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THE MULTISET CHROMATIC NUMBER OF A GRAPH

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Abstract. A vertex coloring of a graph G is a multiset coloring if the multisets of colors of the neighbors of every two adjacent vertices are different. The minimum k for which G has a multiset k-coloring is the multiset chromatic number $\chi_m(G)$ of G. For every graph G, $\chi_m(G)$ is bounded above by its chromatic number $\chi(G)$. The multiset chromatic number is determined for every complete multipartite graph as well as for cycles and their squares, cubes, and fourth powers. It is conjectured that for each $k \geqslant 3$, there exist sufficiently large integers n such that $\chi_m(C_n^k) = 3$. It is determined for which pairs k, n of integers with $1 \leqslant k \leqslant n$ and $n \geqslant 3$, there exists a connected graph G of order n with $\chi_m(G) = k$. For $k = n - 2$, all such graphs G are determined.

Keywords: vertex coloring, multiset coloring, neighbor-distinguishing coloring

MSC 2010: 05C15

1. Introduction

In a proper coloring of a graph G, a color is assigned to each vertex of G so that adjacent vertices are assigned distinct colors. Thus a coloring that is not necessarily proper permits adjacent vertices to be assigned the same color. Hence a proper coloring distinguishes the two vertices in every pair of adjacent vertices. In general, a vertex coloring of a graph in which every two adjacent vertices are assigned distinct colors is referred to as a *neighbor-distinguishing* coloring. Therefore, every proper coloring is neighbor-distinguishing. The minimum number of colors in a proper coloring of G is, of course, the *chromatic number* $\chi(G)$. Neighbor-distinguishing vertex colorings can be defined in other ways however and possibly use fewer than $\chi(G)$ colors.

Edge colorings of graphs, whether proper or not, have been introduced that use the multisets of colors of the incident edges of each vertex in a graph G for the purpose of distinguishing all vertices of G or of distinguishing every two adjacent vertices of G. The papers by Burris [3] and Chartrand, Escuadro, Okamoto, and Zhang [4] deal with the former (vertex-distinguishing edge colorings), while the papers by Addario-Berry, Aldred, Dalal, and Reed [1], Karoński, Łuczak, and Thomason [8], and Escuadro, Okamoto, and Zhang [7] deal with the latter. Furthermore, vertex colorings (proper or not) of a graph G have been introduced that use the multisets of colors of the neighboring vertices of each vertex for the purpose of distinguishing all vertices of G. These concepts have been studied by Chartrand, Lesniak, VanderJagt, and Zhang [5], Radcliffe and Zhang [9], and Anderson, Barrientos, Brigham, Carrington, Kronman, Vitray, and Yellen [2]. In this paper we use multisets of colors to introduce and study a neighbor-distinguishing vertex coloring. We refer to the book [6] for graph theory notation and terminology not described in this paper.

For a connected graph G, let $c\colon V(G) \to \{1, 2, \ldots, k\}$ be a not necessarily proper k-coloring of the vertices of G for some positive integer k (where then adjacent vertices may be colored the same). The coloring c is called a *multiset coloring* if for every pair u, v of adjacent vertices of G, the multisets $M(u)$ and $M(v)$ of the colors of the neighbors of u and v differ, that is, there exists a color i such that the number of neighbors of u colored i and the number of neighbors of v colored i are not the same. Each multiset $M(v)$ of colors of the neighbors of a vertex v of G can be represented by a k-vector. The *color code* of a vertex v of G is the k-vector

$$\mathrm{code}(v) = (a_1, a_2, \ldots, a_k) = a_1 a_2 \ldots a_k,$$

where a_i is the number of occurrences of i in $M(v)$, that is, the number of vertices adjacent to v that are colored i for $1 \leqslant i \leqslant k$. Therefore,

$$\sum_{i=1}^{k} a_i = \deg v.$$

Thus a vertex coloring (not necessarily proper) of a graph G is a multiset coloring if every two adjacent vertices have distinct color codes. Hence every multiset coloring of a graph G is neighbor-distinguishing. The *multiset chromatic number* $\chi_m(G)$ of G is the minimum positive integer k for which G has a multiset k-coloring.

Suppose that c is a proper vertex k-coloring of a graph G. If u is a vertex of G and $c(u) = i$ for some integer i ($1 \leqslant i \leqslant k$), then the i-th coordinate of the color code of u is 0. On the other hand, if v is a neighbor of u, then the i-th coordinate of the color code of v is at least 1, implying that $\mathrm{code}(u) \neq \mathrm{code}(v)$ for every two adjacent vertices u and v in G. Hence every proper coloring of G is a multiset coloring. Therefore, for every graph G,

$$\chi_m(G) \leqslant \chi(G). \tag{1}$$

Suppose that a coloring c of a graph G is given (where adjacent vertices may be assigned the same color). If u and v are vertices (adjacent or nonadjacent) of a graph G such that $\deg u \neq \deg v$, then necessarily $\mathrm{code}(u) \neq \mathrm{code}(v)$. On the other hand, if G contains two adjacent vertices u and v with $\deg u = \deg v$, then in order for c to be a multiset coloring, c must assign at least two distinct colors to the vertices of G. Thus we have the following observation.

Observation 1.1. Let G be a graph. Then $\chi_m(G) = 1$ if and only if every two adjacent vertices of G have distinct degrees.

Since every nonempty bipartite graph has chromatic number 2, the following is an immediate consequence of (1) and Observation 1.1.

Proposition 1.2. *If G is a bipartite graph, then*

$$\chi_m(G) = \begin{cases} 1 & \text{if every two adjacent vertices of } G \text{ have distinct degrees,} \\ 2 & \text{otherwise.} \end{cases}$$

As an illustration, we determine the multiset chromatic number of the Petersen graph P. Since the Petersen graph has chromatic number 3, it follows that $\chi_m(P) \leqslant 3$. However, Figure 1 shows a multiset 2-coloring of P. By Observation 1.1 then, $\chi_m(P) = 2$.

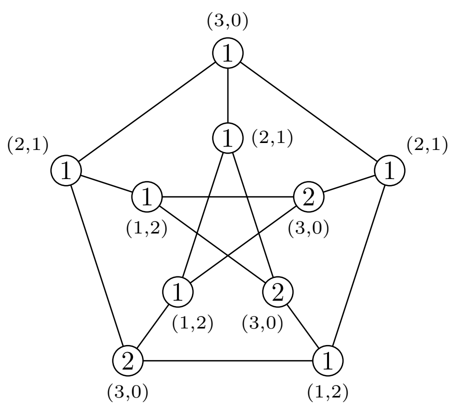

Figure 1: A multiset 2-coloring of the Petersen graph

For a vertex v in a graph G, let $N(v)$ be the neighborhood of v (the set of all vertices adjacent to v in G). The following observation is often useful.

Observation 1.3. If u and v are two adjacent vertices in a graph G such that $N(u) - \{v\} = N(v) - \{u\}$, then $c(u) \neq c(v)$ for every multiset coloring c of G.

