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Mathematica Slovaca, Vol. 49 (1999), No. 4, 417--424

Persistent URL: http://dml.cz/dmlcz/136757

Math. Slovaca, **49** (1999), No. 4, 417–424

Mathematica Slovaca

Dedicated to the memory
of Professor Štefan Schwarz

RADICAL CLASSES OF COMPLETE LATTICE ORDERED GROUPS

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(*Communicated by Tibor Katriňák*)

ABSTRACT. In this paper the relations between radical classes of generalized Boolean algebras and radical classes of complete lattice ordered groups are investigated.

The notion of a radical class of lattice ordered groups was introduced and studied in [7]; cf. also [1], [2], [3], [8], [9], [11].

Radical classes of generalized Boolean algebras were dealt with in [10].

Let $\mathcal{R}$ and $\mathcal{A}$ be the collection of all radical classes of lattice ordered groups or the collection of all radical classes of generalized Boolean algebras, respectively. Both these collections are partially ordered by the class-theoretical inclusion.

We denote by $\mathcal{R}_c$ the class of all $X \in \mathcal{R}$ such that, whenever $G \in X$, then G is a complete lattice ordered group. Then, in fact, $\mathcal{R}_c$ is an interval of $\mathcal{R}$ containing the least element of $\mathcal{R}$.

Further let $\mathcal{A}_c$ be the class of all $Y \in \mathcal{A}$ such that, whenever L is a member of Y, then each interval of L is a complete lattice.

In the present paper the following result will be proved:

(**A**) There exists an isomorphism φ of the partially ordered collection $\mathcal{A}$ into $\mathcal{R}$ such that

(i) $\varphi(\mathcal{A})$ is a convex subcollection of $\mathcal{R}$;

(ii) $\varphi(\mathcal{A})$ contains the least element of $\mathcal{R}$;

(iii) $\varphi(\mathcal{A}_c) \subseteq \mathcal{R}_c$.

AMS Subject Classification (1991): Primary 06F20, 06E99.
Key words: complete lattice ordered group, generalized Boolean algebra, radical class.

Supported by grant GA SAV 1230/96.

In [7; Proof of Proposition 3.3] a proper class of atoms of $\mathcal{R}$ was constructed. No member of this class belongs to $\mathcal{R}_c$. The question arises whether there exists a proper class of atoms of $\mathcal{R}$ which belong to $\mathcal{R}_c$.

By applying (**A**) and the results of [10] we prove that the answer to this question is positive.

1. Preliminaries

We recall shortly the basic relevant notions. For lattice ordered groups we apply the usual terminology and notation; cf., e.g., [4].

Let $\mathcal{G}$ be the class of all lattice ordered groups. For $G \in \mathcal{G}$ we denote by $C(G)$ the system of all convex ℓ-subgroups of G; this system is partially ordered by the set-theoretical inclusion. Then $C(G)$ is a complete lattice. It is obvious that the operation $\wedge$ in $C(G)$ coincides with the set-theoretical operation $\cap$.

1.1. Definition. A subclass $X \neq \emptyset$ of $\mathcal{G}$ is said to be a *radical class of lattice ordered groups* if it satisfies the following conditions:

1) X is closed with respect to isomorphisms;
2) if $G_1 \in X$ and $G_2 \in C(G_1)$, then $G_2 \in X$;
3) if $G \in \mathcal{G}$, $\emptyset \neq \{G_i\}_{i\in I} \subseteq X \cap C(G)$, then $\bigvee_{i\in I} G_i \in X$.

A lattice L with the least element 0 is called a *generalized Boolean algebra* if, whenever $x \in L$, then the interval $[0, x]$ of L is a Boolean algebra. A sublattice L_1 of a generalized Boolean algebra L will be said to be a *subalgebra* of L if $0 \in L_1$. Let $C(L)$ be the set of all convex subalgebras of L with the partial order defined analogously as in $C(G)$ above. Then $C(L)$ is a complete lattice.

Let $\mathcal{B}$ be the class of all generalized Boolean algebras. The following definition is analogous to 1.1.

