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SOME NEW SEQUENCE SPACES DEFINED BY A MODULUS FUNCTION

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ABSTRACT. In this paper we introduce and examine some properties of new sequence spaces defined using a modulus function.

Introduction

Let w denote the set of all complex sequences $x = (x_k)$. Let $p = (p_k)$ be a sequence of real numbers such that $p_k > 0$ for all k and $\sup_k p_k = H < \infty$. This assumption is made throughout the rest of this paper.

Let l_∞ be the set of all real or complex sequences $x = (x_k)$ with the norm $\|x\| = \sup_k |x_k| < \infty$. A linear functional L on l_∞ is said to be a *Banach limit* (Banach [1]) if it has the properties:

(i) $L(x) \ge 0$ if $x \ge 0$, that is when the sequence $x = (x_k)$ has $x_k \ge 0$ for all k,

(ii) $L(e) = 1$, where $e = (1,1,1,\dots)$,

(iii) $L(\mathrm{D}x) = L(x)$, where the shift operator D is defined by $(\mathrm{D}x)_n = x_{n+1}$.

Let B be the set of all Banach limits on l_∞. A sequence x is said to be *almost convergent to a number* s if $L(x) = s$ for all $L \in B$. Let $\hat{c}$ denote the set of all almost convergent sequences. Lorentz [2] proved that

$$\hat{c} = \left\{ x : \lim_k \frac{1}{k+1} \sum_{i=0}^{k} x_{m+i} \text{ exists, uniformly in } m \right\}.$$

Ruckle [3], used the idea of a modulus function f (see Definition 1 below) to construct the sequence space

$$L(f) = \left\{ x \in w : \sum_k f(|x_k|) \right\}.$$

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This space is an FK-space and Ruckle proved that the intersection of all such $L(f)$ spaces is Φ, where Φ denotes the space of all finite sequences.

In the present note we introduce some new sequence spaces by using a modulus function f and examine some properties of these sequence spaces.

Main results

DEFINITION 1. ([3]) A function $f\colon [0,\infty) \to [0,\infty)$ is called a *modulus* if

(i) $f(x) = 0$ if and only if $x = 0$,
(ii) $f(x+y) \le f(x) + f(y)$,
(iii) f is increasing,
(iv) f is continuous from the right at 0.

DEFINITION 2. Let f be a modulus and $A = (a_{nk})$ be a nonnegative matrix. We define

$$[w_0(A,p,f,s)] = \Big\{x \in w:\ \lim_n \sum_k a_{nk} k^{-s} \big[f\big(|t_{km}(x)|\big)\big]^{p_k} = 0,\ s \ge 0,\ \text{uniformly in } m\Big\},$$

$$[w(A,p,f,s)] = \Big\{x \in w:\ \lim_n \sum_k a_{nk} k^{-s} \big[f\big(|t_{km}(x - Le)|\big)\big]^{p_k} = 0,\ s \ge 0,\ \text{uniformly in } m \text{ for some } L\Big\},$$

$$[w_\infty(A,p,f,s)] = \Big\{x \in w:\ \sup_{n,m} \sum_k a_{nk} k^{-s} \big[f\big(|t_{km}(x)|\big)\big]^{p_k} < \infty,\ s \ge 0\Big\},$$

where $e = (1,1,1,\dots)$.

When $f(x) = x$, we have the following sequence space:

$$[w(A,p,f,s)] = \Big\{x \in w:\ \lim_n \sum_k a_{nk} k^{-s} |t_{km}(x - Le)|^{p_k} = 0,\ \text{for some } L,\ s \ge 0,\ \text{uniformly in } m\Big\}.$$

When $A = (a_{nk}) = (C,1)$ Cesaro matrix, $s = 0$ and $f(x) = x$ in the space $[w(A,p,f,s)]$, we have the following sequence space which is a generalization of the sequence space $[w(p)]$ which was defined by Das and Sahoo [4]:

$$[w(p)] = \Big\{x \in w:\ \lim_n \frac{1}{n} \sum_{k=1}^{n} |t_{km}(x - Le)|^{p_k} = 0,\ \text{uniformly in } m\Big\}.$$