2. The multiset chromatic number of complete multipartite graphs

We have noted that for each vertex coloring of a graph G, every two vertices with different degrees have distinct color codes. From this, it follows that determining the multiset chromatic number of G is most interesting and most challenging when G has many vertices of the same degree. We now initiate a study of graphs having this property, especially regular graphs. It is a consequence of Observation 1.3 that $\chi_m(K_n) = n$. By (1) a graph G of order n has multiset chromatic number n if and only if $G = K_n$.

By Proposition 1.2, for the complete bipartite graph $K_{s,t}$,

$$\chi_m(K_{s,t}) = \begin{cases} 1 & \text{if } s \neq t, \\ 2 & \text{if } s = t. \end{cases}$$

We now determine the multiset chromatic numbers of all complete multipartite graphs, beginning with the regular complete multipartite graphs, that is, those complete multipartite graphs all of whose partite sets are of the same cardinality. If every partite set of a complete k-partite graph G has n vertices, then we write $G = K_{k(n)}$, where then $K_{n(1)} = K_n$ and $K_{1(n)} = \overline{K}_n$.

For positive integers l and n,

$$f(l,n) = \binom{n+l-1}{l-1}$$

is the number of n-element multisubsets of an l-element set. We now determine the multiset chromatic number of all regular complete multipartite graphs.

Theorem 2.1. *For positive integers k and n, the multiset chromatic number of the regular complete k-partite graph $K_{k(n)}$ is the unique positive integer l for which*

$$f(l-1,n) < k \leqslant f(l,n).$$

Proof. Denote the partite sets of $G = K_{k(n)}$ by $U_1, U_2, \ldots, U_k$, where then $|U_i| = n$ for each i with $1 \leqslant i \leqslant k$. We first claim that $\chi_m(G) \geqslant l$. Assume, to the contrary, that $\chi_m(G) \leqslant l-1$. Then there exists a multiset $(l-1)$-coloring c of G. Let $A = \{1, 2, \ldots, l-1\}$ denote the set of colors used by c and let S be the set of all n-element multisubsets of the set A. Thus $|S| = f(l-1,n)$. For $1 \leqslant i \leqslant k$, let S_i be the n-element multisubset of A that is used to color the vertices of U_i. Since $k > f(l-1,n)$, it follows that $S_i = S_j$ for some pair i,j of distinct integers with

$1 \leqslant i, j \leqslant k$. However then for $u \in U_i$ and $v \in U_j$, it follows that $\text{code}(u) = \text{code}(v)$, which is impossible. Thus, as claimed, $\chi_m(G) \geqslant l$.

Next, we show that $\chi_m(G) \leqslant l$. Let $B = \{1, 2, \ldots, l\}$. Since $k \leqslant f(l, n)$, there exist k distinct multisubsets $B_1, B_2, \ldots, B_k$ of B. For each i $(1 \leqslant i \leqslant k)$, assign the colors in the multiset B_i to the vertices of U_i. Let u and v be two adjacent vertices of G. Then $u \in U_i$ and $v \in U_j$ for distinct integers i and j with $1 \leqslant i, j \leqslant k$. Let B' be the multiset of colors of the vertices in $V(G) - (U_i \cup U_j)$. Since $M(u) = B_j \cup B'$, $M(v) = B_i \cup B'$, and $B_i \neq B_j$, it follows that $M(u) \neq M(v)$. Hence this l-coloring is a multiset coloring and so $\chi(G) \leqslant l$. □

We now consider more general complete multipartite graphs. We denote a complete multipartite graph containing k_i partite sets of cardinality n_i by

$$K_{k_1(n_1), k_2(n_2), \ldots, k_t(n_t)}.$$

Theorem 2.2. *Let $G = K_{k_1(n_1), k_2(n_2), \ldots, k_t(n_t)}$, where $n_1, n_2, \ldots, n_t$ are t distinct positive integers. Then*

$$\chi_m(G) = \max\{\chi_m(K_{k_i(n_i)}) \colon 1 \leqslant i \leqslant t\}.$$

Proof. Let $l_i = \chi_m(K_{k_i(n_i)})$ for $1 \leqslant i \leqslant t$. Assume, without loss of generality, that

$$l_1 = \max\{\chi_m(K_{k_i(n_i)}) \colon 1 \leqslant i \leqslant t\}.$$

We first show that $\chi_m(G) \leqslant l_1$. For each integer i with $1 \leqslant i \leqslant t$, let c_i be a multiset l_i-coloring of the subgraph $K_{k_i(n_i)}$ in G using the colors in $\{1, 2, \ldots, l_i\}$. We can now define a multiset l_1-coloring c of G by

$$c(x) = c_i(x) \ \text{ if } \ x \in V(K_{k_i(n_i)}) \ \text{ for } \ 1 \leqslant i \leqslant t.$$

Thus $\chi_m(G) \leqslant l_1$. Next, we show that $\chi_m(G) \geqslant l_1$. Assume, to the contrary, that $\chi_m(G) = l \leqslant l_1 - 1$. Let c' be a multiset l-coloring of G. Then c' induces a coloring c'_1 of the subgraph $K_{k_1(n_1)}$ in G such that $c'_1(x) = c'(x)$ for all $x \in V(K_{k_1(n_1)})$. Since c'_1 uses at most l colors and $\chi_m(K_{k_1(n_1)}) = l_1 > l$, it follows that c'_1 is not a multiset coloring of $K_{k_1(n_1)}$ and so there exist two adjacent vertices u and v in $K_{k_1(n_1)}$ having the same code with respect to c'_1. Since u and v are both adjacent to every vertex in $V(G) - V(K_{k_1(n_1)})$, it follows that u and v have the same code in G with respect to c', which is a contradiction. □

In particular, if $k_1 = k_2 = \ldots = k_t = 1$, then $K_{k_i(n_i)} = K_{1(n_i)} = \overline{K}_{n_i}$ for $1 \leqslant i \leqslant t$. Since $\chi_m(\overline{K}_{n_i}) = 1$ for $1 \leqslant i \leqslant t$, it follows that $\chi_m(K_{n_1,n_2,\ldots,n_t}) = 1$, where $n_1, n_2, \ldots, n_t$ are t distinct positive integers.

By (1), if G is a graph with $\chi_m(G) = a$ and $\chi(G) = b$, then $a \leqslant b$. In fact, each pair a, b of positive integers with $a \leqslant b$ is realizable as the multiset chromatic number and chromatic number, respectively, for some connected graph.