1.2. Definition. A nonempty subclass Y of $\mathcal{B}$ is said to be a *radical class of generalized Boolean algebras* if the following conditions are satisfied:

1') Y is closed with respect to isomorphisms;
2') if $L_1 \in Y$ and $L_2 \in C(L_1)$, then $L_2 \in Y$;
3') if $L \in \mathcal{B}$, $\emptyset \neq \{L_i\}_{i\in I} \subseteq Y \cap C(L)$, then $\bigvee_{i\in I} L_i \in Y$.

The meaning of the symbols $\mathcal{R}$, $\mathcal{R}_c$, $\mathcal{A}$ and $\mathcal{A}_c$ is as in the introduction above.

2. Carathéodory functions

Let G be a lattice ordered group.

An element $0 \leqq x \in G$ is called *singular* if, whenever, y is an element of G with $0 \leqq y \leqq x$, then $y \wedge (x - y) = 0$. (Cf. [4].) An equivalent definition consists in the condition that the interval $[0, x]$ of G is a Boolean algebra. The lattice ordered group G is said to be singular if for each $0 < z \in G$ there exists $x \in G$ such that x is singular and $0 < x \leqq z$. We denote by $S(G)$ the set of all singular elements of G.

For a Boolean algebra B we denote by $E(B)$ the vector lattice of all elementary Carathéodory functions on B (cf. [5], [6]); we recall the definition of $E(B)$ as given in [6]. $E(B)$ is constructed as follows: it is the set of all forms

$$f = a_1 b_1 + \cdots + a_n b_n \,, \tag{1}$$

where $a_i \neq 0$ are reals, $b_i \in B$, $b_i > 0$, $b_{i(1)} \wedge b_{i(2)} = 0$ for any distinct $i(1), i(2) \in \{1, 2, \dots, n\}$, and of the "empty form". If g is another such form,

$$g = a'_1 b'_1 + \cdots + a'_m b'_m \,,$$

then f, g are considered as equal if $\bigvee\limits_{i=1}^{n} b_i = \bigvee\limits_{j=1}^{m} b'_j$ and if $a_i = a'_j$ whenever $b_i \wedge b'_j \neq 0$. For $b, b' \in B$ let $b -_1 b'$ be the relative complement of $b \wedge b'$ in the interval $[0, b]$. The operation $+$ in $E(B)$ is defined by

$$f + g = \sum_{i=1}^{n} \sum_{j=1}^{m} (a_i + a'_j)(b_i \wedge b'_j) + \sum_{i=1}^{n} a_i \left(b_i -_1 \left(\bigvee_{j=1}^{m} b'_j \right) \right) + \sum_{j=1}^{m} a'_j \left(b'_j -_1 \left(\bigvee_{i=1}^{n} b_i \right) \right),$$

where in the summations only those terms are taken into account in which $a_i + a'_j \neq 0$ and the elements $b_i \wedge b'_j$, $b_i -_1 \left(\bigvee\limits_{j=1}^{m} b'_j \right)$, $b'_j -_1 \left(\bigvee\limits_{i=1}^{n} b_i \right)$ are non-zero. The multiplication by a real $a \neq 0$ is defined by $af = (aa_1)b_1 + \cdots + (aa_n)b_n$, and if f is the empty form, then af is the empty form as well. Also, $0f$ is the empty form for each $f \in E(B)$. The form (1) is strictly positive if $a_i > 0$ for $i = 1, 2, \dots, n$.

Now let us denote by $E_0(B)$ the subset of $E(B)$ consisting of the empty form f_0 and of all forms (1) such that all a_i are integers. Then $E_0(B)$ is a lattice ordered group.

For each $0 \neq b \in B$ the element b will be identified with the form $f = 1b$; further, the element 0 of B will be identified with f_0. Under this convention, B is a subset of $E_0(B)$.

The following lemma is easy to verify; the proof will be omitted.

2.1. Lemma. *Let B be a Boolean algebra. Then $S(E_0(B)) = B$.*

2.2. Lemma. *Let B be a Boolean algebra, $b_1 \in B$, and let B_1 be the interval $[0, b_1]$ of B. Then $E_0(B_1)$ is a convex ℓ-subgroup of $E_0(B)$.*

Proof. This is an immediate consequence of the definition of $E_0(B)$. □

Let L be a generalized Boolean algebra. For each $b \in L$ we construct the lattice ordered group $E_0([0, b])$. Put

$$E_0'(L) = \bigcup_{b \in L} E_0([0, b]) .$$

If L is a Boolean algebra, then in view of 2.2 we have $E_0'(L) = E_0(L)$.

