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SOME INEQUALITIES RELATED TO THE STAM INEQUALITY

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(Invited)

Abstract. Zamir showed in 1998 that the Stam classical inequality for the Fisher information (about a location parameter)

$$1/I(X+Y) \geqslant 1/I(X) + 1/I(Y)$$

for independent random variables X, Y is a simple corollary of basic properties of the Fisher information (monotonicity, additivity and a reparametrization formula). The idea of his proof works for a special case of a general (not necessarily location) parameter. Stam type inequalities are obtained for the Fisher information in a multivariate observation depending on a univariate location parameter and for the variance of the Pitman estimator of the latter.

Keywords: Fisher information, location parameter, Pitman estimators

MSC 2010: 62F11, 62B10

1. Introduction

Here basic properties (monotonicity, additivity and a reparametrization formula) of the Fisher information are presented and, following Zamir [10], the Stam inequality is obtained as a direct corollary of these properties.

Let $\mathcal{P} = \{P_\theta,\, \theta \in \Theta\}$ be a parametric family of probability distributions of a random element X taking values in a measurable space $(\mathcal{X}, \mathcal{A})$, the parameter space Θ being an open set of $\mathbb{R}$. For the purpose of this paper, the following simplified version of the concept of a regular statistical experiment suffices. A triple $\mathcal{E} = (\mathcal{X}, \mathcal{A}, \mathcal{P})$ is called a *regular statistical experiment* (consisting in an observation of X) if

(a) all P_θ are given by densities $p(x;\theta) = \mathrm{d}P_\theta/\mathrm{d}\mu$ with respect to a measure μ,

(b) $p(x;\theta)$ is continuously differentiable in $\theta \in \Theta$ for μ-almost all $x \in \mathcal{X}$ and

(c) the Fisher information on θ in X (or in $\mathcal{E}$),

$$I(X;\theta) = I_X(\theta) = \int \Big(\frac{\partial p(x;\theta)}{\partial\theta}\Big)^2 / p(x;\theta)\,\mathrm{d}\mu(x),$$

is finite (the integration is over the set $\{x\colon\ p(x;\theta) > 0\}$). In Ibragimov and Khas'minskij [2], the class of regular statistical experiments is larger than the one we have defined. In particular, they need only mean square differentiability in θ of the density $p(x;\theta)$.

The following well-known properties of the Fisher information hold for regular experiments.

1) *Monotonicity.* If $S\colon\ (\mathcal{X},\mathcal{A}) \to (\mathcal{S},\mathcal{B})$ is a statistic, $Q_\theta(B) = P_\theta(S \in B) = P_\theta(S^{-1}B)$, $B \in \mathcal{B}$ (or, in other terms, $\mathcal{E}_S = (\mathcal{S},\mathcal{B},\ \mathcal{Q} = \{Q_\theta,\ \theta \in \Theta\})$ is a subexperiment of $\mathcal{E}$), then

$$I(S;\theta) \leqslant I(X;\theta), \quad \theta \in \Theta.$$

2) *Additivity.* If X_i, $i = 1,2$, are random elements taking values in $(\mathcal{X}_i,\mathcal{A}_i)$ which are independent for each θ, i.e., for all $A_i \in \mathcal{A}_i$, $i = 1,2$,

$$P_\theta(X_1 \in A_1,\ X_2 \in A_2) = P_\theta(X_1 \in A_1)P_\theta(X_2 \in A_2), \quad \theta \in \Theta,$$

and $X = (X_1, X_2)$, then

$$I(X;\theta) = I(X_1;\theta) + I(X_2;\theta).$$

3) *Reparametrization formula.* If g is a differentiable function, then for $\xi = g(\theta)$

$$I(X;\theta) = |g'(\theta)|^2 I(X;\xi)\big|_{\xi=g(\theta)}.$$

Note in passing that if $p(x;\theta) > 0$, then $I(T;\theta) = I(X;\theta)$ implies sufficiency of a statistic T; without positivity of $p(x;\theta)$ this does not hold in general, as shown in Kagan and Shepp [5].