When $A = (a_{nk}) = (C,1)$ Cesaro matrix, $s = 0$ and $p_k = 1$ for all k, we have the following sequence spaces, which were defined by E s i [5]:

$$[w,f]_0 = \left\{ x \in w : \lim_n \frac{1}{n} \sum_{k=1}^{n} f\big(|t_{km}(x)|\big) = 0 , \text{ uniformly in } m \right\},$$

$$[w,f] = \left\{ x \in w : \lim_n \frac{1}{n} \sum_{k=1}^{n} f\big(|t_{km}(x - Le)|\big) = 0 , \text{ uniformly in } m \right\},$$

$$[w,f]_\infty = \left\{ x \in w : \sup_{n,m} \frac{1}{n} \sum_{k=1}^{n} f\big(|t_{km}(x)|\big) < \infty \right\}.$$

We now establish a number of useful theorems.

THEOREM 1. *$[w_0(A,p,f,s)]$, $[w(A,p,f,s)]$ and $[w_\infty(A,p,f,s)]$ are linear spaces over the complex field $\mathbb{C}$.*

P r o o f . We consider only $[w_0(A,p,f,s)]$. The others can be treated similarly. We have

$$|x_k + y_k|^{p_k} \le C\big(|x_k|^{p_k} + |y_k|^{p_k}\big) , \tag{1}$$

where $C = \max(1, 2^{H-1})$.

Let $x, y \in [w_0(A,p,f,s)]$. For λ, $\mu \in \mathbb{C}$, there exist integers T and K such that $|\lambda| \le T$ and $|\mu| \le K$. From Definition 1(ii) and (1), we write

$$\begin{aligned} &\sum_k a_{nk} k^{-s} \big[f\big(|t_{km}(\lambda x + \mu y)|\big)\big]^{p_k} \\ &\le C \cdot T^H \sum_k a_{nk} k^{-s} \big[f\big(|t_{km}(x)|\big)\big]^{p_k} + C \cdot K^H \sum_k a_{nk} k^{-s} \big[f\big(|t_{km}(y)|\big)\big]^{p_k} . \end{aligned}$$

For $n \to \infty$, since $x, y \in [w_0(A,p,f,s)]$, we have $\lambda x + \mu y \in [w_0(A,p,f,s)]$. Thus $[w_0(A,p,f,s)]$ is linear space over $\mathbb{C}$. □

THEOREM 2. *Let A be a nonnegative regular matrix and f be a modulus, then*

$$[w_0(A,p,f,s)] \subset [w(A,p,f,s)] \subset [w_\infty(A,p,f,s)] .$$

P r o o f . The first inclusion is trivial. We now show that $[w(A,p,f,s)] \subset [w_\infty(A,p,f,s)]$. Let $x \in [w(A,p,f,s)]$. By Definition 1(ii) and (1),

$$\begin{aligned} &\sum_k a_{nk} k^{-s} \big[f\big(|t_{km}(x)|\big)\big]^{p_k} \\ &= \sum_k a_{nk} k^{-s} \big[f\big(|t_{km}(x - Le + Le)|\big)\big]^{p_k} \\ &\le C \sum_k a_{nk} k^{-s} \big[f\big(|t_{km}(x - Le)|\big)\big]^{p_k} + C \sum_k a_{nk} k^{-s} \big[f\big(|L|\big)\big]^{p_k} . \end{aligned}$$

There exists an integer K_L such that $|L| \le K_L$. Hence we have

$$\sum_k a_{nk} k^{-s} \left[f\left(|t_{km}(x)|\right)\right]^{p_k}$$
$$\le C \sum_k a_{nk} k^{-s} \left[f\left(|t_{km}(x - Le)|\right)\right]^{p_k} + C\left[K_L f(1)\right]^H \sum_k a_{nk} k^{-s} .$$

