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János T. Tóth; László Zsilinszky

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ON A TYPICAL PROPERTY OF FUNCTIONS

JÁNOS T. TÓTH — LÁSZLÓ ZSILINSZKY

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ABSTRACT. Let s be the space of all real sequences endowed with the Fréchet metric ϱ. Consider the space $\mathcal{F}$ of all functions $f\colon \mathbb{R} \to \mathbb{R}$ with the uniform topology. Denote by $\mathcal{U}$ the class of all functions $f \in \mathcal{F}$ for which the set $\Big\{\{a_i\}_i \in s\,;\ \sum\limits_i f(a_i) \text{ converges}\Big\}$ is σ-superporous in (s,ϱ). Then $\mathcal{U}$ is residual in $\mathcal{F}$, both $\mathcal{U}$ and $\mathcal{F}\setminus\mathcal{U}$ are dense-in-itself and $\mathcal{U}$ is a Baire space in the relative topology.

Introduction

Let (s,ϱ) be the metric space of all real sequences with the *Fréchet metric*

$$\varrho(a,b) = \sum_{i=1}^{\infty} 2^{-i} \cdot \frac{|a_i - b_i|}{1 + |a_i - b_i|}\,, \qquad \text{where} \quad a = \{a_i\}_i\,,\ \ b = \{b_i\}_i \in s\,.$$

Denote by $B(a,r)$ the open ball centred at $a \in s$ with radius $r > 0$ in (s,ϱ). Let $E \subset s$, $a \in s$ and $r > 0$. Define

$$\gamma(a,r,E) = \sup\bigl\{r' > 0\,;\ \exists a' {\in} s\ \ B(a',r') \subset B(a,r) \setminus E\bigr\}\,.$$

We say that E is *porous at* a if

$$\limsup_{r\to 0^+} \frac{\gamma(a,r,E)}{r} > 0\,.$$

Further, the set $E \subset s$ is said to be *superporous at* $a \in s$ (see [7], [8]), if $E \cup F$ is porous at a whenever $F \subset s$ is porous at a. We say that E is *superporous* if it is superporous at each of its points, further E is σ-superporous if it is a countable union of superporous sets.

Denote by $\mathbb{Q}$ the set of all rational numbers, by χ_M the characteristic function of $M \subset \mathbb{R}$, and by $\overline{\mathbb{R}}$ the set $\mathbb{R} \cup \{\pm\infty\}$.

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It is known that the set of all real sequences $\{a_i\}_i$ such that $\sum_i a_i$ converges constitutes a meager set in (s,ϱ) ([2], [6]). It is not hard to generalize this result realizing that the set

$$A(f)=\Big\{\{a_i\}_i\in s\,;\ \sum_i f(a_i)\ \text{converges}\Big\}$$

is meager in (s,ϱ) for every nonvanishing continuous function $f\colon\mathbb{R}\to\mathbb{R}$. In fact, these sets are even "poorer" since, as we will show, $A(f)$ is σ-superporous for a broad class of functions f. More precisely, if $\mathcal{U}$ stands for the class of all functions $f\colon\mathbb{R}\to\mathbb{R}$ (not necessarily continuous) for which $A(f)$ is σ-superporous in s, then $\mathcal{U}$ constitutes a residual set in the space $(\mathcal{F},d)$ of all real functions of one real variable with the sup-metric $d(f,g)=\min\Big\{1,\ \sup_{x\in\mathbb{R}}|f(x)-g(x)|\Big\}$, where $f,g\in\mathcal{F}$. Besides, we will investigate various topological properties of $A(f)$ in (s,ϱ) and of $\mathcal{U}$ in $(\mathcal{F},d)$.

Properties of $A(f)$

First we examine the density of $A(f)$.

THEOREM 1. *The set $A(f)$ is either empty or dense in (s,ϱ).*

Proof. Suppose $A(f)\neq\emptyset$ and $\{b_i\}_i\in A(f)$. Let $a=\{a_i\}_i\in s$ and $\varepsilon>0$. Choose $j\in\mathbb{N}$ such that $2^{-j}<\varepsilon$. Put $c_i=a_i$ for $i\le j$, and $c_i=b_i$ for $i>j$. Then evidently $c=\{c_i\}_i\in A(f)$ and $\varrho(a,c)<\varepsilon$. □