Multivariate versions of 1)–3) are also well known.

1′) If $\boldsymbol{\theta}$ is an m-variate parameter, $\boldsymbol{\theta} \in \Theta$, an open set in $\mathbb{R}^m$, and $I(X;\boldsymbol{\theta})$ is the $m \times m$ matrix of Fisher information on $\boldsymbol{\theta}$ in X, then for any statistic S,

$$I(S;\boldsymbol{\theta}) \leqslant I(X;\boldsymbol{\theta}),$$

i.e., $I(X;\boldsymbol{\theta}) - I(S;\boldsymbol{\theta})$ is a positive semi-definite matrix.

2′) The additivity property has the same form as in the case of a univariate parameter.

3′) If $\boldsymbol{\xi} = g(\boldsymbol{\theta})$ where g is a differentiable mapping of an open set $\Theta \subset \mathbb{R}^m$ into an open set $\boldsymbol{\xi} \subset \mathbb{R}^k$ with Jacobian

$$H = \Big(\frac{\partial g_i}{\partial \theta_j}\Big), \quad i = 1, \dots, k; \;\; j = 1, \dots, m,$$

then

$$I(X; \boldsymbol{\theta}) = H^{\mathrm{T}} I(X; \boldsymbol{\xi})\big|_{\boldsymbol{\xi} = g(\boldsymbol{\theta})} H$$

where T stands for transposition.

Let us turn to the case when the distribution of X is absolutely continuous and θ a location parameter so that the density $p(x; \theta) = p(x - \theta)$. Now the Fisher information does not depend on θ,

$$\begin{aligned} I(X; \theta) &= \int_{x\colon\, p(x-\theta)>0} \{p'(x-\theta)/p(x-\theta)\}^2 p(x-\theta)\,\mathrm{d}x \\ &= \int_{x\colon\, p(x)>0} \{p'(x)/p(x)\}^2 p(x)\,\mathrm{d}x. \end{aligned} \tag{1}$$

In what follows, $I(X)$ will denote the Fisher information on θ in an observation with density $p(x - \theta)$. For independent X_1, X_2 with densities $p_1(x)$, $p_2(x)$, respectively, $I(X_1 + X_2)$ denotes the Fisher information on θ in an observation with density $p(x - \theta)$ where $p(x) = (p_1 * p_2)(x)$.

As a direct corollary of 1), for independent X_1, X_2,

$$I(X_1 + X_2) \leqslant \min\{I(X_1), I(X_2)\}.$$

In Stam [9] a much stronger inequality was proved,

$$\frac{1}{I(X_1 + X_2)} \geqslant \frac{1}{I(X_1)} + \frac{1}{I(X_2)} \tag{2}$$

that is closely linked to the Shannon classical inequality for the differential entropy $H(X)$: for independent X_1, X_2,

$$\mathrm{e}^{2H(X_1+X_2)} \geqslant \mathrm{e}^{2H(X_1)} + \mathrm{e}^{2H(X_2)}.$$

Recently, Madiman and Barron [7] proved a much stronger version of (2): for independent $X_1, \dots, X_n$,

$$\frac{1}{I(X_1 + \ldots + X_n)} \geqslant \frac{1}{\binom{n-1}{m-1}} \sum_{\mathbf{s}} \frac{1}{I(\sum_{i \in \mathbf{s}} X_i)}, \tag{3}$$

where the summation is over all combinations $\mathbf{s}$ of m elements chosen from $\{1,\ldots,n\}$.

One of the corollaries of (3) is the monotone decreasing in n of the information $I((X_1+\ldots+X_n)/\sqrt{n}) = nI(X_1+\ldots+X_n)$ contained in the normalized sum of independent identically distributed $X_1, X_2, \ldots$.

Let us turn now to Zamir's proof of (2) based on properties 1)–3) of the Fisher information.