Let $x, y \in E_0'(L)$ and let $\circ \in \{+, \wedge, \vee\}$. There exist $b_1, b_2 \in L$ such that $x \in E_0([0, b_1])$ and $y \in E_0([0, b_2])$. Hence $x, y \in E_0([0, b_1 \vee b_2])$. We put

$$x \circ y = x \circ_{12} y ,$$

where $\circ_{12}$ is the corresponding operation in the lattice ordered group $E_0([0, b_1 \vee b_2])$. By means of 2.2 we easily verify that $\circ$ is a correctly defined operation in $E_0'(L)$. Hence $E_0'(L)$ is a lattice ordered group and all $E_0([0, b])$ are convex ℓ-subgroups of $E_0'(L)$. From this and from 2.1 we obtain:

2.3. Lemma. $S(E_0'(L)) = L$.

Further, 2.2 yields:

2.3.1. Lemma. *Let $L_1 \in \mathcal{B}$ and let $L_2 \in C(L_1)$. Then $E_0'(L_2)$ is a convex ℓ-subgroup of $E_0'(L_1)$.*

2.4. Lemma. *Let B be a complete Boolean algebra. Then the lattice ordered group $E_0(B)$ is complete.*

Proof. Let $Z \neq \emptyset$ be an upper bounded subset of $(E_0(B))^+$. We have to verify that Z has the supremum in $E_0(B)$. The case $Z = \{0\}$ being trivial, it suffices to suppose that $Z \neq \{0\}$. Hence there is $f \in E_0(B)$ such that $f > 0$ and $z \leqq f$ for each $z \in Z$. Let f be as in (1). Put

$$b_0 = b_1 \vee b_2 \vee \cdots \vee b_n , \qquad a_0 = \max\{a_1, a_2, \ldots, a_n\} .$$

Then $a_0 b_0 \in E_0(B)$ and $Z \subseteq [0, a_0 b_0]$. Since B is complete, the lattice $[0, b_0]$ is complete as well. In view of the fact that the class of all complete lattice ordered groups is a radical class (cf. [7]) we infer that the interval $[0, a_0 b_0]$ must be complete. Hence there exists the least upper bound z_0 of Z in $[0, a_0 b_0]$ and clearly z_0 is the supremum of Z in $E_0(B)$. Therefore $E_0(B)$ is a complete lattice ordered group. □

2.5. Lemma. *Suppose that for each $b \in L$, the Boolean algebra $[0,b]$ is complete. Then $E'_0(L)$ is a complete lattice ordered group.*

Proof. Again, it suffices to verify that if X is a nonempty upper bounded subset of $(E'_0(L))^+$, then $\sup X$ does exist in $E'_0(L)$. Let X have the mentioned properties. There is x_0 in $E'_0(L)$ such that x_0 is an upper bound of X. In view of the definition of $E'_0(L)$ there is $b \in L$ such that $x_0 \in E_0([0,b])$. Since $E_0([0,b])$ is a convex ℓ-subgroup of $E'_0(L)$ we infer that $X \subseteq E_0([0,b])$. Now the assertion of the lemma follows from 2.4. □

The following lemma is an immediate consequence of the definition of $E'_0(L)$.

2.6. Lemma. *The lattice ordered group $E'_0(L)$ is generated by its subset L.*

3. A mapping $\mathcal{A} \to \mathcal{R}$

For $Y \in \mathcal{A}$ we put

$$\varphi(Y) = \{E'_0(L) : L \in Y\}.$$

Hence $\varphi(Y) \subseteq \mathcal{G}$. Moreover, if $G \in \varphi(Y)$, then G is abelian.