Define the following sets for $f\in\mathcal{F}$ and $p,q\in\mathbb{N}$:

$$A_{pq}(f)=\Big\{\{a_i\}_i\in s\,;\ \forall m,n>q,\ m<n\ \ |f(a_{m+1})+\dots+f(a_n)|\le\frac{1}{p}\Big\}.$$

LEMMA 1. *Suppose $\alpha>0$ and $x_0\in\mathbb{R}$. Let $f_0(x)=\max\Big\{0,\ 2-\frac{1}{\alpha}|x-x_0|\Big\}$, $x\in\mathbb{R}$. Then $A_{pq}(f_0)$ is superporous for every $p,q\in\mathbb{N}$.*

Proof. Let $a\in A_{pq}(f_0)$. Suppose $F\subset s$ is an arbitrary set porous at a. Then we have a number $\beta>0$ such that for all $n\ge q$ there exist r_n, r'_n such that $\beta r_n<r'_n<r_n<2^{-n}$ and $a'\in s$ for which

$$B(a',r'_n)\subset B(a,r_n)\setminus F\,. \tag{1}$$

Denote $m_n=\min\{k\in\mathbb{N};\ 2^{-k}<r'_n\}$ and $\varepsilon_n=2^{-m_n}$. Then we have

$$m_n>q \qquad\text{and}\qquad r'_n>\varepsilon_n\ge\frac{r'_n}{2}\,. \tag{2}$$

Define $b \in s$ as follows:

$$b_i = a'_i \qquad \text{if } i \neq m_n + 1\,,$$

$$b_{m_n+1} = \begin{cases} x_0 & \text{if } a'_{m_n+1} \notin \left(x_0 - \frac{\alpha}{4},\, x_0 + \frac{\alpha}{4}\right), \\ x_0 + \frac{\alpha}{2} & \text{if } a'_{m_n+1} \in \left(x_0 - \frac{\alpha}{4},\, x_0 + \frac{\alpha}{4}\right). \end{cases}$$

Then we get

$$\frac{\varepsilon_n}{2} > \varrho(a',b) = 2^{-m_n-1} \cdot \frac{|a'_{m_n+1} - b_{m_n+1}|}{1 + |a'_{m_n+1} - b_{m_n+1}|} \geq 2^{-m_n-1} \cdot \frac{\frac{\alpha}{4}}{1 + \frac{\alpha}{4}} = \frac{\alpha}{4+\alpha} \cdot \frac{\varepsilon_n}{2}\,,$$

and, by (2), we have

$$\frac{\varepsilon_n}{2} > \varrho(a',b) \geq \frac{\alpha}{4+\alpha} \cdot \frac{\varepsilon_n}{2} \geq \frac{\alpha}{4(4+\alpha)} \cdot r'_n\,. \tag{3}$$

Put $\delta = \dfrac{\alpha}{4+\alpha} \cdot \varrho(a',b)$ and choose an arbitrary $c \in B(b,\delta)$. Then we get $\dfrac{\varepsilon_n}{2} \cdot \dfrac{|c_{m_n+1} - b_{m_n+1}|}{1 + |c_{m_n+1} - b_{m_n+1}|} \leq \varrho(c,b) < \delta$, thus in view of (3)

$$|c_{m_n+1} - b_{m_n+1}| < \frac{\dfrac{2\delta}{\varepsilon_n}}{1 - \dfrac{2\delta}{\varepsilon_n}} < \frac{\dfrac{2}{\varepsilon_n} \cdot \dfrac{\varepsilon_n}{2} \cdot \dfrac{\alpha}{4+\alpha}}{1 - \dfrac{2}{\varepsilon_n} \cdot \dfrac{\varepsilon_n}{2} \cdot \dfrac{\alpha}{4+\alpha}} = \frac{\alpha}{4}\,,$$

consequently, $c_{m_n+1} \in \left(x_0 - \frac{3\alpha}{4},\, x_0 + \frac{3\alpha}{4}\right)$ (see the definition of b_{m_n+1}).

