using a well-known technique we obtain these roots:

$$u_{j,1} = 0.1668, \quad u_{j,2} = 0.4366, \quad u_{j,3} = 0.7041.$$

Now, using the relation (9) we have

$$m_1 = 1.00159, \quad m_2 = 6.74971, \quad m_3 = 22.7916,$$

next, we must try to determine the coefficients S_i, $i = 1, 2, 3$ from (23).

Example 2. For the case $k = 2$, the computation can be simplified considerably and from values in the Table we can rewrite (22) in the form

$$(1 - q_1)u^2 + (1 + 3q_1 - q_2)u + 3q_2 = 0$$

and, analogously, for $k = 4$ we have

$$(-3q_3 + 3q_2)u^4 + (-3q_4 + 20q_3 - 8q_2 - 3q_1 - 1)u^3 + (20q_4 - 43q_3 - q_2 + 2q_1 + 2)u^2$$
$$+(-43q_4 + 30q_3 + 10q_2 + 5q_1 + 3)u + 30q_4 = 0.$$

It is clear that the foregoing results can be generalized also for $k > 4$.

4. Conclusion

This paper deals with a new method of numerical inversion of the Laplace transform. Thus, for the determination of the approximation (4) two problems must be considered:

(i) the numerical calculation of the coefficients S_i and m_i, $i = 1, 2, \ldots, k$,
(ii) the efficient evaluation of q_i, $i = 0, 1, \ldots, k$ and the roots of the polynomial equation (22).

The advantage of using the mentioned method thus becomes evident. Owing to the introduction of minimizing the quadratic deviation (19) and solving only one polynomial equation (22), a much larger class of Laplace transforms can be efficiently inverted.

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