3.1. Lemma. *$\varphi(Y)$ is closed with respect to isomorphisms.*

Proof. This is a consequence of the fact that Y is closed with respect to isomorphisms. □

3.2. Lemma. *Let $G_1 \in \varphi(Y)$ and let $G_2 \in C(G_1)$. Then $G_2 \in \varphi(Y)$.*

Proof. There exists $L_1 \in Y$ such that $G_1 = E'_0(L_1)$. Hence $L_1 \subseteq G_1$. Moreover, L_1 is a convex subset of G_1^+. Put $L_2 = L_1 \cap G_2$. Then $L_2 \in C(L_1)$, thus $L_2 \in Y$. From 2.6 we obtain that $E'_0(L_2)$ is a convex ℓ-subgroup of G_2. Let $g_2 \in G_2^+$. Then $g_2 \in (E'_0(L_1))^+$, hence g_2 has the form (1) for some $b_i \in L_1$ and some positive integers a_i $(i = 1, 2, \ldots, n)$. But then, in view of convexity of G_2, we have $b_1, \ldots, b_n \in G_2$, thus $b_1, \ldots, b_n \in L_2$ and so $g_2 \in E'_0(L_2)$. Therefore $G_2^+ \subseteq E'_0(L_2)$, implying that $G_2 \subseteq E'_0(L_2)$. Hence $G_2 = E'_0(L_2)$ and $G_2 \in \varphi(Y)$. □

3.3. Lemma. *Let $G \in \mathcal{G}$, $G_i \in \varphi(Y) \cap C(G)$ $(i \in I)$, $\bigvee_{i \in I} G_i = G$. Then $G \in \varphi(Y)$.*

Proof. For each $i \in I$ there exists $L_i \in Y$ such that $G_i = E'_0(L_i)$. Hence $L_i \subseteq G_i$ for each $i \in I$. Put $L = \bigvee_{i \in I} L_i$. Then $L \subseteq G$. Also, L is a generalized Boolean algebra, $L_i \in C(L)$ for each $i \in I$. Hence $L \in Y$. We denote by G_0 the

ℓ-subgroup of G which is generated by the set L. Then $E_0'(L)$ is an ℓ-subgroup of G_0. Further, since G_i are convex in G and L_i are convex in G_i we obtain that G_0 is convex in G. Similarly as in the proof of 3.2 we now conclude that $G_0 = E_0'(L)$.

Let $0 \leqq g \in G$. There are $i(1), \dots, i(n) \in I$ and $g_{i(j)} \in G_{i(j)}^+$ $(j = 1, 2, \dots, n)$ such that $g = g_{i(1)} + \dots + g_{i(n)}$. Then in view of $G_i = E_0'(L_i)$ we get that $g \in E_0'(L)$. Hence $g \in G_0$. Therefore $G = G_0 \in \varphi(Y)$. $\square$

As a consequence of 3.1, 3.2 and 3.3 we obtain:

3.4. Lemma. *$\varphi(Y)$ is a member of $\mathcal{R}$.*

3.5. Lemma. *If all members of Y are conditionally complete, then $\varphi(Y) \subseteq \mathcal{R}_c$.*

P r o o f. This is a consequence of 3.4 and 2.5. $\square$

The definition of φ yields:

3.6. Lemma. *If $Y_1, Y_2 \in \mathcal{A}$, $Y_1 \leqq Y_2$, then $\varphi(Y_1) \leqq \varphi(Y_2)$.*

Let Y be as above. Put $\varphi(Y) = X$. Further, let $X_1 \in \mathcal{R}$, $X_1 \leqq X$. Denote

$$Y_1 = \{L_1 \in \mathcal{B} :\ G_1 = E_0'(L_1) \text{ for some } G_1 \in X_1\}.$$

Then $Y_1 \neq \emptyset$ and it is closed with respect to isomorphisms.

3.7. Lemma. *Let $L_1 \in Y_1$, $L_2 \in C(L_1)$. Then $L_2 \in Y_1$.*

P r o o f. Denote $G_2 = E_0'(L_2)$. From 2.3.1 we obtain that $G_2 \in C(G_1)$, thus $G_2 \in X_1$. Hence $L_1 \in Y_1$. $\square$

3.8. Lemma. *Let $L \in \mathcal{B}$, $L_i \in C(L) \cap Y_1$ $(i \in I)$, $\bigvee_{i \in I} L_i = L$. Then $L \in Y_1$.*

