Jiří Rachůnek; Dana Šalounová

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CLASSES OF FUZZY FILTERS OF RESIDUATED LATTICE ORDERED MONOIDS

Jiří Rachůnek, Olomouc, Dana Šalounová, Ostrava

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Abstract. The logical foundations of processes handling uncertainty in information use some classes of algebras as algebraic semantics. Bounded residuated lattice ordered monoids (Rl-monoids) are common generalizations of BL-algebras, i.e., algebras of the propositional basic fuzzy logic, and Heyting algebras, i.e., algebras of the propositional intuitionistic logic. From the point of view of uncertain information, sets of provable formulas in inference systems could be described by fuzzy filters of the corresponding algebras. In the paper we investigate implicative, positive implicative, Boolean and fantastic fuzzy filters of bounded Rl-monoids.

Keywords: residuated l-monoid, non-classical logics, basic fuzzy logic, intuitionistic logic, filter, fuzzy filter, BL-algebra, MV-algebra, Heyting algebra

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1. Introduction

As is well known, while information processing dealing with certain information is based on the classical two-valued logic, non-classical logics including logics behind fuzzy reasoning handle information with various facets of uncertainty such as fuzziness, randomness, vagueness, etc.

The classical two-valued logic has Boolean algebras as an algebraic semantics. Similarly, for important non-classical logics there are algebraic semantics in the form of classes of algebras. Using these classes, one can obtain an algebraization of inference systems that handle various kinds of uncertainty. The sets of provable formulas in inference systems are described by filters, and from the point of view of uncertain information, by fuzzy filters of the corresponding algebras.

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BL-algebras were introduced by P. Hájek as an algebraic counterpart of the basic fuzzy logic BL [7]. Omitting the requirement of pre-linearity in the definition of a BL-algebra, one obtains the definition of a bounded commutative residuated lattice ordered monoid (Rl-monoid). Nevertheless, bounded commutative Rl-monoids are a generalization not only of BL-algebras but also of Heyting algebras which are an algebraic counterpart of the intuitionistic propositional logic. Therefore, bounded commutative Rl-monoids could be taken as an algebraic semantics of a more general logic than Hájek's fuzzy logic. It is known that every BL-algebra (and consequently every MV-algebra [3]) is a subdirect product of linearly ordered BL-algebras. Moreover, a bounded commutative Rl-monoid is a subdirect product of linearly ordered Rl-monoids if and only if it is a BL-algebra [17]. On the other hand, bounded commutative Rl-monoids which need not be BL-algebras can be constructed from BL-algebras by means of other natural operations, e.g. by means of pasting, i.e. ordinal sums.

In both the BL-algebras and bounded commutative Rl-monoids, filters coincide with deductive systems of those algebras and are exactly the kernels of their congruences. Various types of filters of BL-algebras (Boolean deductive systems, implicative filters, positive implicative filters, fantastic filters) were studied in [23], [9] and [15]. Generalizations of these kinds of filters were introduced and investigated in [18] and [20].

Fuzzy ideals (or, in the dual form, fuzzy filters) of MV-algebras were introduced and developed in [11], [12], and their generalizations for pseudo MV-algebras in [14] and [5]. Moreover, fuzzy filters of bounded Rl-monoids were recently introduced and studied in [21]. Some related results one can also find in [25].

In the paper we further develop the theory of fuzzy filters of bounded commutative Rl-monoids. We introduce and investigate implicative fuzzy filters, positive implicative fuzzy filters, Boolean fuzzy filters and fantastic fuzzy filters of bounded commutative Rl-monoids and describe their mutual connection, as well as their relations to the corresponding filters.

For concepts and results concerning MV-algebras, BL-algebras and Heyting algebras see for instance [3], [7], [1].

2. Preliminaries

A *bounded commutative* Rl-*monoid* is an algebra $M = (M; \odot, \vee, \wedge, \rightarrow, 0, 1)$ of type $\langle 2, 2, 2, 2, 0, 0\rangle$ satisfying the following conditions:

(Rl1) $(M; \odot, 1)$ is a commutative monoid.

(Rl2) $(M; \vee, \wedge, 0, 1)$ is a bounded lattice.