Let w_1, w_2 be positive numbers with $w_1+w_2=1$ and let observations X_i' be of the form

$$X_i' = w_i\theta + X_i, \quad i=1,2$$

with $\theta\in\mathbb{R}$ as a parameter and X_1, X_2 independent with $X_i\sim p_i(x)$, $i=1,2$. By virtue of 3),

$$I(X_i';\theta) = w_i^2 I(X_i), \quad i=1,2.$$

Consider now a statistic

$$S(X_1',X_2') = X_1'+X_2' = \theta + X_1 + X_2.$$

Due to 1) and 2),

$$I(X_1+X_2) = I(X_1'+X_2') \leqslant I(X_1') + I(X_2') = w_1^2 I(X_1) + w_2^2 I(X_2). \tag{4}$$

Choosing

$$w_i = \frac{1/I(X_i)}{1/I(X_1)+1/I(X_2)}, \quad i=1,2,$$

one immediately gets from (4) the Stam inequality

$$\frac{1}{I(X_1+X_2)} \geqslant \frac{1}{I(X_1)} + \frac{1}{I(X_2)}.$$

If $\mathbf{X}$, $\mathbf{X}^{\mathrm{T}} = (X_1,\ldots,X_s)$ is an m-variate random vector with density $p(\mathbf{x}-\boldsymbol{\theta}) = p(x_1-\theta_1,\ldots,x_m-\theta_m)$ depending on an m-variate location parameter $\boldsymbol{\theta}\in\mathbb{R}^m$, the matrix $I(\mathbf{X})$ of the Fisher information on $\boldsymbol{\theta}$ in $\mathbf{X}$ does not depend on $\boldsymbol{\theta}$,

$$I(\mathbf{X}) = (I_{ij})_{i,j=1,\ldots,m}, \quad I_{ij} = \int_{\mathbf{x}:\, p(\mathbf{x})>0} \frac{1}{p}\Big(\frac{\partial p}{\partial x_i}\Big)\Big(\frac{\partial p}{\partial x_j}\Big)\,\mathrm{d}\mathbf{x},$$

and is positive definite (the matrix $I(X;\theta)$ of the Fisher information on a general m-variate parameter, not necessarily location, is only non-negative definite). Indeed, take a nonzero $\mathbf{c}\in\mathbb{R}^m$ and consider a random vector $\tilde{\mathbf{X}}$ with density $p(x_1-c_1\theta,\ldots,x_m-c_m\theta)$. Plainly, $I(\tilde{\mathbf{X}};\theta) = \mathbf{c}^{\mathrm{T}} I(\mathbf{X})\mathbf{c}$ and due to 1), $I(\tilde{\mathbf{X}};\theta) \geqslant$

$I(\tilde{X}_j;\theta)$. The density of the jth component $\tilde{X}_j$ of $\mathbf{X}$ is $p_j(x_j - c_j\theta)$ so that $I(\tilde{X}_j;\theta) > 0$ if $c_j \neq 0$. Hence $I(\mathbf{X})$ is positive definite.

Now let W_1, W_2 be $(m \times m)$ matrices with $W_1 + W_2 = I_m$, the $(m \times m)$ identity matrix. Set

$$\mathbf{X}_i' = W_i\theta + \mathbf{X}_i, \quad i = 1, 2,$$

where $\mathbf{X}_1$, $\mathbf{X}_2$ are independent m-variate random vectors, $\mathbf{X}_i \sim p_i(\mathbf{x})$, $i = 1, 2$ and $\boldsymbol{\theta} \in \mathbb{R}^m$. By virtue of 1′)–3′),

$$\begin{aligned} I(\mathbf{X}_1 + \mathbf{X}_2) = I(\mathbf{X}_1' + \mathbf{X}_2') &\leqslant I(\mathbf{X}_1';\boldsymbol{\theta}) + I(\mathbf{X}_2';\boldsymbol{\theta}) \\ &= W_1^{\mathrm{T}} I(\mathbf{X}_1) W_1 + W_2^{\mathrm{T}} I(\mathbf{X}_2) W_2. \end{aligned} \tag{5}$$