Proposition 2.3. *For each pair a, b of positive integers with $a \leqslant b$, there exists a connected graph G such that $\chi_m(G) = a$ and $\chi(G) = b$.*

Proof. If $a = b$, let $G = K_a$ and then $\chi_m(G) = \chi(G) = a$. Thus, we may assume that $a < b$. Let G be a complete b-partite graph with partite sets $V_1, V_2, \ldots, V_b$, where $|V_i| = 1$ for $1 \leqslant i \leqslant a$ and $2 \leqslant |V_{a+1}| < |V_{a+2}| < \ldots < |V_b|$. Then $\chi(G) = b$. It remains to show that $\chi_m(G) = a$. Let $U = V_1 \cup V_2 \cup \ldots \cup V_a$. By Observation 1.3, if c is a multiset coloring of G, then $c(x) \neq c(y)$ for every two distinct vertices x and y in U, which implies that $\chi_m(G) \geqslant a$. On the other hand, the coloring that assigns color i to the vertex in V_i for $1 \leqslant i \leqslant a$ and color 1 to the remaining vertices of G is a multiset a-coloring of G. Therefore, $\chi_m(G) = a$. □

3. The multiset chromatic numbers of powers of cycles

In addition to regular complete multipartite graphs, another well-known and large class of regular (and vertex-transitive) graphs are the powers of cycles. For a connected graph G of order n and an integer k with $1 \leqslant k < n$, the *k-th power* G^k of G is that graph with $V(G^k) = V(G)$ such that $uv \in E(G^k)$ if and only if $1 \leqslant d_G(u, v) \leqslant k$. Thus $G^1 = G$ and $G^k = K_n$ if $k \geqslant \operatorname{diam}(G)$. We begin with the cycles themselves and show that their multiset chromatic number equals their chromatic number.

Proposition 3.1. *For each integer $n \geqslant 3$, $\chi_m(C_n) = \chi(C_n)$.*

Proof. Since C_n is 2-regular, $\chi_m(C_n) \geqslant 2$ by Observation 1.1. If n is even, then $\chi_m(C_n) = 2$ by Proposition 1.2. If n is odd, then $\chi_m(C_n) = 2$ or $\chi_m(C_n) = 3$. We claim that $\chi_m(C_n) = 3$. Assume, to the contrary, that there exists a multiset 2-coloring $c\colon V(C_n) \to \{1, 2\}$. Let $C_n\colon v_1, v_2, \ldots, v_n, v_1$ and consider the cyclic color sequence

$$s\colon c(v_1), c(v_2), \ldots, c(v_n), c(v_1).$$

Necessarily, the sequence s has an even number of maximal subsequences consisting of terms of the same color. Observe that s cannot contain a maximal subsequence of s consisting of exactly two terms or of four or more terms of the same color.

Therefore, every maximal subsequence of s consisting of terms of the same color has length 1 or 3 and so has odd length, which is impossible since n is odd. Thus, as claimed, $\chi_m(C_n) = 3$ if n is odd. $\square$

Since $C_{2k}^{k-1} = K_{k(2)}$, we have the following by Theorem 2.1.

Proposition 3.2. *For each integer $k \geqslant 2$,*

$$\chi_m(C_{2k}^{k-1}) = \left\lceil \frac{-1 + \sqrt{8k+1}}{2} \right\rceil.$$

We now determine the multiset chromatic numbers of the squares of cycles.

Proposition 3.3. *For each integer $n \geqslant 3$,*

$$\chi_m(C_n^2) = \begin{cases} n & \text{if } 3 \leqslant n \leqslant 5, \\ 2 & \text{if } n \equiv 0 \pmod 6, \\ 3 & \text{otherwise.} \end{cases}$$

Proof. For $3 \leqslant n \leqslant 5$, observe that $C_n^2 = K_n$ and so $\chi_m(C_n^2) = n$; while $\chi_m(C_6^2) = 2$ by Proposition 3.2. For $n \geqslant 7$, let $C_n \colon v_1, v_2, \ldots, v_n, v_1$. Since C_n^2 is 4-regular, $\chi_m(C_n^2) \geqslant 2$. Suppose first that $6 \mid n$. Define the 2-coloring $c \colon V(C_n^2) \to \{1, 2\}$ by

$$c(v_i) = \begin{cases} 1 & \text{if } i \equiv 1, 2, 4 \pmod 6, \\ 2 & \text{if } i \equiv 3, 5, 0 \pmod 6. \end{cases}$$

Since

$$\operatorname{code}(v_i) = \begin{cases} (1,3) & \text{if } i \equiv 1 \pmod 3, \\ (2,2) & \text{if } i \equiv 2 \pmod 3, \\ (3,1) & \text{if } i \equiv 0 \pmod 3, \end{cases}$$

it follows that c is a multiset 2-coloring. Thus, $\chi_m(C_n^2) = 2$.

It now remains to show that if $n \geqslant 7$ and $6 \nmid n$, then $\chi_m(C_n^2) = 3$. Suppose that there exists a multiset 2-coloring $c \colon V(C_n^2) \to \{1, 2\}$. First, we claim that no vertex of C_n^2 can have color code $(4, 0)$, for suppose that $\operatorname{code}(v_3) = (4, 0)$. Then $c(v_1) = c(v_2) = c(v_4) = c(v_5) = 1$. Thus $c(v_3) = 2$, for otherwise $\{\operatorname{code}(v_2), \operatorname{code}(v_3), \operatorname{code}(v_4)\} \in \{(3,1), (4,0)\}$, which is impossible. Necessarily, $\{\operatorname{code}(v_2), \operatorname{code}(v_4)\} = \{(2,2), (3,1)\}$, say $\operatorname{code}(v_2) = (2,2)$ and $\operatorname{code}(v_4) = (3,1)$. Thus $c(v_6) = 1$. This implies that $\operatorname{code}(v_5) = (2,2)$, $c(v_7) = 2$, and $\operatorname{code}(v_6) \in \{(2,2), (3,1)\}$, which cannot occur. Therefore, as claimed, no vertex of C_n^2 can have color code $(4, 0)$. Similarly, no vertex of C_n^2 can have color code $(0, 4)$.

Since every vertex of C_n^2 has one of the three color codes $(3,1)$, $(2,2)$, and $(1,3)$, these color codes must occur cyclically about the vertices of C_n. Thus $3 \mid n$. Since $6 \nmid n$, it follows that n is odd. Suppose that n_j vertices ($j = 1,2$) are colored j in C_n^2, where $n = n_1 + n_2$. By summing the number of occurrences of the color j in the multiset $M(v_i)$ for $1 \leqslant i \leqslant n$, we obtain $n = 2n_1 = 2n_2$, which is impossible since n is odd. Hence $\chi_m(C_n^2) \geqslant 3$.

We now show that $\chi_m(C_n^2) \leqslant 3$ by defining a multiset 3-coloring of C_n^2. For $7 \leqslant n \leqslant 11$, construct C_n^2 from

$$C_n \colon\ u_1, u_2, \ldots, u_n, u_1$$

and let $c_n^* \colon V(C_n^2) \to \{1,2,3\}$ be the coloring such that if

$$s_n^* \colon\ c_n^*(u_1), c_n^*(u_2), \ldots, c_n^*(u_n)$$

is a color sequence of the vertices of C_n^2, then

$$\begin{aligned}
s_7^* &\colon\ 1,1,2,3,1,2,2,\\
s_8^* &\colon\ 1,1,2,3,3,1,2,2,\\
s_9^* &\colon\ 1,1,2,1,2,3,1,2,2,\\
s_{10}^* &\colon\ 1,1,2,3,3,3,3,1,2,2,\\
s_{11}^* &\colon\ 1,1,2,3,3,3,3,3,1,2,2.
\end{aligned}$$

(See Figure 2.) Observe that c_n^* is a multiset 3-coloring. Hence $\chi_m(C_n^2) \leqslant 3$ for $7 \leqslant n \leqslant 11$.