(Rl3) $x \odot y \leqslant z$ if and only if $x \leqslant y \to z$ for any $x, y, z \in M$.

(Rl4) $x \odot (x \to y) = x \wedge y$ for any $x, y \in M$.

In the sequel, by an Rl-*monoid* we will mean a *bounded commutative* Rl-*monoid.*

On any Rl-monoid M let us define a unary operation negation $^-$ by $x^- := x \to 0$ for any $x \in M$.

R e m a r k 2.1. In fact, bounded commutative Rl-monoids can be also recognized as *commutative residuated lattices* [24], [6] *satisfying the divisibility condition* or as *divisible integral residuated commutative* l-*monoids* [10] or as *bounded integral commutative generalized* BL-*algebras* [2], [13], [4].

The above mentioned algebras can be characterized in the class of all Rl-monoids as follows: An Rl-monoid M is

a) BL-algebra if and only if M satisfies the identity of pre-linearity $(x \to y) \vee (y \to x) = 1$;

b) an MV-algebra if and only if M fulfils the double negation law $x^{--} = x$;

c) a Heyting algebra if and only if the operations "$\odot$" is idempotent.

When doing calculations, we will use the following list of basic rules for bounded Rl-monoids.

Lemma 2.2 [19], [22]. *In any bounded commutative* Rl-*monoid* M *we have for any* $x, y, z \in M$:

(1) $1 \to x = x$,

(2) $x \leqslant y \iff x \to y = 1$,

(3) $x \odot y \leqslant x \wedge y$,

(4) $x \leqslant y \to x$,

(5) $(x \odot y) \to z = x \to (y \to z) = y \to (x \to z)$,

(6) $(x \vee y) \to z = (x \to z) \wedge (y \to z)$,

(7) $x \to (y \wedge z) = (x \to y) \wedge (x \to z)$,

(8) $x \leqslant x^{--}$, $x^- = x^{---}$,

(9) $x \leqslant y \Longrightarrow y^- \leqslant x^-$,

(10) $(x \odot y)^- = y \to x^- = y^{--} \to x^- = x \to y^- = x^{--} \to y^-$,

(11) $x \leqslant y \Longrightarrow z \to x \leqslant z \to y$, $y \to z \leqslant x \to z$,

(12) $x \to y \leqslant y^- \to x^-$,

(13) $x \vee y \leqslant ((x \to y) \to y) \wedge ((y \to x) \to x)$,

(14) $x \to y \leqslant (y \to z) \to (x \to z)$,

(15) $x \to y \leqslant (z \to x) \to (z \to y)$.

A non-empty subset F of an Rl-monoid M is called a *filter* of M if

(F1) $x, y \in F$ imply $x \odot y \in F$;

(F2) $x \in F$, $y \in M$, $x \leqslant y$ imply $y \in F$.

A subset D of an Rl-monoid M is called a *deductive system* of M if

(i) $1 \in D$;

(ii) $x \in D$, $x \to y \in D$ imply $y \in D$.

Proposition 2.3 [4]. *Let H be a non-empty subset of an* Rl*-monoid M. Then H is a filter of M if and only if H is a deductive system of M.*

Filters of commutative Rl-monoids are exactly the kernels of their congruences. If F is a filter of M, then F is the kernel of the unique congruence $\Theta(F)$ such that $\langle x, y\rangle \in \Theta(F)$ if and only if $(x \to y) \wedge (y \to x) \in F$ for any $x, y \in M$. Hence we will consider quotient Rl-monoids M/F of Rl-monoids M by their filters F.

3. Fuzzy filters of Rl-monoids

Let $[0,1]$ be the closed unit interval of reals and let $M \neq \emptyset$ be a set. Recall that a *fuzzy set* in M is any function $\nu\colon M \longrightarrow [0,1]$.

A fuzzy set ν in an Rl-monoid M is called a *fuzzy filter* of M if any $x, y \in M$ satisfy

(f1) $\nu(x \odot y) \geqslant \nu(x) \wedge \nu(y)$,

(f2) $x \leqslant y \Longrightarrow \nu(x) \leqslant \nu(y)$.

By (f2), it follows immediately that

(f3) $\nu(1) \geqslant \nu(x)$ for every $x \in M$.