Choosing in (5)

$$W_i = (I(\mathbf{X}_i))^{-1}\{(I(\mathbf{X}_1))^{-1} + (I(\mathbf{X}_2))^{-1}\}^{-1}, \quad i = 1, 2$$

one gets

$$I(\mathbf{X}_1 + \mathbf{X}_2) \leqslant \{(I(\mathbf{X}_1))^{-1} + (I(\mathbf{X}_2))^{-1}\}^{-1}$$

whence, by taking the inverse of both sides, the multivariate Stam inequality follows:

$$(I(\mathbf{X}_1 + \mathbf{X}_2))^{-1} \geqslant (I(\mathbf{X}_1))^{-1} + (I(\mathbf{X}_2))^{-1}. \tag{6}$$

The matrices $I(\mathbf{X}_1)$ and $I(\mathbf{X}_2)$ are not assumed commutative. This proof of (6) is due to Zamir [10]. The authors' contribution is an observation that the matrix of the Fisher information on a multivariate location parameter is positive definite so that there is no need in assuming the information matrices nonsingular.

Note that in Kagan and Landsman [3] another inequality for the matrices of the Fisher information first proved analytically in Carlen [1], was shown to be a direct corollary of 1) and 2).

2. The case of a general parameter

Let X_1, X_2 be independent random variables with densities $p_1(x;\theta_1)$, $p_2(x;\theta_2)$ depending on general (not necessarily location) parameters θ_1, θ_2 belonging to the same parameter set $\Theta = (a, b)$, $a \leqslant 0$, $b > 0$ such that $\alpha\Theta \subset \Theta$ for any α, $0 < \alpha < 1$.

To get a version of the Stam inequality for $X_1 \sim p_1(x;\theta_1)$, $X_2 \sim p_2(x;\theta_2)$, we need a number of assumptions.

First, the Fisher information $I(X_1;\theta_1)$ on θ_1 in X_1 and $I(X_2;\theta_2)$ on θ_2 in X_2 is assumed finite, positive and constant in the parameters,

$$0 < I(X_i;\theta_i) = I_i < \infty, \quad i = 1, 2. \tag{7}$$

The condition (7) plainly holds in the case of location parameters θ_1, θ_2 but it is much more general. If X has a density $p(x;\eta)$ and a new parameter η is introduced by $\eta = g(\theta)$ so that $\tilde{p}(x;\theta) = p(x;g(\theta))$, then $I(X;\theta) = |g'(\theta)|^2 I(X;\eta)\big|_{\eta=g(\theta)}$ whence one can construct many families with a constant Fisher information. For example, if X has a Pareto density

$$p(x;\eta) = (\eta-1)/x^{\eta}, \quad x \geqslant 1$$

with $\eta > 1$ as a parameter, the reparametrization $\eta = \mathrm{e}^{\theta}+1$ stabilizes the information on θ.

Second, let $S = S(X_1, X_2)$ be a statistic taking values in a measurable space $(\mathcal{S}, \mathcal{B})$. It is assumed that the density $p(s;\theta_1,\theta_2)$ of S depends on the parameters only through $\theta_1+\theta_2$,

$$p(s;\theta_1,\theta_2) = p(s;\theta_1+\theta_2), \quad s \in \mathcal{S}, \tag{8}$$

so that the distribution of S depends on a univariate parameter $\theta = \theta_1+\theta_2$. If $p_i(x;\theta_i) = p_i(x-\theta_i)$, $i = 1,2$ and $S(X_1+X_2) = X_1+X_2$, (8) is plainly satisfied.

Theorem 1. *Under the conditions* (7), (8), *the Fisher information* $I(S;\theta)$ *on* θ *in* S *satisfies the inequality*

$$\frac{1}{I(S;\theta)} \geqslant \frac{1}{I_1} + \frac{1}{I_2}. \tag{9}$$

P r o o f. Take positive w_1, w_2 with $w_1+w_2 = 1$ and set $\theta_1 = w_1\theta$, $\theta_2 = w_2\theta$. Then $\theta_1+\theta_2 = \theta$. By 3), $I(X_i;\theta) = w_i^2 I_i$, $i = 1,2$ and by 1) and 2),

$$I(S;\theta) \leqslant I(X_1;\theta) + I(X_2;\theta) = w_1^2 I_1 + w_2^2 I_2.$$

Choosing

$$w_i = \frac{1/I_i}{1/I_1+1/I_2}, \quad i = 1,2$$

leads to (9). □

R e m a r k. Zamir's idea works in some cases when versions of (7), (8) hold. Here is an example in which the dependence of the distribution of S on $\theta_1+\theta_2$ is replaced with the dependence of its distribution on $\theta_1\theta_2$ where both θ_1 and θ_2 are positive.