Observe now that $|f_0(c_{m_n+1})| > 1 \geq \dfrac{1}{p}$, so

$$c \in s \setminus A_{pq}(f_0)\,. \tag{4}$$

Using (3), we have $\varepsilon_n - \varrho(a',b) > \dfrac{\varepsilon_n}{2} > \dfrac{\alpha}{4+\alpha} \cdot \dfrac{\varepsilon_n}{2} > \delta$, therefore $B(b,\delta) \subset B(a',\varepsilon_n) \subset B(a',r'_n)$. In virtue of (4) and (1), there holds

$$B(b,\delta) \subset B(a',r'_n) \setminus A_{pq}(f_0) \subset B(a,r_n) \setminus \left(F \cup A_{pq}(f_0)\right).$$

It means that $\gamma\big(a,r_n,F \cup A_{pq}(f_0)\big) \geq \delta \geq \left(\dfrac{\alpha}{4+\alpha}\right)^2 \cdot \dfrac{r'_n}{4} > \left(\dfrac{\alpha}{4+\alpha}\right)^2 \dfrac{\beta}{4} \cdot r_n$, thus

$$\limsup_{r \to 0^+} \frac{\gamma\big(a,r,F \cup A_{pq}(f_0)\big)}{r} \geq \left(\frac{\alpha}{4+\alpha}\right)^2 \cdot \frac{\beta}{4} > 0\,.$$

Therefore $F \cup A_{pq}(f_0)$ is porous at a. □

THEOREM 2. *Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function for which there exists $x_0 \in \overline{\mathbb{R}}$ such that*

$$\liminf_{x \to x_0} |f(x)| > 0 . \tag{5}$$

Then $A(f)$ is σ-superporous in (s, ϱ).

P r o o f. First consider $x_0 \in \mathbb{R}$. Then by (5), there exist $h > 0$ and $\alpha > 0$ such that

$$|f(x)| \geq h \tag{6}$$

for all $x \in (x_0 - 2\alpha,\ x_0 + 2\alpha)$. Let $a \in A(f)$. By (6), the interval $(x_0 - 2\alpha,\ x_0 + 2\alpha)$ contains only a finite number of terms of a. Thereby $a \in A(f_0)$, where f_0 is defined in Lemma 1. Hence $A(f) \subset A(f_0)$. It suffices to observe that $A(f_0) = \bigcap_p \bigcup_q A_{pq}(f_0)$ and use Lemma 1.

If $x_0 = \pm\infty$, then, by (5), one can easily find $x_0' \in \mathbb{R}$ and $\alpha > 0$ such that (6) is fulfilled for every $x \in (x_0' - 2\alpha,\ x_0' + 2\alpha)$, which converts this case to the previous one. □

R e m a r k 1. It is worth noticing which classes of functions fulfil (5). Some examples follow:

(i) *Functions that are lower (upper) semicontinuous at an $x_0 \in \mathbb{R}$ such that $f(x_0) > 0$ ($f(x_0) < 0$).* This can be inferred from the definition of semicontinuous functions and Theorem 2.

(ii) *Nonvanishing functions with closed graph* (in the product topology – cf. [3], [4]). To show this, recall that each function $f \in \mathcal{F}$ having closed graph is a Baire 1 function (cf. [3; Theorem 1']). Thus the set of its continuity points C_f is dense in $\mathbb{R}$ ([5; p. 235]). It means that every $x \in \mathbb{R}$ is a limit of a sequence $x_i \in C_f$ $(i \in \mathbb{N})$. Thus, by [4; Theorem 1], $f(x_i) \to f(x)$ as $i \to \infty$. Consequently, $|f(x_0)| > 0$ for some $x_0 \in C_f$, since otherwise $f \equiv 0$. Hence, we have $\liminf_{x \to x_0} |f(x)| = |f(x_0)| > 0$.

(iii) *Nonvanishing, monotone functions.* That is clear from Theorem 2 since, if f is nonvanishing and increasing (decreasing), then (5) holds for $x_0 = +\infty$ $(x_0 = -\infty)$. □

Properties of $\mathcal{U}$

Introduce an auxiliary set

$$\mathcal{U}_0 = \left\{ f \in \mathcal{F};\ f \text{ satisfies (5) for some } x_0 \in \overline{\mathbb{R}} \right\}.$$

We have

LEMMA 2. *The set $\mathcal{U}_0$ is dense and open in $(\mathcal{F}, d)$, thus $\mathcal{F} \setminus \mathcal{U}_0$ is nowhere dense in $(\mathcal{F}, d)$.*

P r o o f. Choose $f \in \mathcal{U}_0$. Then for some $x_0 \in \overline{\mathbb{R}}$ there exists $h > 0$ and a neighbourhood I of x_0 such that (6) holds for each $x \in I$. Put $\varepsilon_0 = \frac{h}{2}$. For every $g \in B(f, \varepsilon_0)$ we get that $|g(x)| \geq |f(x)| - |f(x) - g(x)| \geq h - \varepsilon_0 = \varepsilon_0 > 0$ for each $x \in I$. Consequently $g \in \mathcal{U}_0$, thus $\mathcal{U}_0$ is open in $\mathcal{F}$.