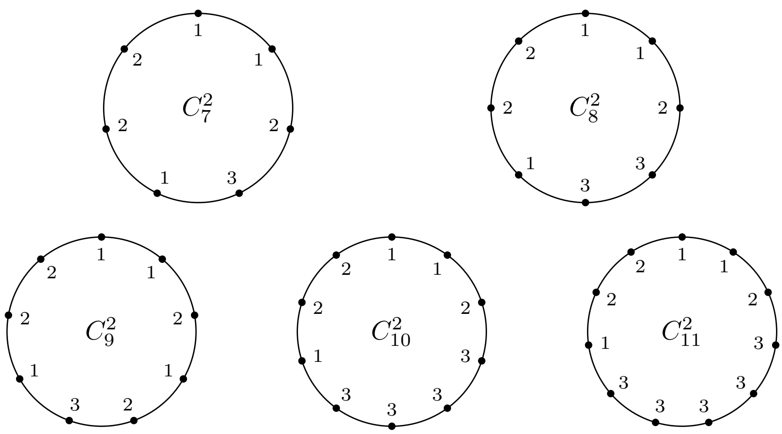

Figure 2: Multiset 3-colorings of C_n^2 for $7 \leqslant n \leqslant 11$

For $n \geqslant 13$ and $6 \nmid n$, let q, r be the unique pair of positive integers such that $n = 6q + r$, where $7 \leqslant r \leqslant 11$. Let

$$C_n\colon\ v_1, v_2, \ldots, v_n, v_1$$

and consider a coloring $c\colon\ V(C_n^2) \to \{1,2,3\}$ given by

$$c(v_i) = \begin{cases} 1 & \text{if } 1 \leqslant i \leqslant 6q \text{ and } i \equiv 1,2,4 \pmod 6, \\ 2 & \text{if } 1 \leqslant i \leqslant 6q \text{ and } i \equiv 3,5,0 \pmod 6, \\ c_r^*(u_{i-6q}) & \text{if } 6q+1 \leqslant i \leqslant 6q+r. \end{cases}$$

In other words, the color sequence

$$s_n\colon\ c(v_1), c(v_2), \ldots, c(v_n)$$

of the vertices of C_n^2 for $n = 6q + r \geqslant 13$ is

$$s_n\colon\ 1,1,2,1,2,2,\ldots,1,1,2,1,2,2,s_r^*.$$

Then

$$\operatorname{code}(v_i) = \begin{cases} (1,3,0) & \text{if } 1 \leqslant i \leqslant 6q \text{ and } i \equiv 1 \pmod 3, \\ (2,2,0) & \text{if } 1 \leqslant i \leqslant 6q \text{ and } i \equiv 2 \pmod 3, \\ (3,1,0) & \text{if } 1 \leqslant i \leqslant 6q \text{ and } i \equiv 0 \pmod 3, \\ \operatorname{code}_{c_r^*}(u_{i-6q}) & \text{if } 6q+1 \leqslant i \leqslant 6q+r. \end{cases}$$

Hence c is a multiset 3-coloring of C_n^2, that is, $\chi_m(C_n^2) \leqslant 3$ for $n \geqslant 7$ and $6 \nmid n$. This completes the proof. □

Observe that for $n \geqslant 6$,

$$\chi_m(C_n^2) = \begin{cases} \chi(C_n^2) & \text{if } n \equiv 3 \pmod 6, \\ \chi(C_n^2) - 1 & \text{otherwise.} \end{cases}$$

We now determine the multiset chromatic numbers of the cubes of cycles.

Proposition 3.4. *For each integer $n \geqslant 3$,*

$$\chi_m(C_n^3) = \begin{cases} n & \text{if } 3 \leqslant n \leqslant 7, \\ 3 & \text{if } n \geqslant 8. \end{cases}$$

Proof. For $3 \leqslant n \leqslant 7$, observe that $\chi_m(C_n^3) = \chi_m(K_n) = n$. Furthermore, $C_8^3 = K_{4(2)}$ and so $\chi_m(C_8^3) = 3$ by Proposition 3.2. We now show that $\chi_m(C_n^3) \geqslant 3$

for $n \geqslant 9$. Assume, to the contrary, that there exists a multiset 2-coloring c of C_n^3, where $C_n\colon v_1, v_2, \ldots, v_n, v_1$.

We claim that no vertex of C_n^3 can be labeled with the color code $(6,0)$, for suppose that $\text{code}(v_4) = (6,0)$. Then neither $\text{code}(v_3)$ nor $\text{code}(v_5)$ can be $(6,0)$. Necessarily, $\{\text{code}(v_3), \text{code}(v_5)\} = \{(4,2),(5,1)\}$, say $\text{code}(v_3) = (4,2)$ and $\text{code}(v_5) = (5,1)$. This implies that $c(v_4) = 2$ and $c(v_8) = 1$. Thus $\text{code}(v_6) \in \{(4,2),(5,1)\}$, which is impossible. Hence, as claimed, no vertex of C_n^3 can be labeled with the color code $(6,0)$. Similarly, no vertex of C_n^3 can be labeled with the color code $(0,6)$.

Therefore, every four consecutive vertices of C_n must be labeled in C_n^3 with four distinct color codes in the set $\{(1,5),(2,4),(3,3),(4,2),(5,1)\}$. Thus some vertex of C_n^3 has the color code $(5,1)$ or $(1,5)$, say $\text{code}(v_6) = (5,1)$. We may therefore assume that $c(v_3) = c(v_4) = c(v_5) = 1$. Since $\text{code}(v_5) \notin \{(5,1),(6,0)\}$ and at least one of $c(v_7)$ and $c(v_8)$ is 1, it follows that $\text{code}(v_5) \in \{(3,3),(4,2)\}$. Similarly, since $\text{code}(v_7) \notin \{(5,1),(6,0)\}$ and at least one of $c(v_8)$ and $c(v_9)$ is 1, it follows that $\text{code}(v_7) \in \{(3,3),(4,2)\}$. Therefore, $\{\text{code}(v_5), \text{code}(v_7)\} = \{(3,3),(4,2)\}$. Then $\text{code}(v_4) \notin \{(3,3),(4,2),(5,1),(6,0)\}$, implying that $\text{code}(v_4) = (2,4)$ and so $c(v_1) = c(v_2) = c(v_6) = c(v_7) = 2$. However, this now implies that $\text{code}(v_5) = (3,3)$ and $\text{code}(v_3) \in \{(2,4),(3,3)\}$, which is impossible.

Consequently, no vertex of C_n^3 can be labeled with the color code $(5,1)$ or, similarly, with $(1,5)$ either. This is impossible. Therefore, $\chi_m(C_n^3) \geqslant 3$ for $n \geqslant 9$.