P r o o f. There are $G_i \in X_1$ with $G_i = E_0'(L_i)$ $(i \in I)$. Put $G = E_0'(L)$. Then $G_i \in C(G)$ for each $i \in I$. Denote $G_0 = \bigvee_{i \in I} G_i$. Hence $G_0 \in C(G)$. Let $0 \leqq g \in G$. In view of the definition of G, the element g can be expressed in the form (1), where $b_j \in \bigcup_{i \in I} L_i$ and a_j is a positive integer for $j = 1, 2, \dots, n$. Thus $g \in G_0$ and hence $G_0 = G$. But G_0 belongs to X_1 and therefore L belongs to Y_1. $\square$

3.9. Lemma. *Let Y_1 be as above. Then*

(i) *$Y_1 \in \mathcal{A}$;*
(ii) *$Y_1 \leqq Y$;*
(iii) *$\varphi(Y_1) = X_1$.*

Proof.

(i) is a consequence of 3.7, 3.8 and of the fact that Y_1 is closed with respect to isomorphisms.

The relation (ii) is obvious.

(iii) is implied by the definition of Y_1. □

3.10. LEMMA. *The mapping φ is an isomorphism of $\mathcal{A}$ into $\mathcal{R}$.*

Proof. From the definition of φ and from 2.3 we get that if $Y \in \mathcal{A}$ and $\varphi(Y) = X$, then

$$Y = \{S(G) : G \in X\}.$$

This shows that φ is a monomorphism and that

$$\varphi(Y_1) \leqq \varphi(Y_2) \implies Y_1 \leqq Y_2.$$

Hence in view of 3.6, φ is an isomorphism. □

Proof of (**A**). Let X^0 and Y^0 be the class of all one-element lattice ordered groups or the class of all one-element generalized Boolean algebras, respectively. Then clearly X^0 is the least element of $\mathcal{R}$, Y^0 is the least element of $\mathcal{A}$ and we have $\varphi(Y_0) = X_0$. Hence (ii) from (**A**) holds. The remaining parts of (**A**) were proved in 3.10, 3.9 and 3.5. □

3.11. THEOREM. *There exists a proper class of atoms of $\mathcal{R}$ which belong to $\mathcal{R}_c$.*

Proof. According to [10] there exists a proper class A_1 of atoms of $\mathcal{A}$ such that each member of A_1 belongs to $\mathcal{A}_c$. Put $A_2 = \varphi(A_1)$. In view of Theorem (**A**), A_2 is a proper class of atoms of $\mathcal{R}$ and each member of A_2 belongs to $\mathcal{R}_c$. □

REFERENCES

[1] CONRAD, P.: *K-radical classes of lattice ordered groups.* In: Proc. Conf. Carbondale. Lecture Notes in Math. 848, Springer, New York, 1981, pp. 186–207.

[2] TON, DAO-RONG: *Product radical classes of ℓ-groups*, Czechoslovak Math. J. **42** (1992), 129–142.

[3] DARNEL, M.: *Closure operations on radicals of lattice ordered groups*, Czechoslovak Math. J. **37** (1987), 51–64.

[4] DARNEL, M. R.: *Theory of Lattice-Ordered Groups*, M. Dekker, New York-Basel-Hong Kong, 1995.

[5] GOFMAN, C.: *Remarks on lattice ordered groups and vector lattices. I. Carathéodory functions*, Trans. Amer. Math. Soc. **88** (1958), 107–120.

[6] JAKUBÍK, J.: *Cardinal properties of lattice ordered groups*, Fund. Math. **74** (1972), 85–98.

[7] JAKUBÍK, J.: *Radical mappings and radical classes of lattice ordered groups.* In: Sympos. Math. 21, Academic Press, New York-London, 1977, pp. 451–477.

[8] JAKUBÍK, J.: *Products of radical classes of lattice ordered groups*, Acta Math. Univ. Comenian. **39** (1980), 31–41.

[9] JAKUBÍK, J.: *On K-radicals of lattice ordered groups*, Czechoslovak Math. J. **33** (1983), 149–163.

[10] JAKUBÍK, J.: *Radical classes of generalized Boolean algebras*, Czechoslovak Math. J. **48** (1998), 253–268.

[11] MEDVEDEV, N. YA.: *On the lattice of radicals of a finitely generated ℓ-group*, Math. Slovaca **33** (1983), 185–188. (Russian)

Received January 3, 1997
Revised January 15, 1998

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