To show the density of $\mathcal{U}_0$ in $\mathcal{F}$, choose $f \in \mathcal{F}$ and $\varepsilon > 0$. Put $I = (0,1)$. Define $M = \left\{x \in \mathbb{R};\ \text{either } x \in X \setminus I, \text{ or } x \in I \text{ and } |f(x)| \geq \frac{\varepsilon}{4}\right\}$ and $M' = \mathbb{R} \setminus M$. Define a function $g = f \cdot \chi_M + \frac{\varepsilon}{4} \cdot \chi_{M'}$. Then $|f(x) - g(x)| = \left|f(x) - \frac{\varepsilon}{4}\right| \cdot \chi_{M'}(x) \leq \left(|f(x)| + \frac{\varepsilon}{4}\right) \cdot \chi_{M'}(x) \leq \frac{\varepsilon}{2}$ for all $x \in \mathbb{R}$. Further for $x \in I$ we have $|g(x)| = |f(x)| \cdot \chi_M(x) + \frac{\varepsilon}{4} \cdot \chi_{M'}(x) \geq \frac{\varepsilon}{4} > 0$. Accordingly $g \in \mathcal{U}_0 \cap B(f, \varepsilon)$. □

Since $(\mathcal{F}, d)$ is a complete metric space, the following theorem is meaningful:

THEOREM 3. *The set $\mathcal{U}$ is residual in $(\mathcal{F}, d)$.*

P r o o f. It is an easy consequence of Lemma 2 and the fact that $\mathcal{U}_0 \subset \mathcal{U}$ (see Theorem 2). □

R e m a r k 2. In connection with the inclusion $\mathcal{U}_0 \subset \mathcal{U}$ notice that $\mathcal{U}_0 \neq \mathcal{U}$. Indeed, we will show that $\chi_{\mathbb{R} \setminus \mathbb{Q}} \in \mathcal{U} \setminus \mathcal{U}_0$.

In favour of this, introduce the set $A_k(x) = \{\{a_i\}_i \in s;\ a_k = x\}$ for every $k \in \mathbb{N}$, $x \in \mathbb{R}$. Choose $a \in A_k(x)$ ($k \in \mathbb{N}$, $x \in \mathbb{R}$) and a set $F \subset s$ which is porous at a. Then there exist $\beta > 0$, sequences $r_n, r'_n > 0$ and $a' \in s$ such that $r_n \searrow 0$, $\beta r_n < r'_n < r_n < 2^{-k+1}$ and

$$B(a', r'_n) \subset B(a, r_n) \setminus F. \tag{7}$$

Define the sequence $b = \{b_i\}_i \in s$ as follows:

$$b_i = a'_i \qquad \text{if } i \neq k,$$
$$b_k = \begin{cases} a'_k - \dfrac{2^{k-1} r'_n}{1 - 2^{k-1} r'_n} & \text{if } a'_k < x, \\[2ex] a'_k + \dfrac{2^{k-1} r'_n}{1 - 2^{k-1} r'_n} & \text{if } a'_k \geq x. \end{cases}$$

Put $\delta = \frac{r'_n}{2}$. Then $\varrho(b, a') = \delta$, thus

$$B(b, \delta) \subset B(a', r'_n). \tag{8}$$

Furthermore, if $c \in B(b,\delta)$, then $\frac{r'_n}{2} > \varrho(b,c) \geq 2^{-k} \cdot \frac{|b_k - c_k|}{1+|b_k - c_k|}$, so $|b_k - c_k| < \frac{2^{k-1}r'_n}{1-2^{k-1}r'_n}$. Therefore $c_k \neq x$ since, according to the definition of b, we have $|b_k - x| \geq \frac{2^{k-1}r'_n}{1-2^{k-1}r'_n}$. In view of (7), (8), it means that

$$B(b,\delta) \subset B(a',r'_n) \setminus A_k(x) \subset B(a,r_n) \setminus \big(F \cup A_k(x)\big).$$

Consequently, we get $\gamma\big(a, r_n, F \cup A_k(x)\big) \geq \delta > \frac{\beta}{2} r_n$. Hence

$$\limsup_{r \to 0^+} \frac{\gamma\big(a, r, F \cup A_k(x)\big)}{r} \geq \frac{\beta}{2} > 0.$$