Since A is regular and $x \in \left[w(A,p,f,s)\right]$, we get $x \in \left[w_\infty(A,p,f,s)\right]$ and this completes the proof. □

THEOREM 3. *Let A be a nonnegative regular matrix and $M = \max(1,H)$. $\left[w_0(A,p,f,s)\right]$ and $\left[w(A,p,f,s)\right]$ are complete linear topological spaces paranormed by G, where*

$$G(x) = \sup_{n,m} \left(\sum_k a_{nk} k^{-s} \left[f\left(|t_{km}(x)|\right)\right]^{p_k} \right)^{\frac{1}{M}} .$$

Proof. From Theorem 2, $G(x)$ exists for each $x \in \left[w(A,p,f,s)\right]$. Clearly $G(0) = 0$, $G(x) = G(-x)$, where $0 = (0,0,0,\dots)$. By Minkowski's inequality,

$$\left(\sum_k a_{nk} k^{-s} \left[f\left(|t_{km}(x+y)|\right)\right]^{p_k} \right)^{\frac{1}{M}}$$
$$\le \left(\sum_k a_{nk} k^{-s} \left[f\left(|t_{km}(x)|\right)\right]^{p_k} \right)^{\frac{1}{M}} + \left(\sum_k a_{nk} k^{-s} \left[f\left(|t_{km}(y)|\right)\right]^{p_k} \right)^{\frac{1}{M}} ,$$

whence we obtain that $G(x+y) \le G(x) + G(y)$. We now show that the scalar multiplication is continuous. From this $\lambda \to 0$, $x \to 0$ imply $G(\lambda x) \to 0$ and also $x \to 0$, λ fixed imply $G(\lambda x) \to 0$. We now show that $\lambda \to 0$, x fixed imply $G(\lambda x) \to 0$.

Let $x \in \left[w(A,p,f,s)\right]$, then as $n \to \infty$,

$$S_{mn} = \sum_k a_{nk} k^{-s} \left[f\left(|t_{km}(x - Le)|\right)\right]^{p_k} \to 0, \quad \text{uniformly in } m.$$

For $|\lambda| < 1$, we have

$$\left(\sum_k a_{nk} k^{-s} \left[f\left(|t_{km}(\lambda x)|\right)\right]^{p_k} \right)^{\frac{1}{M}}$$
$$= \left(\sum_k a_{nk} k^{-s} \left[f\left(|t_{km}(\lambda x - \lambda L + \lambda L)|\right)\right]^{p_k} \right)^{\frac{1}{M}}$$
$$\le \left(\sum_k a_{nk} k^{-s} \left[f\left(|t_{km}(\lambda x - \lambda L)|\right) + f\left(|t_{km}(\lambda L)|\right)\right]^{p_k} \right)^{\frac{1}{M}} .$$

By Minkowski's inequality

$$\Big(\sum_k a_{nk}k^{-s}\big[f\big(|t_{km}(\lambda x)|\big)\big]^{p_k}\Big)^{\frac{1}{M}}$$
$$\le \Big(\sum_k a_{nk}k^{-s}\big[f\big(|t_{km}(\lambda x-\lambda L)|\big)\big]^{p_k}\Big)^{\frac{1}{M}} + \Big(\sum_k a_{nk}k^{-s}\big[f\big(|t_{km}(\lambda L)|\big)\big]^{p_k}\Big)^{\frac{1}{M}}$$
$$\le \Big(\sum_{k>N} a_{nk}k^{-s}\big[f\big(|t_{km}(\lambda x)|\big)\big]^{p_k}\Big)^{\frac{1}{M}} + \Big(\sum_{k\le N} a_{nk}k^{-s}\big[f\big(|t_{km}(\lambda x)|\big)\big]^{p_k}\Big)^{\frac{1}{M}}$$
$$+ \Big(\sum_k a_{nk}k^{-s}\big[f\big(|t_{km}(\lambda L)|\big)\big]^{p_k}\Big)^{\frac{1}{M}}.$$