To verify that $\chi_m(C_n^3) = 3$, it remains to show that there is a multiset 3-coloring of C_n^3 for every $n \geqslant 9$. For $8 \leqslant n \leqslant 13$, construct C_n^3 from

$$C_n\colon u_1, u_2, \ldots, u_n, u_1$$

and let $c_n^*\colon V(C_n^3) \to \{1,2,3\}$ be the coloring such that if

$$s_n^*\colon c_n^*(u_1), c_n^*(u_2), \ldots, c_n^*(u_n)$$

is a color sequence of the vertices of C_n^3, then

$$\begin{aligned}
&s_8^*\colon 1,1,2,1,1,2,3,3,\\
&s_9^*\colon 1,1,2,2,3,3,2,3,3,\\
&s_{10}^*\colon 1,1,2,2,3,3,1,2,3,3,\\
&s_{11}^*\colon 1,1,2,2,3,3,1,1,2,3,3,\\
&s_{12}^*\colon 1,1,2,2,3,3,1,1,2,2,3,3,\\
&s_{13}^*\colon 1,1,2,2,3,1,2,3,1,1,2,3,3.
\end{aligned}$$

(See Figure 3.) Observe that c_n^* is a multiset 3-coloring and so $\chi_m(C_n^3) \leqslant 3$ for $8 \leqslant n \leqslant 13$.

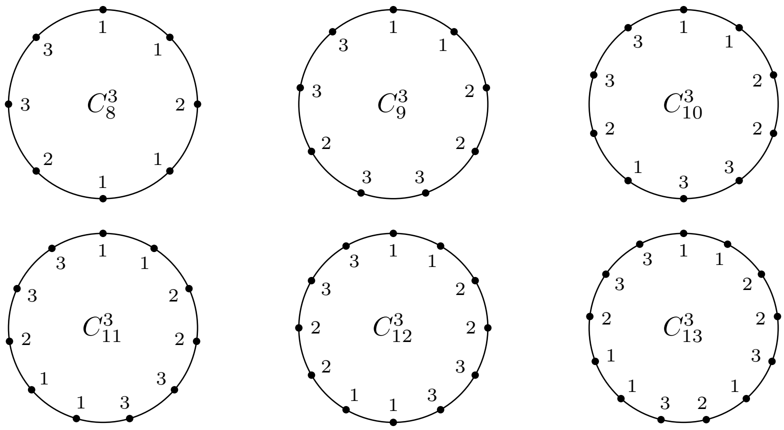

Figure 3: Multiset 3-colorings of C_n^3 for $8 \leqslant n \leqslant 13$

For $n \geqslant 14$, let q, r be the unique pair of positive integers such that $n = 6q + r$, where $8 \leqslant r \leqslant 13$. Let

$$C_n\colon\ v_1, v_2, \ldots, v_n, v_1$$

and consider a coloring $c\colon\ V(C_n^3) \to \{1, 2, 3\}$ given by

$$c(v_i) = \begin{cases} 1 & \text{if } 1 \leqslant i \leqslant 6q \text{ and } i \equiv 1, 2 \pmod 6, \\ 2 & \text{if } 1 \leqslant i \leqslant 6q \text{ and } i \equiv 3, 4 \pmod 6, \\ 3 & \text{if } 1 \leqslant i \leqslant 6q \text{ and } i \equiv 5, 0 \pmod 6, \\ c_r^*(u_{i-6q}) & \text{if } 6q + 1 \leqslant i \leqslant 6q + r. \end{cases}$$

In other words, the color sequence

$$s_n\colon\ c(v_1), c(v_2), \ldots, c(v_n)$$

of the vertices of C_n^3 for $n = 6q + r \geqslant 14$ is

$$s_n\colon\ 1, 1, 2, 2, 3, 3, \ldots, 1, 1, 2, 2, 3, 3, s_r^*.$$

Then

$$\text{code}(v_i) = \begin{cases} (1, 3, 2) & \text{if } 1 \leqslant i \leqslant 6q \text{ and } i \equiv 1 \pmod 6, \\ (1, 2, 3) & \text{if } 1 \leqslant i \leqslant 6q \text{ and } i \equiv 2 \pmod 6, \\ (2, 1, 3) & \text{if } 1 \leqslant i \leqslant 6q \text{ and } i \equiv 3 \pmod 6, \\ (3, 1, 2) & \text{if } 1 \leqslant i \leqslant 6q \text{ and } i \equiv 4 \pmod 6, \\ (3, 2, 1) & \text{if } 1 \leqslant i \leqslant 6q \text{ and } i \equiv 5 \pmod 6, \\ (2, 3, 1) & \text{if } 1 \leqslant i \leqslant 6q \text{ and } i \equiv 0 \pmod 6, \\ \text{code}_{c_r^*}(u_{i-6q}) & \text{if } 6q + 1 \leqslant i \leqslant 6q + r. \end{cases}$$

Hence c is a multiset 3-coloring of C_n^3, that is, $\chi_m(C_n^3) \leqslant 3$ for $n \geqslant 14$. Therefore, $\chi_m(C_n^3) = 3$ for $n \geqslant 8$. □

We next determine the multiset chromatic numbers of the fourth powers of cycles.

Proposition 3.5. *For each integer $n \geqslant 3$,*

$$\chi_m(C_n^4) = \begin{cases} n & \text{if } 3 \leqslant n \leqslant 9, \\ 3 & \text{if } n \geqslant 10. \end{cases}$$

Proof. For $3 \leqslant n \leqslant 9$, observe that $\chi_m(C_n^4) = \chi_m(K_n) = n$. We now show that $\chi_m(C_n^4) \geqslant 3$ for $n \geqslant 10$. Assume, to the contrary, that there exists a multiset 2-coloring c of C_n^4, where $C_n\colon\ v_1, v_2, \ldots, v_n, v_1$.

We first show that no vertex of C_n^4 can be labeled with the color code $(8,0)$, for suppose that $\mathrm{code}(v_5) = (8,0)$. Then necessarily $\{\mathrm{code}(v_4), \mathrm{code}(v_6)\} = \{(6,2),(7,1)\}$, say $\mathrm{code}(v_4) = (6,2)$ and $\mathrm{code}(v_6) = (7,1)$. Then $c(v_5) = 2$ and $c(v_{10}) = 1$. However, this implies that $\mathrm{code}(v_7) \in \{(6,2),(7,1),(8,0)\}$, which is impossible. Therefore, as claimed, no vertex of C_n^4 can be labeled with the color code $(8,0)$. Similarly, no vertex of C_n^4 can be labeled with the color code $(0,8)$.

Next we show that no vertex of C_n^4 can be labeled with the color code $(7,1)$. Assume, to the contrary, that $\mathrm{code}(v_5) = (7,1)$. Then without loss of generality, we may assume that $c(v_i) = 1$ for $1 \leqslant i \leqslant 4$. Each of the vertices v_4 and v_6 is adjacent to at least five vertices that are assigned the color 1 and so $\{\mathrm{code}(v_4), \mathrm{code}(v_6)\} = \{(5,3),(6,2)\}$. Then since v_3 is adjacent to at least four vertices that are assigned the color 1, it follows that $\mathrm{code}(v_3) = (4,4)$, which in turn implies that $\mathrm{code}(v_2) = (3,5)$. Therefore, we have $c(v_5) = c(v_6) = 2$ and $c(v_7) = c(v_8) = c(v_9) = 1$. However, this implies that $\mathrm{code}(v_7) \in \{(4,4),(5,3),(6,2)\}$, which cannot occur. Therefore, there is no vertex in C_n^4 that is labeled with $(7,1)$ or, similarly, with $(1,7)$ either.

