

Anthony Donald Keedwell

Corrigendum to “Realizations of Loops and Groups defined by short identities”

Commentationes Mathematicae Universitatis Carolinae, Vol. 50 (2009), No. 4, 639--640

Persistent URL: <http://dml.cz/dmlcz/137454>

Terms of use:

© Charles University in Prague, Faculty of Mathematics and Physics, 2009

Institute of Mathematics of the Academy of Sciences of the Czech Republic provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these *Terms of use*.



This paper has been digitized, optimized for electronic delivery and stamped with digital signature within the project *DML-CZ: The Czech Digital Mathematics Library* <http://project.dml.cz>

Corrigendum to “Realizations of Loops and Groups defined by short identities”

A.D. KEEDWELL

In a recent paper [1], we considered all those quasigroup identities (or “laws”) which force a quasigroup to be a loop or group and, for a subset of these, provided proofs.

In several cases, we appealed to the Belousov/Taylor theorem (which says that any quasigroup which satisfies a balanced identity (or law) is isotopic to a group) but without making it clear that the quasigroup is itself a group only if it has an identity element (that is, only if it is a loop). We now provide the omitted part of these proofs using the notation of [1].

Theorem 1. *A quasigroup which satisfies any one of the balanced laws $x \cdot yz = xz \cdot y$ (L8.4), $x \cdot yz = y \cdot zx$ (L9.2) or $xy \cdot z = y \cdot zx$ (L9.3) has a two-sided identity element (and is an abelian group).*

PROOF: Put $y = f_z$ in (L8.4). This gives $xz = xz \cdot f_z$, whence $f_z = e_{xz}$. But, for all $w \in Q$, there exists $x \in Q$ such that $xz = w$. Thus, $f_z = e_w$ for all $z, w \in Q$. Since both z and w are arbitrary, this implies that all left local identities are equal to all right local identities and so there is an universal two-sided identity.

Next put $x = e_z$ in (L9.2). This gives $e_z \cdot yz = yz$ so $e_z = f_{yz}$. Given any $w \in Q$, there exists a unique $y \in Q$ such that $yz = w$. Thus, $e_z = f_w$ for all $z, w \in Q$. Since both z and w are arbitrary, this implies that all left local identities are equal to all right local identities and so, as before, there is an universal two-sided identity.

Finally, put $x = e_z$ in (L9.3). This gives $e_z y \cdot z = yz$ and so $e_z y = y$ by right cancellation of z . Thus, $e_z = f_y$ for all $z, y \in Q$. Hence, a quasigroup which satisfies (L9.3) has a universal two-sided identity. □

We also give a complete proof of Theorem 5.6(C) of [1].

Theorem 2. *A quasigroup which satisfies the law $x(yx \cdot z) = yz$ (L5.3) is a group.*

PROOF: Put $x = z$ in L5.3. This gives $z(yz \cdot z) = yz$. But, given $z \in Q$, we may choose $y \in Q$ such that $yz = w$ for any element $w \in Q$. Thus, $z(wz) = w$ for all $w, z \in Q$. We note that

$$z(wz) = w \Leftrightarrow R_z L_z = \text{Id} \Leftrightarrow L_z R_z = \text{Id} \Leftrightarrow (zw)z = w.$$

Now multiply *L5.3* on the right by x . We get $[x(yx \cdot z)]x = (yz)x$ and so, using the equality just proved, $yx \cdot z = yz \cdot x$. This equality is a balanced identity and so (Q, \cdot) is group isotopic¹. To show that it is a group, we must show the existence of a two-sided identity element.

First put $x = e_y$ in the relation $yx \cdot z = yz \cdot x$. This gives $yz = yz \cdot e_y$ and so $e_y = e_{yz}$. Since, for any $w \in Q$, there exists an element z for which $w = yz$, $e_y = e_w$ for all $y, w \in Q$. Thus, e_y is a universal right identity.

Now, put $y = x$ in *L5.3*. We get $x(xx \cdot z) = xz$ so, by left cancellation of x , $xx \cdot z = z$ and so $xx = f_z$ for all $x, z \in Q$. Therefore, in particular, $f_z = e_y e_y = e_y$ for all $y, z \in Q$. That is, the identity is two-sided as required. \square

Remark. In Theorem 5.4 of [1], it is necessary to use the method of the footnote of that paper to show the existence of an identity element before using the Belousov/Taylor theorem because the balanced identity $x \cdot zy = z \cdot xy$ is again not one of those which forces the quasigroup to be a loop².

REFERENCES

- [1] Keedwell A.D., *Realizations of Loops and Groups defined by short identities*, Comment. Math. Univ. Carolin. **50** (2009), no. 3, 373–383.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF SURREY, GUILDFORD,
SURREY GU2 7XH, UNITED KINGDOM

(Received September 7, 2009)

¹Although this identity is balanced, it is not equivalent to one of those (listed in Theorem 1 above) which forces the quasigroup to have a two-sided identity element.

²In fact, it is the dual of the one which appears in Theorem 2 above.