Let $\varepsilon > 0$ and choose N such that for each n, m and $k > N$ implies $S_{mn} < \varepsilon/2$. For each N, by continuity of f, as $\lambda \to 0$,

$$\Big(\sum_{k\le N} a_{nk}k^{-s}\big[f\big(|t_{km}(\lambda x)|\big)\big]^{p_k}\Big)^{\frac{1}{M}} + \Big(\sum_k a_{nk}k^{-s}\big[f\big(|t_{km}(\lambda L)|\big)\big]^{p_k}\Big)^{\frac{1}{M}} \to 0.$$

Then choose $\delta < 1$ such that $|\lambda| < \delta$ implies

$$\Big(\sum_{k\le N} a_{nk}k^{-s}\big[f\big(|t_{km}(\lambda x)|\big)\big]^{p_k}\Big)^{\frac{1}{M}} + \Big(\sum_k a_{nk}k^{-s}\big[f\big(|t_{km}(\lambda L)|\big)\big]^{p_k}\Big)^{\frac{1}{M}} < \frac{\varepsilon}{2}.$$

Hence we have

$$\Big(\sum_k a_{nk}k^{-s}\big[f\big(|t_{km}(\lambda x)|\big)\big]^{p_k}\Big)^{\frac{1}{M}} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

and $G(\lambda x) \to 0$ $(\lambda \to 0)$. Thus $\big[w(A,p,f,s)\big]$ is paranormed linear topological space by G.

Now, we show that $\big[w(A,p,f,s)\big]$ is complete with respect to its paranorm topology.

Let (x^i) be a Cauchy sequence in $\big[w(A,p,f,s)\big]$. Then we write $G(x^i - x^j) \to 0$, $i,j \to \infty$. i.e., as $i,j \to \infty$, for all n and m, we write

$$G(x^i - x^j) = \sup_{n,m}\Big(\sum_k a_{nk}k^{-s}\big[f\big(|t_{km}(x^i - x^j)|\big)\big]^{p_k}\Big)^{\frac{1}{M}} \to 0. \qquad (2)$$

Hence for each n, m and k, as $i,j \to \infty$, we have

$$k^{-s}\big[f\big(|t_{km}(x^i - x^j)|\big)\big]^{p_k} \to 0$$

and by continuity of f

$$\lim_{i,j\to\infty} k^{-s}\left[f\left(|t_{km}(x^i - x^j)|\right)\right]^{p_k} = k^{-s}\left[f\left(\lim_{i,j\to\infty}|t_{km}(x^i - x^j)|\right)\right]^{p_k}.$$

It follows that

$$\lim_{i,j\to\infty}|t_{km}(x^i - x^j)| = 0$$

for each k and m. In particular

$$\lim_{i,j\to\infty}|t_{0m}(x^i - x^j)| = \lim_{i,j\to\infty}|(x^i - x^j)| = 0$$

for each fixed m. Hence (x^i) is a Cauchy sequence in $\mathbb{C}$. Since $\mathbb{C}$ is complete, there exists $x \in \mathbb{C}$ such that $x^i \to x$ coordinatewise as $i \to \infty$. It follows from (2) that given $\varepsilon > 0$, there exists i_0 such that

$$\left(\sum_k a_{nk}k^{-s}\left[f\left(|t_{km}(x^i - x^j)|\right)\right]^{p_k}\right)^{\frac{1}{M}} < \varepsilon \tag{3}$$

for all n, m and $i,j > i_0$. Since for any fixed natural number U, we have from (3),

$$\left(\sum_{k\le U} a_{nk}k^{-s}\left[f\left(|t_{km}(x^i - x^j)|\right)\right]^{p_k}\right)^{\frac{1}{M}} < \varepsilon \tag{4}$$

for all n, m and $i,j > i_0$, by taking $j \to \infty$ in the above expression we obtain

$$\left(\sum_{k\le U} a_{nk}k^{-s}\left[f\left(|t_{km}(x^i - x)|\right)\right]^{p_k}\right)^{\frac{1}{M}} < \varepsilon$$

for all n, m and $i > i_0$. Since U is arbitrary, by letting $U \to \infty$ we obtain

$$\left(\sum_k a_{nk}k^{-s}\left[f\left(|t_{km}(x^i - x)|\right)\right]^{p_k}\right)^{\frac{1}{M}} < \varepsilon$$

for all n, m and $i > i_0$, that is $G(x^i - x) \to 0$ as $i \to \infty$, and thus $x^i \to x$ as $i \to \infty$.