Hence every vertex of C_n^4 has one of the five color codes $(2,6)$, $(3,5)$, $(4,4)$, $(5,2)$, and $(6,2)$. Furthermore, since $\omega(C_n^4) = 5$, these five color codes must occur cyclically about the vertices of C_n. Thus $5 \mid n$.

If $n = 10$, then observe that $\chi_m(C_{10}^4) = \chi_m(K_{5(2)}) = 3$ by Proposition 3.2, a contradiction. Hence suppose that $n \geqslant 15$. Without loss of generality, assume that $\mathrm{code}(v_5) = \mathrm{code}(v_{10}) = (6,2)$. If $c(v_i) = 1$ for $1 \leqslant i \leqslant 4$, then observe that $\mathrm{code}(v_i) \neq (2,6)$ for $1 \leqslant i \leqslant 5$, which is impossible. Similarly, it is impossible that $c(v_i) = 1$ for $6 \leqslant i \leqslant 9$ and for $11 \leqslant i \leqslant 14$. Therefore,

$$\{c(v_i)\colon\ 1 \leqslant i \leqslant 4\} = \{c(v_i)\colon\ 6 \leqslant i \leqslant 9\} = \{c(v_i)\colon\ 11 \leqslant i \leqslant 14\} = \{1,1,1,2\}$$

as multisets. However, this implies that each of the four vertices v_i $(6 \leqslant i \leqslant 9)$ is adjacent to at least three vertices that are colored 1, implying that $\mathrm{code}(v_i) \neq (2,6)$ for $5 \leqslant i \leqslant 10$. This is a contradiction. Therefore, $\chi_m(C_n^4) \geqslant 3$ for $n \geqslant 10$.

To verify that $\chi_m(C_n^4) = 3$, it remains to show that there is a multiset 3-coloring of C_n^4 for each $n \geqslant 10$. For $10 \leqslant n \leqslant 15$, construct C_n^4 from

$$C_n\colon\ u_1, u_2, \ldots, u_n, u_1$$

and let $c_n^*\colon\ V(C_n^4) \to \{1,2,3\}$ be the coloring so that if

$$s_n^*\colon\ c_n^*(u_1), c_n^*(u_2), \ldots, c_n^*(u_n)$$

is a color sequence of the vertices of C_n^4, then

$$\begin{aligned}
&s_{10}^*\colon\ 1,1,2,2,3,3,2,2,3,3,\\
&s_{11}^*\colon\ 1,1,2,2,3,3,1,2,2,3,3,\\
&s_{12}^*\colon\ 1,1,2,2,3,3,1,1,2,2,3,3,\\
&s_{13}^*\colon\ 1,1,2,2,3,3,1,1,1,2,2,3,3,\\
&s_{14}^*\colon\ 1,1,2,2,3,3,3,1,1,2,2,2,3,3,\\
&s_{15}^*\colon\ 1,1,2,2,2,3,3,2,1,1,2,2,2,3,3.
\end{aligned}$$

(See Figure 4.) Observe that c_n^* is a multiset 3-coloring and so $\chi_m(C_n^4) \leqslant 3$ for $10 \leqslant n \leqslant 15$.

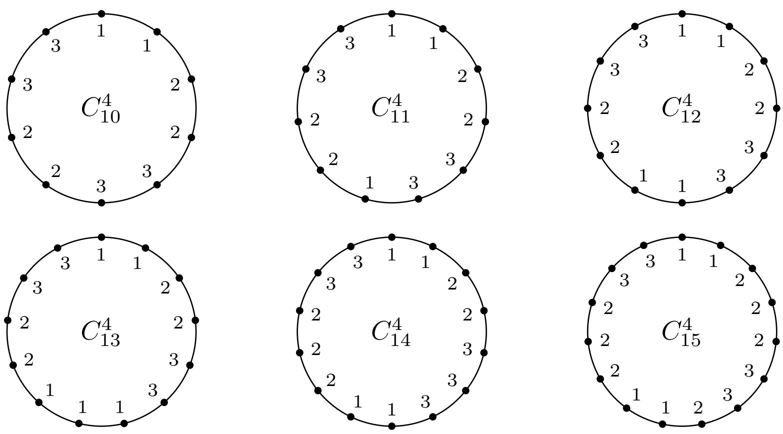

Figure 4: Multiset 3-colorings of C_n^4 for $10 \leqslant n \leqslant 15$

For $n \geqslant 16$, let q, r be the unique pair of positive integers such that $n = 6q + r$, where $10 \leqslant r \leqslant 15$. Let

$$C_n\colon\ v_1, v_2, \ldots, v_n, v_1$$

and consider a coloring $c\colon V(C_n^4) \to \{1,2,3\}$ given by

$$c(v_i) = \begin{cases} 1 & \text{if } 1 \leqslant i \leqslant 6q \text{ and } i \equiv 1,2 \pmod 6,\\ 2 & \text{if } 1 \leqslant i \leqslant 6q \text{ and } i \equiv 3,4 \pmod 6,\\ 3 & \text{if } 1 \leqslant i \leqslant 6q \text{ and } i \equiv 5,0 \pmod 6,\\ c_r^*(u_{i-6q}) & \text{if } 6q+1 \leqslant i \leqslant 6q+r. \end{cases}$$

In other words, the color sequence

$$s_n\colon\ c(v_1), c(v_2), \ldots, c(v_n)$$

of the vertices of C_n^4 for $n = 6q + r \geqslant 16$ is

$$s_n\colon\ 1,1,2,2,3,3,\ldots,1,1,2,2,3,3,s_r^*.$$

Then

$$\operatorname{code}(v_i) = \begin{cases} (1,4,3) & \text{if } 1 \leqslant i \leqslant 6q \text{ and } i \equiv 1 \pmod 6,\\ (1,3,4) & \text{if } 1 \leqslant i \leqslant 6q \text{ and } i \equiv 2 \pmod 6,\\ (3,1,4) & \text{if } 1 \leqslant i \leqslant 6q \text{ and } i \equiv 3 \pmod 6,\\ (4,1,3) & \text{if } 1 \leqslant i \leqslant 6q \text{ and } i \equiv 4 \pmod 6,\\ (4,3,1) & \text{if } 1 \leqslant i \leqslant 6q \text{ and } i \equiv 5 \pmod 6,\\ (3,4,1) & \text{if } 1 \leqslant i \leqslant 6q \text{ and } i \equiv 0 \pmod 6,\\ \operatorname{code}_{c_r^*}(u_{i-6q}) & \text{if } 6q+1 \leqslant i \leqslant 6q+r. \end{cases}$$

Hence c is a multiset 3-coloring of C_n^4, that is, $\chi_m(C_n^4) \leqslant 3$ for $n \geqslant 16$. This completes the proof. □

An upper bound for a more general class of powers of cycles is presented next.