Let independent random variables X_1, X_2 have densities $\theta_1 p_1(\theta_1 x)$, $\theta_2 p_2(\theta_2 x)$ depending on scale parameters $\theta_1, \theta_2 \in \mathbb{R}_+$. If the distributions of X_1 and X_2 are concentrated on $\mathbb{R}_+$ or $\mathbb{R}_-$, the setup is reduced to that of location parameters. This assumption is not made here.

Let $T(X_1, X_2) = X_1 X_2$. It is easily seen that the distribution of T depends on θ_1, θ_2 only through the scale parameter $\theta = \theta_1\theta_2$,

$$p(t;\theta) = \theta p(\theta x).$$

Simple calculations show that

$$I(X_i;\theta_i) = \theta_i^{-2} I(X_i;1), \;\; i = 1,2; \quad I(T;\theta) = \theta^{-2} I(T;1).$$

Now set $\theta_1 = \theta^{\gamma_1}$, $\theta_2 = \theta^{\gamma_2}$ with $\gamma_i > 0$, $\gamma_1 + \gamma_2 = 1$. Then $\theta_1\theta_2 = \theta$ and

$$I(X_i;\theta) = (\gamma_i \theta^{\gamma_i - 1})^2 I(X_i;\theta_i) = \gamma_i^2 \theta^{-2} I(X_i;1), \quad i = 1,2.$$

One has

$$I(T;\theta) \leqslant I(X_1;\theta) + I(X_2;\theta)$$

whence

$$I(T;1) \leqslant \gamma_1^2 I(X_1;1) + \gamma_2^2 I(X_2;1).$$

Choosing

$$\gamma_i = \frac{(I(X_i;1))^{-1}}{(I(X_1;1))^{-1} + (I(X_2;1))^{-1}}$$

one gets a Stam type inequality for the Fisher information on a scale parameter: for independent X_1, X_2 one has

$$\frac{1}{I(X_1X_2;\theta)} \geqslant \frac{1}{I(X_1;\theta)} + \frac{1}{I(X_2;\theta).}$$

Unfortunately, the proof does not work when the distribution of S depends on an arbitrary (univariate) function $h(\theta_1, \theta_2)$.

3. Relation to the Pitman estimators

Let $\mathbf{X}^{\mathrm{T}} = (X_1, \ldots, X_m) \sim p(x_1 - \theta, \ldots, x_m - \theta) = p(\mathbf{x} - \theta \cdot \mathbf{1})$ where $\mathbf{1}^{\mathrm{T}} = (1, \ldots, 1)$ is an m-variate vector with all the components 1, be an m-variate random vector whose distribution depends on a univariate location parameter θ. If $\mathbf{I}$ is the matrix of the Fisher information on $\boldsymbol{\theta}^{\mathrm{T}} = (\theta_1, \ldots, \theta_m)$ in an observation with density $p(\mathbf{x} - \boldsymbol{\theta})$, then the Fisher information I on θ in $\mathbf{X}$ is

$$I = \mathbf{1}^{\mathrm{T}} \mathbf{I} \mathbf{1}.$$

Let now $\mathbf{X}_1$, $\mathbf{X}_2$ be independent random vectors, $\mathbf{X}_1 \sim p_1(\mathbf{x} - \theta \cdot \mathbf{1})$, $\mathbf{X}_2 \sim p_2(\mathbf{x} - \theta \cdot \mathbf{1})$, $p(\mathbf{x}) = (p_1 * p_2)(\mathbf{x})$ and let I_1, I_2, I denote the Fisher observation on the univariate parameter θ contained in $\mathbf{X}_1$, $\mathbf{X}_2$, $\mathbf{X}$, respectively. As an immediate corollary of Theorem 1, one gets