Also, for each i, there exists L^i with

$$\sum_k a_{nk}k^{-s}\left[f\left(|t_{km}(x^i - L^i e)|\right)\right]^{p_k} \to 0 \qquad (n\to\infty) \tag{5}$$

uniformly in m. From the regularity of A, Definition 1(ii) and (5), we have $f\left(|L^i e - L^j e|\right) \to 0$ as $i,j \to \infty$ and (L^i) is a Cauchy sequence in $\mathbb{C}$. So (L^i) converges, say, to L. Consequently we get

$$\sum_k a_{nk}k^{-s}\left[f\left(|t_{km}(x - Le)|\right)\right]^{p_k} \to 0 \qquad (n\to\infty)$$

uniformly in m. So that $x \in \left[w(A,p,f,s)\right]$ and the space is complete. □

Using the same technique of Theorem 4 of Maddox [6], it is easy to prove the following theorem.

THEOREM 4. *Let A be a nonnegative regular matrix,* $\inf p_k > 0$ *and f be a modulus, then*

$$[w_0(A,p,s)] \subset [w_0(A,p,f,s)],$$
$$[w(A,p,s)] \subset [w(A,p,f,s)],$$
$$[w_\infty(A,p,s)] \subset [w_\infty(A,p,f,s)].$$

THEOREM 5. *Let A be a nonnegative regular matrix,* $\inf p_k > 0$ *and f be a modulus. If $\beta = \lim_t (f(t)/t) > 0$ then,* $[w(A,p,s)] = [w(A,p,f,s)]$.

Proof. In Theorem 4, it was shown that $[w(A,p,s)] \subset [w(A,p,f,s)]$. We must show that $[w(A,p,f,s)] \subset [w(A,p,s)]$. For any modulus function, the existence of a positive limit for given β is proved in Maddox [7; Proposition 1]. Now, let $\beta > 0$ and let $x \in [w(A,p,f,s)]$. Since $\beta > 0$, for every $t > 0$, we write $f(t) \geq \beta t$. From this inequality, it is easy to see that $x \in [w(A,p,s)]$. This completes the proof. □

Some information on multipliers for $[w_\infty(A,p,f,s)]$ is given in Theorem 6 (i). For any set E of sequences, we denote by $M(E)$ the space $\{a \in w : a \cdot x \in E$ for $x \in E\}$.

THEOREM 6. *Let A be a nonnegative regular matrix and f be a modulus, then*

(i) $l_\infty \subset M([w_\infty(A,p,f,s)]) \subset [w_\infty(A,p,f,s)]$,

(ii) $\inf p_k > 0$ *and* $x_k \to L$ *imply* $x_k \to L[w(A,p,f,s)]$,

(iii) $s_1 \leq s_2$ *implies* $[w(A,p,f,s_1)] \subset [w(A,p,f,s_2)]$.

Proof.

(i) Let a $\in l_\infty$. This implies $|a_k| \leq K$ for some $K > 0$ and all k. Hence $x \in [w_\infty(A,p,f,s)]$ implies

$$\begin{aligned}\sum_k a_{nk} k^{-s} [f(|t_{km}(ax)|)]^{p_k} &\leq \sum_k a_{nk} k^{-s} [K f(|t_{km}(x)|)]^{p_k} \\ &\leq K^H \sum_k a_{nk} k^{-s} [f(|t_{km}(x)|)]^{p_k}\end{aligned}$$

which gives the first inclusion. The second inclusion follows from the fact $e = (1,1,1,\dots) \in [w_\infty(A,p,f,s)]$.