Proposition 3.6. *Let $p \geqslant 2$ be an integer. If $(3p) \mid n$ and $n \geqslant 6p$, then*

$$\chi_m(C_n^k) \leqslant 3$$

for $2p - 1 \leqslant k \leqslant \left\lfloor \frac{1}{2}(5p-1) \right\rfloor$.

P r o o f. Suppose that $n = 3pl$, where $l \geqslant 2$ is an integer. Construct C_n^k from

$$\begin{aligned} C_n\colon\ & u_{1,1}, u_{1,2}, \ldots, u_{1,p}, v_{1,1}, v_{1,2}, \ldots, v_{1,p}, w_{1,1}, w_{1,2}, \ldots, w_{1,p},\\ & u_{2,1}, u_{2,2}, \ldots, u_{2,p}, v_{2,1}, v_{2,2}, \ldots, v_{2,p}, w_{2,1}, w_{2,2}, \ldots, w_{2,p}, \ldots\\ & u_{l,1}, u_{l,2}, \ldots, u_{l,p}, v_{l,1}, v_{l,2}, \ldots, v_{l,p}, w_{l,1}, w_{l,2}, \ldots, w_{l,p}, u_{1,1} \end{aligned}$$

and consider a 3-coloring $c\colon V(C_n^k) \to \{1,2,3\}$ defined by

$$c(x) = \begin{cases} 1 & \text{if } x = u_{j,i} \ (1 \leqslant i \leqslant p, 1 \leqslant j \leqslant l), \\ 2 & \text{if } x = v_{j,i} \ (1 \leqslant i \leqslant p, 1 \leqslant j \leqslant l), \\ 3 & \text{if } x = w_{j,i} \ (1 \leqslant i \leqslant p, 1 \leqslant j \leqslant l). \end{cases}$$

We show that c is a multiset coloring of C_n^k. By symmetry, observe that

$$\begin{aligned} \operatorname{code}(u_{j_1,i}) &= \operatorname{code}(u_{j_2,i}), \\ \operatorname{code}(v_{j_1,i}) &= \operatorname{code}(v_{j_2,i}), \\ \operatorname{code}(w_{j_1,i}) &= \operatorname{code}(w_{j_2,i}) \end{aligned}$$

for $1 \leqslant i \leqslant p$ and $1 \leqslant j_1, j_2 \leqslant l$. Hence we only consider the codes of $u_{1,i}$, $v_{1,i}$, and $w_{1,i}$ for $1 \leqslant i \leqslant p$. Furthermore, since $k < 3p$, it suffices to show that each of the $3p$ vertices $u_{1,1}, u_{1,2}, \ldots, u_{1,p}, v_{1,1}, v_{1,2}, \ldots, v_{1,p}, w_{1,1}, w_{1,2}, \ldots, w_{1,p}$ has a distinct code.

If $k = 2p-1, 2p$, then for $1 \leqslant i \leqslant p$,

$$\begin{aligned} \operatorname{code}(u_{1,i}) &= (p-1, k+1-i, k-p+i), \\ \operatorname{code}(v_{1,i}) &= (k-p+i, p-1, k+1-i), \\ \operatorname{code}(w_{1,i}) &= (k+1-i, k-p+i, p-1) \end{aligned}$$

and observe that the $3p$ codes are different.

If $2p+1 \leqslant k \leqslant \left\lfloor \frac{1}{2}(5p-1) \right\rfloor$, then

$$\begin{aligned} \operatorname{code}(u_{1,i}) &= \begin{cases} (k-p-i, 2p, k-p+i) & \text{if } 1 \leqslant i \leqslant k-2p, \\ (p-1, k+1-i, k-p+i) & \text{if } k-2p+1 \leqslant i \leqslant 3p-k, \\ (k-2p-1+i, k+1-i, 2p) & \text{if } 3p-k+1 \leqslant i \leqslant p, \end{cases} \\ \operatorname{code}(v_{1,i}) &= \begin{cases} (k-p+i, k-p-i, 2p) & \text{if } 1 \leqslant i \leqslant k-2p, \\ (k-p+i, p-1, k+1-i) & \text{if } k-2p+1 \leqslant i \leqslant 3p-k, \\ (2p, k-2p-1+i, k+1-i) & \text{if } 3p-k+1 \leqslant i \leqslant p, \end{cases} \\ \operatorname{code}(w_{1,i}) &= \begin{cases} (2p, k-p+i, k-p-i) & \text{if } 1 \leqslant i \leqslant k-2p, \\ (k+1-i, k-p+i, p-1) & \text{if } k-2p+1 \leqslant i \leqslant 3p-k, \\ (k+1-i, 2p, k-2p-1+i) & \text{if } 3p-k+1 \leqslant i \leqslant p \end{cases} \end{aligned}$$

and again the $3p$ codes are all different. Therefore, c is a multiset 3-coloring of C_n^k and so $\chi_m(C_n^k) \leqslant 3$. □

For example, for $l \geqslant 2$,

$$\begin{aligned}
\chi_m(C_{6l}^k) &\leqslant 3 \quad \text{for } k = 3, 4,\\
\chi_m(C_{9l}^k) &\leqslant 3 \quad \text{for } k = 5, 6, 7,\\
\chi_m(C_{12l}^k) &\leqslant 3 \quad \text{for } k = 7, 8, 9,\\
\chi_m(C_{15l}^k) &\leqslant 3 \quad \text{for } k = 9, 10, 11, 12,\\
\chi_m(C_{18l}^k) &\leqslant 3 \quad \text{for } k = 11, 12, 13, 14,\\
\chi_m(C_{21l}^k) &\leqslant 3 \quad \text{for } k = 13, 14, 15, 16, 17.
\end{aligned}$$

Based on the information above, we have the following conjecture.

Conjecture 3.7. For every integer $k \geqslant 3$, there exists an integer $f(k)$ such that $\chi_m(C_n^k) = 3$ for all $n \geqslant f(k)$.

From what we have seen, $f(k) = 2k + 2$ for $k = 3, 4$; however, we believe that $f(k) > 2k + 2$ for sufficiently large k.

4. Graphs with prescribed order and multiset chromatic number

We have seen that if G is a connected graph of order n and $\chi_m(G) = k$, then $1 \leqslant k \leqslant n$. Furthermore, $\chi_m(G) = n$ if and only if $G = K_n$. We now determine all pairs k, n of positive integers that are realizable as the multiset chromatic number and the order, respectively, for some connected graph.