$$\frac{1}{I} \geqslant \frac{1}{I_1} + \frac{1}{I_2}. \tag{10}$$

This inequality is independent of the multivariate Stam inequality

$$\mathbf{I}^{-1} \geqslant \mathbf{I}_1^{-1} + \mathbf{I}_2^{-1} \tag{11}$$

where $\mathbf{I}_1$, $\mathbf{I}_2$, $\mathbf{I}$ are the matrices of the Fisher information on the m-variate parameter $\boldsymbol{\theta}$ contained in $\mathbf{X}_1 \sim p_1(\mathbf{x} - \boldsymbol{\theta})$, $\mathbf{X}_2 \sim p_2(\mathbf{x} - \boldsymbol{\theta})$, $\mathbf{X} \sim p(\mathbf{x} - \boldsymbol{\theta})$.

Inequality (10) has its analog in terms of Pitman estimators; no regularity type conditions, even absolute continuity, are required from the distributions.

Let

$$\mathbf{x}'^{\mathrm{T}}_1 = (x'_{11}, \ldots, x'_{1m}), \ \ldots, \ \mathbf{x}'^{\mathrm{T}}_n = (x'_{n1}, \ldots, x'_{nm}),$$
$$\mathbf{x}''^{\mathrm{T}}_1 = (x''_{11}, \ldots, x''_{1m}), \ \ldots, \ \mathbf{x}''^{\mathrm{T}}_n = (x''_{n1}, \ldots, x''_{nm})$$

be independent samples from distributions $F_1(\mathbf{x} - \theta \cdot \mathbf{1})$ and $F_2(\mathbf{x} - \theta \cdot \mathbf{1})$ and let

$$\mathbf{x}^{\mathrm{T}}_1 = (x_{11}, \ldots, x_{1m}), \ \ldots, \ \mathbf{x}^{\mathrm{T}}_n = (x_{n1}, \ldots, x_{nm})$$

be a sample from $F(\mathbf{x} - \theta \cdot \mathbf{1})$ where $F = F_1 * F_2$.

Set

$$\bar{x}'_1 = (x'_{11} + \ldots + x'_{n1})/n, \ \ldots, \ \bar{x}'_m = (x'_{1m} + \ldots + x'_{nm})/n$$

and

$$\bar{x}' = (\bar{x}'_1 + \ldots + \bar{x}'_m)/m$$

with

$$\bar{x}_1'', \ldots, \bar{x}_m'', \ \bar{x}'', \ \bar{x}_1, \ldots, \bar{x}_m, \ \bar{x}$$

defined similarly for the other two samples.

Let σ', σ'', σ be the σ-algebras generated by $x_{11}' - \bar{x}'$, …, $x_{nm}' - \bar{x}'$; $x_{11}'' - \bar{x}''$, …, $x_{nm}'' - \bar{x}''$; $x_{11}' - \bar{x}' + x_{11}'' - \bar{x}''$, …, $x_{nm}' - \bar{x}' + x_{nm}'' - \bar{x}''$, respectively. Plainly, σ is a subalgebra of the σ-algebra generated by

$$x_{11}' - \bar{x}', \ \ldots, \ x_{nm}' - \bar{x}', \ x_{11}'' - \bar{x}'', \ \ldots, \ x_{nm}'' - \bar{x}''.$$

The latter is usually denoted $\sigma' \vee \sigma''$ so that $\sigma \subset \sigma' \vee \sigma''$.