Lemma 3.1. *Let ν be a fuzzy filter of an* Rl*-monoid M. Then for any $x, y \in M$ we have*

(i) $\nu(x \vee y) \geqslant \nu(x) \wedge \nu(y)$,

(ii) $\nu(x \wedge y) = \nu(x) \wedge \nu(y)$,

(iii) $\nu(x \odot y) = \nu(x) \wedge \nu(y)$.

Proof. For any $x, y \in M$ we have $x \odot y \leqslant x \wedge y \leqslant x \vee y$. Then (f2) and (f1) imply $\nu(x \vee y) \geqslant \nu(x \odot y) \geqslant \nu(x) \wedge \nu(y)$. Since $x \odot y \leqslant x \wedge y \leqslant x, y$, it follows by (f1) and (f2) that $\nu(x) \wedge \nu(y) \leqslant \nu(x \odot y) \leqslant \nu(x \wedge y) \leqslant \nu(x) \wedge \nu(y)$. $\square$

Theorem 3.2. *A fuzzy set ν in an* Rl*-monoid M is a fuzzy filter of M if and only if it satisfies* (f1) *and*

(f4) $\nu(x \vee y) \geqslant \nu(x)$ *for any $x, y \in M$.*

Proof. If ν is a fuzzy filter of an Rl-monoid M then $x \leqslant x \vee y$ implies $\nu(x) \leqslant \nu(x \vee y)$.

Conversely, if ν satisfies (f1) and (f4) and $x \leqslant y$, then $\nu(y) = \nu(x \vee y) \geqslant \nu(x)$. Hence ν is a fuzzy filter of M. $\square$

Theorem 3.3. *Let ν be a fuzzy set in an* Rl-*monoid M. Then the following conditions are equivalent.*

(1) *ν is a fuzzy filter of M.*

(2) *ν satisfies* (f3) *and for all $x, y \in M$,*

$$(*) \qquad \nu(y) \geqslant \nu(x) \wedge \nu(x \to y).$$

Proof. (1)$\Rightarrow$(2): Let ν be a fuzzy filter of M and let $x, y \in M$. Then, by Lemma 3.1(iii), $\nu(y) \geqslant \nu(x \wedge y) = \nu((x \to y) \odot x) = \nu(x \to y) \wedge \nu(x)$. Hence ν satisfies the condition (2).

(2)$\Rightarrow$(1): Let ν be a fuzzy set in M satisfying (f3) and ($*$). Let $x, y \in M$, $x \leqslant y$. Then $x \to y = 1$. Thus $\nu(y) \geqslant \nu(x) \wedge \nu(1) = \nu(x)$, hence (f2) holds.

Further, since $x \leqslant y \to (x \odot y)$, by ($*$) and (f2) we get $\nu(x \odot y) \geqslant \nu(y) \wedge \nu(y \to (x \odot y)) \geqslant \nu(y) \wedge \nu(x)$. Therefore (f1) is also satisfied and hence ν is a fuzzy filter of M. □

Let F be a subset of M and let α, $\beta \in [0,1]$ be such that $\alpha > \beta$. Define a fuzzy subset $\nu_F(\alpha, \beta)$ in M by

$$\nu_F(\alpha, \beta)(x) := \begin{cases} \alpha, & \text{if } x \in F, \\ \beta, & \text{otherwise.} \end{cases}$$

In particular, $\nu_F(1, 0)$ is the characteristic function χ_F of F. We will use the notation ν_F instead of $\nu_F(\alpha, \beta)$ for every α, $\beta \in [0,1]$, $\alpha > \beta$.

Let ν be a fuzzy set in M and let $\alpha \in [0,1]$. The set

$$U(\nu; \alpha) := \{x \in M \colon \nu(x) \geqslant \alpha\}$$

is called the *level subset of ν determined by α*.

Kondo and Dudek in [16] formulated and proved the so-called *Transfer Principle* (TP) which can be used to any (general) algebra of any type:

Transfer Principle. *A fuzzy set λ defined in a (general) algebra $\mathcal{A}$ has a property $\mathcal{P}$ if and only if all non-empty level subsets $U(\lambda; \alpha)$ have the property $\mathcal{P}$.*

(A property $\mathcal{P}$ is defined in a standard way by means of terms of algebras. For more information concerning (TP) see [16].)