Proposition 4.1. *Let k and n be integers with $1 \leqslant k \leqslant n$. Then there exists a connected graph G of order n with $\chi_m(G) = k$ if and only if $k \neq n - 1$.*

Proof. For $n = 1, 2$, the result immediately follows. Hence suppose that $n \geqslant 3$. For $k = 1$, let G be a connected graph of order n such that no two adjacent vertices of G have the same degree. Then $\chi_m(G) = 1$. For $k = n$, let $G = K_n$ and so $\chi_m(G) = n$. For $2 \leqslant k \leqslant n - 2$, let $G = K_{1,1,\ldots,1,n-k}$ be the complete $(k+1)$-partite graph such that k partite sets of G are singleton and one partite set of G consists of $n - k$ vertices. Since $n - k \geqslant 2$, it follows that $\chi_m(G) = k$. For the converse, assume, to the contrary, that there is a connected graph G of order n with $\chi_m(G) = n - 1$. Then $G \neq K_n$ and $\chi(G) = n - 1$. Thus G is obtained from K_{n-1} by joining a new vertex to some (but not all) vertices of K_{n-1}. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$, where the subgraph induced by $V(G) - \{v_n\}$ is K_{n-1} and v_n is adjacent to $v_1, v_2, \ldots, v_t$, where $1 \leqslant t \leqslant n-2$. The $(n-2)$-coloring c of G given by

$$c(v_i) = \begin{cases} i & \text{if } 1 \leqslant i \leqslant t, \\ i-1 & \text{if } t+1 \leqslant i \leqslant n-1, \\ n-2 & \text{if } i = n \end{cases}$$

is a multiset coloring and so $\chi_m(G) \leqslant n-2$, which is a contradiction. □

By Proposition 4.1, $\chi_m(G) \leqslant n-2$ if and only if $G \neq K_n$. Let $\mathcal{G}_n$ be the set of connected graphs G of order n with $\chi_m(G) = n-2$. For $3 \leqslant n \leqslant 5$,

$$\begin{aligned} \mathcal{G}_3 &= \{K_3 - e\,(= (K_1 \cup K_1) + K_1)\} \\ \mathcal{G}_4 &= \{K_4 - e, (K_2 \cup K_1) + K_1, C_4, P_4\} \\ \mathcal{G}_5 &= \{K_5 - e, (K_3 \cup K_1) + K_1, C_5\}. \end{aligned}$$

We next present a characterization of connected graphs G of order n with $\chi_m(G) = n-2$ for all $n \geqslant 6$. In order to do this, we first prove a useful lemma.

Lemma 4.2. *If G is a connected graph of order $n \geqslant 6$ and $\Delta(G) \leqslant n-2$, then $\chi_m(G) \leqslant n-3$.*

Proof. Since G is connected and $\Delta(G) \leqslant n-2$, the graph $\overline{G}$ contains $2K_2$ as a subgraph. If $\overline{G}$ contains either $K_2 \cup K_3$ or $3K_2$ as a subgraph, then $\chi(G) \leqslant n-3$ and so $\chi_m(G) \leqslant n-3$. Otherwise, let u_1, u_2, w_1, and w_2 be four distinct vertices in G such that $u_1w_1, u_2w_2 \notin E(G)$ and

$$X = V(G) - \{u_1, u_2, w_1, w_2\} = \{v_1, v_2, \ldots, v_{n-4}\}.$$

Since $\overline{G}$ does not contain $3K_2$, it follows that the subgraph induced by the $n-4$ vertices in X is K_{n-4}.

If there exists a vertex $v \in X$ that is adjacent to both u_1 and w_1 or to both u_2 and w_2, say v_1 is adjacent to both u_1 and w_1, then observe that the $(n-3)$-coloring $c_1\colon V(G) \to \{1, 2, \ldots, n-3\}$ given by

$$c_1(x) = \begin{cases} i & \text{if } x = v_i\ (1 \leqslant i \leqslant n-4), \\ 1 & \text{if } x = u_1, w_1, \\ n-3 & \text{if } x = u_2, w_2 \end{cases}$$

is neighbor-distinguishing. Therefore, $\chi_m(G) \leqslant n-3$.

There is only one case left to consider. For each $i = 1, 2$, suppose that one of u_i and w_i is adjacent to every vertex in X and the other is adjacent to no vertex in X, say u_1 and u_2 are adjacent to every vertex in X and w_1 and w_2 are adjacent to no vertex in X. Therefore, $\deg v = n-3$ for every $v \in X$, while

$$\deg u_i \in \{n-4, n-3, n-2\} \quad \text{and} \quad \deg w_i \in \{1, 2\}.$$

Also observe that $\deg u_i > \deg w_j$ for $1 \leqslant i, j \leqslant 2$ and $|\deg u_1 - \deg u_2| \leqslant 1$. If $\deg u_1 = \deg u_2$, then $u_1 w_2, u_2 w_1 \in E(G)$. Consider the coloring $c_2\colon V(G) \to \{1, 2, \ldots, n-3\}$ defined by

$$c_2(x) = \begin{cases} i & \text{if } x = v_i \ (1 \leqslant i \leqslant n-4) \text{ or } x = w_i \ (i = 1, 2), \\ n-3 & \text{if } x = u_1, u_2. \end{cases}$$

If $\deg u_1 \neq \deg u_2$, then let $u \in \{u_1, u_2\}$ such that $\deg u = n-3$ and consider the coloring $c_3\colon V(G) \to \{1, 2, \ldots, n-3\}$ defined by

$$c_3(x) = \begin{cases} i & \text{if } x = v_i \ (1 \leqslant i \leqslant n-4), \\ n-3 & \text{if } x = u, \\ 1 & \text{otherwise.} \end{cases}$$

Observe that both c_2 and c_3 are multiset colorings and so $\chi_m(G) \leqslant n-3$ in each case. $\square$

Theorem 4.3. *For a connected graph G of order $n \geqslant 6$, $\chi_m(G) = n-2$ if and only if $G \in \{K_n - e, (K_{n-2} \cup K_1) + K_1\}$.*

Proof. Let G be a connected graph of order $n \geqslant 6$. It is clear that if $G \in \{K_n - e, (K_{n-2} \cup K_1) + K_1\}$, then $\chi_m(G) = n-2$.

For the converse, suppose that $\chi_m(G) = n-2$ and let c be a multiset $(n-2)$-coloring of G. Then $G \neq K_n$ and by Lemma 4.2, $\Delta(G) = n-1$. Let $X = \{v_1, v_2, \ldots, v_{n'}\}$ be the set of vertices in G of degree $n-1$ and $Y = V(G) - X$. (Hence $1 \leqslant n' \leqslant n-2$.) Observe that c must assign a unique color to each vertex in X. Let H be the subgraph induced by the $n-n'$ vertices in Y and observe that

$$n-2 = \chi_m(G) \leqslant \max\{n', \chi_m(H)\}.$$

Note that since $G \neq K_n$, it follows that $H \neq K_{n-n'}$.

If $n' = n-2$, then $H = 2K_1$ and $G = K_n - e$. If $n' \leqslant n-3$, then let $H_1, H_2, \ldots, H_s$ be the components of H, where each H_i is a graph of order n_i and $n_1 \geqslant n_2 \geqslant \ldots \geqslant n_s$. Observe that

$$n-2 \leqslant \chi_m(H) = \max\{\chi_m(H_i)\colon 1 \leqslant i \leqslant s\} \leqslant n_1 \leqslant n-s,$$

that is, $s = 1$ or $s = 2$. If $s = 1$, then H is a noncomplete connected graph of order $n - n'$ and so $\chi_m(H) \leqslant (n - n') - 2 < n - 2$, which is impossible. If $s = 2$, then $\chi_m(H) = n_1 = n - 2$. Hence $H_1 = K_{n-2}$ and $H_2 = K_1$, implying that $G = (K_{n-2} \cup K_1) + K_1$. $\square$

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