So we have proved that $A_k(x)$ is superporous at a. It is now sufficient to observe that

$$A(\chi_{\mathbb{R}\setminus\mathbb{Q}}) \subset \bigcup_k \bigcup_n A_k(p_n),$$

where $\mathbb{Q} = \{p_1, \dots, p_n, \dots\}$. □

In virtue of Lemma 2, the set $\mathcal{U}$ ($\supset \mathcal{U}_0$) is dense in $\mathcal{F}$ and, evidently, $\mathcal{U} \neq \mathcal{F}$. Consequently, $\mathcal{U}$ is not closed in $\mathcal{F}$, hence neither is a complete subspace of $(\mathcal{F}, d)$. Nevertheless, we have:

THEOREM 4. *The space $(\mathcal{U}, d)$ is a Baire space.*

P r o o f. By Lemma 2, $\mathcal{U}_0$ is open in the complete metric space $(\mathcal{F}, d)$, thus $(\mathcal{U}_0, d)$ is a Baire space ([1; Proposition 1.14]). Furthermore, $\mathcal{U}_0$ is dense in $\mathcal{U}$ (see Lemma 2), hence $(\mathcal{U}, d)$ is a Baire space as well ([1; Proposition 1.15]). □

THEOREM 5. *Each point of $\mathcal{U}$ ($\mathcal{F}\setminus\mathcal{U}$) is a point of condensation of $\mathcal{U}$ ($\mathcal{F}\setminus\mathcal{U}$).*

P r o o f. Let $0 < \varepsilon < 1$. One can find a nonvanishing function $f \in \mathcal{U}$ ($f \in \mathcal{F} \setminus \mathcal{U}$). Then $f(x_0) \neq 0$ for some $x_0 \in \mathbb{R}$. Define $f_c = f + cf(x_0) \cdot \chi_{\{x_0\}}$ for each $c > 0$. We have $A(f) = A(f_c)$ ($c > 0$). Now, it is easy to check that $f_c \in B(f,\varepsilon) \cap \mathcal{U}$ ($f_c \in B(f,\varepsilon) \cap (\mathcal{F} \setminus \mathcal{U})$), $f_c \neq f$ for every $0 < c < \frac{\varepsilon}{|f(x_0)|}$. □

R e m a r k 3. In the light of Theorems 3–5, the set $U = \bigcup_{f \in \mathcal{U}} A(f)$ would be worth studying. What we know is that $U_0 = \bigcup_{f \in \mathcal{U}_0} A(f)$ is σ-superporous in s. To show this enumerate intervals with rational endpoints as $I_1, I_2, \dots$, further denote the midpoint of I_n by q_n ($n \in \mathbb{N}$). Define the functions $f_n(x) = \big(1 - |q_n - x|\big) \cdot \chi_{I_n}(x)$ for $x \in \mathbb{R}$. Now it suffices to notice that $f_n \in \mathcal{U}_0$ ($n \in \mathbb{N}$) and $U_0 = \bigcup_{n=1}^{\infty} A(f_n)$. □

REFERENCES

[1] HAWORTH, R. C.—McCOY, R. A.: *Baire spaces*, Dissertationes Math. (Rozprawy Mat.) **141** (1977).

[2] KOSTYRKO, P.: *On unconditional convergence of series in Banach lattices*, Math. Slovaca **35** (1985), 67–75.

[3] KOSTYRKO, P.—NEUBRUNN, T.—ŠALÁT, T.: *On the functions, the graphs of which are closed sets II* (Russian), Acta Fac. Rer. Nat. Univ. Comenian. **X** (1965), 51–61.

[4] KOSTYRKO, P.—ŠALÁT, T.: *On the functions, the graphs of which are closed sets* (Russian), Časopis Pěst. Mat. **89** (1964), 426–432.

[5] KURATOWSKI, K.: *Topology I*, PWN, Warszawa, 1958.

[6] LEGÉŇ, A.—ŠALÁT, T.: *On some applications of the category method in the theory of sequence spaces* (Russian), Mat.-Fyz. Čas. SAV **14** (1964), 217–231.

[7] ZAJÍČEK, L.: *Porosity and σ-porosity*, Real Anal. Exchange **13** (1987-88), 314–350.

[8] ZAJÍČEK, L.: *Porosity, $\mathcal{J}$-density topology and abstract density topologies*, Real Anal. Exchange **12** (1986-87), 313–326.

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College of Education
Department of Mathematics
Farská 19
SK - 949 74 Nitra
Slovakia