(ii) Suppose that $x_k \to L$ as $k \to \infty$. This implies $t_{km}(x) \to L$ as $k \to \infty$ uniformly in m. Since f is modulus then

$$\lim_{k\to\infty} [f(|t_{km}(x) - L|)] = f\Big[\lim_{k\to\infty} (|t_{km}(x) - L|)\Big] = 0$$

uniformly in m. Since $\inf p_k = h > 0$ then,

$$\lim_{k\to\infty} \left[f\left(|t_{km}(x) - L|\right)\right]^h = 0$$

uniformly in m. So, for $0 < \varepsilon < 1$, $\exists k_0 \in \mathbb{N}$ for all $k > k_0$ and for all m,

$$\left[f\left(|t_{km}(x) - L|\right)\right]^h < \varepsilon < 1$$

and since $p_k \geq h$ for all k,

$$\left[f\left(|t_{km}(x) - L|\right)\right]^{p_k} \leq \left[f\left(|t_{km}(x) - L|\right)\right]^h < \varepsilon$$

then we get

$$\lim_{k\to\infty} \left[f\left(|t_{km}(x) - L|\right)\right]^{p_k} = 0$$

uniformly in m. Since (k^{-s}) is bounded, we write

$$\lim_{k\to\infty} k^{-s}\left[f\left(|t_{km}(x) - L|\right)\right]^{p_k} = 0$$

uniformly in m. From regularity of A, we have

$$\lim_{k\to\infty} \sum_k a_{nk} k^{-s}\left[f\left(|t_{km}(x) - L|\right)\right]^{p_k} = 0$$

uniformly in m. So that $x \in \left[w(A,p,f,s)\right]$.

(iii) Let $s_1 \leq s_2$. Then $k^{-s_2} < k^{-s_1}$ for all $k \in \mathbb{N}$. Since

$$k^{-s_2}\left[f\left(|t_{km}(x) - L|\right)\right]^{p_k} \leq k^{-s_1}\left[f\left(|t_{km}(x) - L|\right)\right]^{p_k}$$

for all k and m. Hence we have

$$\sum_k a_{nk} k^{-s_2}\left[f\left(|t_{km}(x) - L|\right)\right]^{p_k} \leq \sum_k a_{nk} k^{-s_1}\left[f\left(|t_{km}(x) - L|\right)\right]^{p_k}.$$

Since $x \in \left[w(A,p,f,s_1)\right]$, we get $x \in \left[w(A,p,f,s_2)\right]$. □

THEOREM 7. *Let f and g be two moduli, then*

(i) $\lim_{k\to\infty} \frac{f(x)}{g(x)}$ *implies* $\left[w(A,p,g,s)\right] \subset \left[w(A,p,f,s)\right]$,

(ii) $\left[w(A,p,g,s)\right] \cap \left[w(A,p,f,s)\right] \subset \left[w(A,p,f+g,s)\right]$,

Proof. This is trivial. □

REFERENCES

[1] BANACH, S.: *Théorie des operations linéaries*, Chelsea, New York, 1932.

[2] LORENTZ, G. G.: *A contribution to the theory of divergent sequences*, Acta Math. **80** (1948), 167–190.

[3] RUCKLE, W. H.: *FK spaces in which the sequence of coordinate vectors in bounded*, Canad. J. Math. **25** (1973), 973–978.

[4] DAS, G.—SAHOO, S. K.: *On some sequence spaces*, J. Math. Anal. Appl. **164** (1992), 381–388.

[5] ESI, A.: *Some new Sequence Spaces Defined by a Modulus and Statistical Convergence.* PhD Thesis, Fırat University, Graduate School of Natural and Applied Sciences, Department of Mathematics, 1995 (Unpublished).

[6] MADDOX, I. J.: *Sequence spaces defined by a modulus*, Math. Proc. Cambridge Philos. Soc. **100** (1986), 161–166.

[7] MADDOX, I. J.: *Inclusion between FK spaces and Kuttner's theorem*, Math. Proc. Cambridge Philos. Soc. **101** (1987), 523–527.

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