An estimator $\tilde{\theta}(\mathbf{y}_1, \ldots, \mathbf{y}_n)$ of θ from a sample from $G(\mathbf{y} - \theta \cdot \mathbf{1})$ is called equivariant if for any $c \in \mathbb{R}$

$$\tilde{\theta}(\mathbf{y}_1 + c \cdot \mathbf{1}, \ldots, \mathbf{y}_n + c \cdot \mathbf{1}) = \tilde{\theta}(\mathbf{y}_1, \ldots, \mathbf{y}_n) + c. \tag{12}$$

Assuming $\int |\mathbf{x}|^2 \, \mathrm{d}F_i(\mathbf{x}) < \infty$, $i = 1, 2$, the Pitman estimators t_n', t_n'' of θ (with respect to the quadratic loss function) from samples of size n from $p_1(\mathbf{x} - \theta \cdot \mathbf{1})$ and $p_2(\mathbf{x} - \theta \cdot \mathbf{1})$, i.e., the minimum variance equivariant estimators, can be written as

$$t_n' = \bar{x}' - E(\bar{x}' \mid \sigma'), \quad t_n'' = \bar{x}'' - E(\bar{x}'' \mid \sigma'')$$

and their variances as

$$\operatorname{var}(t_n') = \operatorname{var}(\bar{x}') - \operatorname{var}\{E(\bar{x}' \mid \sigma')\}, \quad \operatorname{var}(t_n'') = \operatorname{var}(\bar{x}'') - \operatorname{var}\{E(\bar{x}'' \mid \sigma'')\}.$$

(All the expectations are taken at $\theta = 0$, though the variances do not depend on θ.) Now

$$\operatorname{var}(t_n) = \operatorname{var}(\bar{x}) - \operatorname{var}\{E(\bar{x} \mid \sigma)\}$$

and using the fact that $(\mathbf{x}_1, \ldots, \mathbf{x}_n)$ is equidistributed with $(\mathbf{x}_1' + \mathbf{x}_1'', \ldots, \mathbf{x}_n' + \mathbf{x}_n'')$, one gets

$$\operatorname{var}(t_n) = \operatorname{var}(\bar{x}' + \bar{x}'') - \operatorname{var}\{E(\bar{x}' + \bar{x}'' \mid \sigma)\}.$$

Since $\sigma \subset \sigma' \vee \sigma''$, one has

$$\operatorname{var}\{E(\bar{x}' + \bar{x}'' \mid \sigma)\} \leqslant \operatorname{var}\{E(\bar{x}' + \bar{x}'' \mid \sigma' \vee \sigma'')\}.$$

Furthermore, $\bar{x}', x_{11}' - \bar{x}', \ldots, x_{nm}' - \bar{x}'$ is independent of $x_{11}'' - \bar{x}'', \ldots, x_{nm}'' - \bar{x}''$ implying

$$E(\bar{x}' \mid \sigma' \vee \sigma'') = E(\bar{x}' \mid \sigma'), \quad E(\bar{x}'' \mid \sigma' \vee \sigma'') = E(\bar{x}'' \mid \sigma'')$$

(see, e.g., Shao [8]). Thus,

$$\operatorname{var}(t_n) \geqslant \operatorname{var}(t'_n) + \operatorname{var}(t''_n).$$

This inequality, holding for any n, may be considered a small sample version of (10). In the case of $m = 1$ it was proved in Kagan [4]. For other connections between the variance of Pitman estimators and the Fisher information see Kagan et al. [6].

As said above, (10) and (11) are independent in the sense that neither is a corollary of the other. However, an inequality connecting I and $\mathbf{I}$ has a simple statistical interpretation.

Let $\mathbf{w}^{\mathrm{T}} = (w_1, \ldots, w_m)$ be a vector with $\mathbf{w}^{\mathrm{T}}\mathbf{1} = 1$. Then, by virtue of the Cauchy inequality,

$$1 = (\mathbf{w}^{\mathrm{T}}\mathbf{1})^2 = (\mathbf{w}^{\mathrm{T}}\mathbf{I}^{-1/2}\mathbf{I}^{1/2}\mathbf{1})^2 \leqslant (\mathbf{w}^{\mathrm{T}}\mathbf{I}^{-1}\mathbf{w})(\mathbf{1}^{\mathrm{T}}\mathbf{I}\mathbf{1})$$

so that

$$I \geqslant \frac{1}{\mathbf{w}^{\mathrm{T}}\mathbf{I}^{-1}\mathbf{w}}. \tag{13}$$