Some of the assertions of our paper will be immediate consequences of (TP) and of its corollaries in [16], hence the proofs of them will be omited. The first of them is

Theorem 3.4. *Let F be a non-empty subset of an* Rl*-monoid M. Then the fuzzy set ν_F is a fuzzy filter of M if and only if F is a filter of M.*

Let ν be a fuzzy set in an Rl-monoid M. Denote by M_ν the set

$$M_\nu := \{x \in M\colon\ \nu(x) = \nu(1)\}.$$

Note that $M_\nu = U(\nu; \nu(1))$, hence M_ν is a special case of a level subset of M.

Theorem 3.5. *If ν is a fuzzy filter of an* Rl*-monoid M, then M_ν is a filter of M.*

Proof. Let ν be a fuzzy filter of M. Let $x, y \in M_\nu$, i.e. $\nu(x) = \nu(1) = \nu(y)$. Then $\nu(x \odot y) \geqslant \nu(x) \wedge \nu(y) = \nu(1)$, hence $\nu(x \odot y) = \nu(1)$, thus $x \odot y \in M_\nu$.

Further, let $x \in M_\nu$, $y \in M$ and $x \leqslant y$. Then $\nu(1) = \nu(x) \leqslant \nu(y)$, hence $\nu(y) = \nu(1)$, and therefore $y \in M_\nu$.

That means M_ν is a filter of M. $\square$

The converse implication to that from Theorem 3.5 is not true in general, not even for pseudo MV-algebras, as was shown in [5, Example 3.9].

Theorem 3.6. *Let ν be a fuzzy set in an* Rl*-monoid M. Then ν is a fuzzy filter of M if and only if its level subset $U(\nu; \alpha)$ is a filter of M or $U(\nu; \alpha) = \emptyset$ for each $\alpha \in [0, 1]$.*

Proof. It follows from (TP). $\square$

Theorem 3.7. *Let ν be a fuzzy subset in an* Rl*-monoid M. Then the following conditions are equivalent.*

(1) *ν is a fuzzy filter of M.*

(2) *$\forall x, y, z \in M;\ x \to (y \to z) = 1 \Longrightarrow \nu(z) \geqslant \nu(x) \wedge \nu(y)$.*

Proof. (1)⇒(2): Let ν be a fuzzy filter of M. Let $x, y, z \in M$ and $x \to (y \to z) = 1$. Then by Theorem 3.3, $\nu(y \to z) \geqslant \nu(x) \wedge \nu(x \to (y \to z)) = \nu(x) \wedge \nu(1) = \nu(x)$.

Moreover, also by Theorem 3.3, $\nu(z) \geqslant \nu(y) \wedge \nu(y \to z)$, hence we obtain $\nu(z) \geqslant \nu(y) \wedge \nu(x)$.

(2)⇒(1): Let a fuzzy set ν in M satisfy the condition (2). Let $x, y \in M$. Since $x \to (x \to 1) = 1$, we have $\nu(1) \geqslant \nu(x) \wedge \nu(x) = \nu(x)$, hence (f3) is satisfied.

Further, since $(x \to y) \to (x \to y) = 1$ we get $\nu(y) \geqslant \nu(x \to y) \wedge \nu(x)$, thus ν satisfies $(*)$, which means, by Theorem 3.3, that ν is a fuzzy filter of M. $\square$

Corollary 3.8. *A fuzzy set ν in an* Rl*-monoid M is a fuzzy filter of M if and only if for all $x, y, z \in M$, $x \odot y \leqslant z$ implies $\nu(z) \geqslant \nu(x) \wedge \nu(y)$.*

4. Implicative fuzzy filters

Let M be an Rl-monoid and F a subset of M. Then F is called an *implicative filter* of M if

(I) $1 \in F$;

(II) $x \to (y \to z) \in F$, $x \to y \in F$ imply $x \to z \in F$ for any $x, y, z \in M$.

By [20], every implicative filter is a filter of M.

A fuzzy set ν in an Rl-monoid M is called an *implicative fuzzy filter* of M if for any $x, y, z \in M$

(1) $\nu(1) \geqslant \nu(x)$;

(2) $\nu(x \to (y \to z)) \wedge \nu(x \to y) \leqslant \nu(x \to z)$.

Proposition 4.1. *Every implicative fuzzy filter of an* Rl*-monoid M is a fuzzy filter of M.*

Proof. Let ν be an implicative fuzzy filter of M. Let $\alpha \in [0, 1]$ be such that $U(\nu; \alpha) \neq \emptyset$. Then for any $x \in U(\nu; \alpha)$ we have $\nu(1) \geqslant \nu(x)$, thus $1 \in U(\nu; \alpha)$.

Let x, $x \to y \in U(\nu; \alpha)$, i.e. $\nu(x)$, $\nu(x \to y) \geqslant \alpha$. Then $\nu(1 \to x)$, $\nu(1 \to (x \to y)) \geqslant \alpha$, hence $\nu(1 \to (x \to y)) \wedge \nu(1 \to x) \geqslant \alpha$, thus by (2), $\nu(1 \to y) \geqslant \alpha$. That means $\nu(y) \geqslant \alpha$, and therefore $y \in U(\nu; \alpha)$. Hence by Theorem 3.6, ν is a fuzzy filter of M. □

Theorem 4.2. *A filter F of an* Rl*-monoid M is implicative if and only if ν_F is an implicative fuzzy filter of M.*

Proof. It follows from (TP). □

Theorem 4.3 ([20, Theorem 3.3])**.** *Let F be a filter of an* Rl*-monoid M. Then the following conditions are equivalent.*

(a) *F is an implicative filter of M.*

(b) *$y \to (y \to x) \in F$ implies $y \to x \in F$ for any $x, y \in M$.*

(c) *$z \to (y \to x) \in F$ implies $(z \to y) \to (z \to x) \in F$ for any $x, y, z \in M$.*

(d) *$z \to (y \to (y \to x)) \in F$ and $z \in F$ imply $y \to x \in F$ for any $x, y, z \in M$.*

(e) *$x \to (x \odot x) \in F$ for any $x \in M$.*

Theorem 4.4. *Let F be a filter of an* Rl-*monoid M. Then the following conditions are equivalent.*

(a) ν_F *is an implicative fuzzy filter of M.*

(b) $\nu_F(y \to (y \to x)) \leqslant \nu_F(y \to x)$ *for any $x, y \in M$.*

(c) $\nu_F(z \to (y \to x)) \leqslant \nu_F((z \to y) \to (z \to x))$ *for any $x, y, z \in M$.*

(d) $\nu_F(z \to (y \to (y \to x))) \wedge \nu_F(z) \leqslant \nu_F(y \to x)$ *for any $x, y, z \in M$.*

(e) $\nu_F(x \to (x \odot x)) = \nu_F(1)$.

Proof. (a)$\Leftrightarrow$(b): Let ν_F be an implicative fuzzy filter of M. Then by Theorem 4.2, F is an implicative filter of M, and hence by Theorem 4.3, $y \to (y \to x) \in F$ implies $y \to x \in F$ for any $x, y \in M$. Let $x, y \in M$ and let $\nu_F(y \to (y \to x)) = \alpha$. Then $\nu_F(y \to x) = \alpha$, and thus ν_F satisfies the condition (b).

Conversely, let ν_F satisfy (b). Let $x, y \in M$ and $y \to (y \to x) \in F$. Then $\nu_F(y \to (y \to x)) = \alpha$, hence also $\nu_F(y \to x) = \alpha$, that means $y \to x \in F$. Therefore by 4.3, F is an implicative fuzzy filter of M.

The proofs of the equivalences (a)$\Leftrightarrow$(c), (a)$\Leftrightarrow$(d) and (a)$\Leftrightarrow$(e) are analogous, and hence they are omitted. $\square$

Theorem 4.5. *Let ν be a fuzzy filter of an* Rl-*monoid M. Then ν is an implicative fuzzy filter of M if and only if $U(\nu; \alpha)$ is an implicative filter for any $\alpha \in [0, 1]$ such that $U(\nu; \alpha) \neq \emptyset$.*

Proof. It follows from (TP). $\square$

As a consequence we obtain the following theorem.

Theorem 4.6. *If ν is a fuzzy filter of an* Rl-*monoid M, then ν is an implicative fuzzy filter of M if and only if $U(\nu; \alpha)$ satisfies any of conditions* (b)–(e) *of Theorem* 4.3 *for each $\alpha \in [0, 1]$ such that $U(\nu; \alpha) \neq \emptyset$.*

Theorem 4.7 ([20, Theorem 3.4])**.** *If F is a filter of an* Rl-*monoid M, then F is an implicative filter if and only if the quotient* Rl-*monoid M/F is a Heyting algebra.*

The following assertion follows from Theorems 4.6 and 4.7.

Theorem 4.8. *If ν is a fuzzy filter of an* Rl-*monoid M, then ν is an implicative fuzzy filter of M if and only if the quotient* Rl-*monoid $M/U(\nu; \alpha)$ is a Heyting algebra for any $\alpha \in [0, 1]$ such that $U(\nu; \alpha) \neq \emptyset$.*

Theorem 4.9 ([20, Theorem 3.10]). *Let M be an* Rl-*monoid. Then the following conditions are equivalent.*

(a) *M is a Heyting algebra.*

(b) *Every filter of M is implicative.*

(c) *$\{1\}$ is an implicative filter of M.*

Theorem 4.10. *Let M be an* Rl-*monoid. Then the following conditions are equivalent.*

(a) *M is a Heyting algebra.*

(b) *Every fuzzy filter of M is implicative.*

(c) *Every fuzzy filter ν of M such that $\nu(1) = 1$ is implicative.*

(d) *$\chi_{\{1\}}$ is an implicative fuzzy filter of M.*

Proof. (a)⇒(b): Let M be a Heyting algebra and ν a fuzzy filter of M. If $\alpha \in [0,1]$ and $U(\nu;\alpha) \neq \emptyset$, then $U(\nu;\alpha)$ is, by Theorem 4.9, an implicative filter of M. Hence by Theorem 4.5, ν is an implicative fuzzy filter of M.

(b)⇒(c), (c)⇒(d): Obvious.

(d)⇒(a): If the fuzzy filter $\chi_{\{1\}} = \nu_{\{1\}}(1,0)$ is implicative, then by Theorem 4.2, $\{1\}$ is an implicative filter of M, and hence by Theorem 4.9, M is a Heyting algebra. □

5. Positive implicative and Boolean fuzzy filters

Let M be an Rl-monoid and F a subset of M. Then F is called a *positive implicative filter* of M if

(I) $1 \in F$;

(III) $x \to ((y \to z) \to y) \in F$ and $x \in F$ imply $y \in F$ for any $x, y, z \in M$.

By [20], every positive implicative filter of M is a filter of M.

A fuzzy set ν in an Rl-monoid M is called a *positive implicative fuzzy filter* of M if for any $x, y, z \in M$,

(1) $\nu(1) \geqslant \nu(x)$;

(3) $\nu(x \to ((y \to z) \to y)) \wedge \nu(x) \leqslant \nu(y)$.

Proposition 5.1. *Every positive implicative fuzzy filter of an* Rl-*monoid M is a fuzzy filter of M.*

Proof. Let ν be a positive implicative fuzzy filter of $M, \alpha \in [0,1]$ and $U(\nu;\alpha) \neq \emptyset$. Then $1 \in U(\nu;\alpha)$.

Further, let x, $x \to y \in U(\nu;\alpha)$, i.e. $\nu(x)$, $\nu(x \to y) \geqslant \alpha$. Then $\nu(x \to ((y \to 1) \to y)) = \nu(x \to (1 \to y)) = \nu(x \to y)$, hence $\nu(x \to ((y \to 1) \to y) \wedge \nu(x) \geqslant \alpha$ and thus by (3), $\nu(y) \geqslant \alpha$. Therefore $y \in U(\nu;\alpha)$.

That means, by Theorem 3.6, ν is a fuzzy filter of M. $\square$

Theorem 5.2. *A filter F of an* Rl*-monoid M is positive implicative if and only if ν_F is a positive implicative fuzzy filter of M.*

Proof. