Jaroslav Hančl
Transcendental sequences

Mathematica Slovaca, Vol. 46 (1996), No. 2-3, 177--179

Persistent URL: http://dml.cz/dmlcz/136667

Mathematica Slovaca
©1996
Mathematical Institute
Slovak Academy of Sciences

Math. Slovaca, **46** (1996), No. 2–3, 177–179

Dedicated to Professor Tibor Šalát
on the occasion of his 70th birthday

TRANSCENDENTAL SEQUENCES[1]

JAROSLAV HANČL

(*Communicated by Stanislav Jakubec*)

ABSTRACT. We introduce the so called transcendental sequence and prove a criterion for a sequence to be transcendental.

There are a lot of papers concerning the irrationality (see, e.g., [2], [3], [4], [5]), or the transcendency (see, e.g., [1]) of infinite series. In a previous paper [5], the author proved a criterion for irrational sequences. In this paper, we prove a theorem concerning the transcendental sequences. A similar method was used by Kostra in [6].

DEFINITION. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of positive real numbers. If for every sequence $\{c_n\}_{n=1}^{\infty}$ of positive integers the number $\sum\limits_{n=1}^{\infty} 1/(a_n c_n)$ is transcendental, then the sequence $\{a_n\}_{n=1}^{\infty}$ is called *transcendental.*

THEOREM. *Let α, β be positive real numbers such that $\alpha > \beta$ and $\{a_n/b_n\}_{n=1}^{\infty}$ be a sequence, where a_n and b_n are positive integers. If*

$$a_n \geq 2^{(3+\alpha)^n} \tag{1}$$

and

$$b_n \leq 2^{(3+\beta)^n} \tag{2}$$

hold for every large positive integer n, then the sequence $\{a_n/b_n\}_{n=1}^{\infty}$ is transcendental.

Proof. It is sufficient to prove the transcendency of the series $H = \sum\limits_{n=1}^{\infty} b_n/a_n$. (If we take a sequence $\{c_n\}_{n=1}^{\infty}$ of positive integers and put $A_n =$

AMS Subject Classification (1991): Primary 11J81.
Key words: positive real number, positive integer, transcendental number, transcendental sequence.

[1] Supported by the grant no. GA 201/93/2122 of the Czech Grant Agency.

$c_n a_n$, then the sequence $\{A_n/b_n\}_{n=1}^{\infty}$ will fulfill (1) and (2) for every large n.) (1) implies that there is a positive real number γ, $\beta < \gamma < \alpha$ such that

$$a_n \geq 2^{(3+\gamma)^n} \tag{3}$$

holds for every large n. Let c be a positive integer such that for every $n > c$ (1) and (2) hold. Then we take a positive integer B such that for every $n \leq c$, $a_n < 2^{(3+\gamma)^B}$. Let the number of a_n such that $a_n < 2^{(3+\gamma)^n}$ be equal to s. The inequality (1) then implies, that there is a positive integer N such that the number of a_n satisfying $a_n \in \langle 2^{(3+\gamma)^B}, 2^{(3+\gamma)^N} \rangle$ is less then or equal to $N-B-s-1$. The number of intervals $\langle 2^{(3+\gamma)^B}, 2^{(3+\gamma)^{B+1}} \rangle, \dots, \langle 2^{(3+\gamma)^{(N-1)}}, 2^{(3+\gamma)^N} \rangle$ is $N-B$. Thus there is a smallest positive integer M, $B < M \leq N$, such that the number of a_n satisfying $a_n \in \langle 2^{(3+\gamma)^B}, 2^{(3+\gamma)^M} \rangle$ is less then $M-B-s$. Because the number M is the smallest number fulfilling the above assumption, for every positive integer K $(B < K < M)$, the number of integers a_n such that $a_n \in \langle 2^{(3+\gamma)^K}, 2^{(3+\gamma)^M} \rangle$ is less then or equal to $M-K-1$. Thus, there is no a_n contained in $\langle 2^{(3+\gamma)^{M-1}}, 2^{(3+\gamma)^M} \rangle$. These conditions imply

$$\begin{aligned}\prod_{a_n \in \langle 0, 2^{(3+\gamma)^{M-1}} \rangle} a_n &= \prod_{a_n \in \langle 0, 2^{(3+\gamma)^B} \rangle} a_n \prod_{a_n \in \langle 2^{(3+\gamma)^B}, 2^{(3+\gamma)^{M-1}} \rangle} a_n \\ &\leq 2^{s(3+\gamma)^B} 2^{\sum_{j=B+s}^{M-1} (3+\gamma)^j} \leq 2^{\sum_{j=B}^{M-1} (3+\gamma)^j} \leq 2^{(3+\gamma)^M/(2+\gamma)}\,.\end{aligned} \tag{4}$$

On the other hand, if B is large enough, then

$$\begin{aligned}\sum_{a_n > 2^{(3+\gamma)^{M-1}}} b_n/a_n &\leq \sum_{\substack{a_n > 2^{(3+\gamma)^{M-1}} \\ n \leq M}} b_n/a_n + \sum_{n>M} b_n/a_n \\ &\leq M 2^{(3+\beta)^M - (3+\gamma)^M} + \sum_{n=M}^{\infty} 2^{(3+\beta)^n - (3+\alpha)^n} \\ &\leq 2^{2(3+\beta)^M - (3+\gamma)^M}\end{aligned} \tag{5}$$

holds. If

$$p/q = \sum_{a_n < 2^{(3+\gamma)^{M-1}}} b_n/a_n\,,$$

then, from (4) and (5), it follows

$$|H - p/q| = \sum_{a_n \geq 2^{(3+\gamma)^{M-1}}} b_n/a_n \leq 2^{2(3+\beta)^M - (3+\gamma)^M} \leq q^{-2-\varepsilon}\,,$$

where $0 < \varepsilon < \gamma$. If we now apply Roth's theorem (see, e.g., [7]), we obtain the transcendency of the number H. □

COROLLARY. *The sequence* $\{2^{4^n}\}_{n=1}^{\infty}$ *is transcendental.*

Remark. The problem remains open whether $\{2^{3^n}\}_{n=1}^{\infty}$ is a transcendental sequence.

REFERENCES

[1] BUNDSCHUH, P.: *A criterion for algebraic independence with some applications*, Osaka J. Math. **25** (1988), 849–858.

[2] ERDŐS, P.: *Some problems and results on the irrationality of the sums of infinite series*, J. Math. Sci. **10** (1975), 1–7.

[3] ERDŐS, P.: *On the irrationality of certain series, problems and results.* In: New Advances in Transcendence Theory (A. Baker, ed.), Cambridge Univ. Press, London-New York, 1988, pp. 102–109.

[4] ERDŐS, P.—GRAHAM, L.: *Old and New Problems in Combinatorial Number Theory.* Monographies de L'Enseignement Math., Université de Genève, Geneva, 1980.

[5] HANČL, J.: *Criterion for irrational sequences*, J. Number Theory **43** (1993), 88–92.

[6] KOSTRA, J.: *On sums of two units*, Abh. Math. Sem. Univ. Hamburg **64** (1994), 11–14.

[7] ROTH, K. F.: *Rational approximations to algebraic numbers*, Mathematika **2** (1955), 1–20.

Received November 7, 1994

Department of Mathematics
University of Ostrava
Bráfova 7
CZ–701 03 Ostrava 1
Czech Republic
E-mail: hancl@osu.cz