Let now $\mathbf{t}_n$ be the Pitman estimator of an m-variate $\boldsymbol{\theta}$ from a sample $(\mathbf{x}_1, \ldots, \mathbf{x}_n)$ from $F(\mathbf{x} - \boldsymbol{\theta})$. If $\int \mathbf{x}^2 \,\mathrm{d}F(\mathbf{x}) < \infty$, $\mathbf{t}_n$ can be written (componentwise) as

$$\mathbf{t}_n = \bar{\mathbf{x}} - E(\bar{\mathbf{x}} \mid x_{11} - \bar{x}_1, \ldots, x_{n1} - \bar{x}_1, \ldots, x_{1m} - \bar{x}_m, \ldots, x_{nm} - \bar{x}_m). \tag{14}$$

Note that the σ-algebra generated by the residuals in (14) is smaller than the σ-algebra generated by $x_{11} - \bar{x}, \ldots, x_{nm} - \bar{x}$ where $\bar{x} = (\bar{x}_1 + \ldots + \bar{x}_m)/m$ (mind the difference between $\bar{\mathbf{x}}$ and $\bar{x}$). The latter occurs in the representation

$$t_n = \bar{x} - E(\bar{x} \mid x_{11} - \bar{x}, \ldots, x_{nm} - \bar{x}) \tag{15}$$

of the Pitman estimator of a univariate θ from a sample $(\mathbf{x}_1, \ldots, \mathbf{x}_n)$ from $F(\mathbf{x} - \theta \cdot \mathbf{1})$ when $\mathbf{w}^{\mathrm{T}}\mathbf{t}_n$ is an equivariant estimator of θ and, thus,

$$\mathbf{w}^{\mathrm{T}} \operatorname{var}(\mathbf{t}_n)\mathbf{w} = \operatorname{var}(\mathbf{w}^{\mathrm{T}}\mathbf{t}_n) \geqslant \operatorname{var}(t_n). \tag{16}$$

This inequality is, in a sense, a small sample version of (13). Indeed, as $n \to \infty$,

$$n \operatorname{var}(\mathbf{t}_n) = \mathbf{I}^{-1}(1 + o(1)), \quad n \operatorname{var}(t_n) = \frac{1}{\mathbf{1}^{\mathrm{T}}\mathbf{I}\mathbf{1}}(1 + o(1)),$$

so that (16) becomes (13). The relation between these two equations is one more illustration of that many results for the Fisher information/information matrix have direct analogs in terms of the variances of the Pitman estimators in small samples.

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References

[1] *E. A. Carlen*: Superadditivity of Fisher's information and logarithmic Sobolev inequalities. J. Funct. Anal. *101* (1991), 194–211.

[2] *I. A. Ibragimov, R. Z. Khas'minskij*: Statistical Estimation. Asymptotic Theory. Springer, New York, 1981.

[3] *A. Kagan, Z. Landsman*: Statistical meaning of Carlen's superadditivity of the Fisher information. Statist. Probab. Lett. *32* (1997), 175–179.

[4] *A. Kagan*: An inequality for the Pitman estimators related to the Stam inequality. Sankhya *A64* (2002), 282–292.

[5] *A. Kagan, L. A. Shepp*: A sufficiency paradox: an insufficient statistic preserving the Fisher information. Amer. Statist. *59* (2005), 54–56.

[6] *A. Kagan, T. Yu, A. Barron, M. Madiman*: Contribution to the theory of Pitman estimators. Submitted.

[7] *M. Madiman, A. Barron*: The monotonicity of information in the central limit theorem and entropy power inequalities. Preprint. Dept. of Statistics, Yale University, 2006.

[8] *J. Shao*: Mathematical Statistics, 2nd ed. Springer, New York, 2003.

[9] *A. J. Stam*: Some inequalities satisfied by the quantities of information of Fisher and Shannon. Inform. and Control *2* (1959), 101–112.

[10] *R. Zamir*: A proof of the Fisher information inequality via a data processing argument. IEEE Trans. Inf. Theory *44* (1998), 1246–1250.

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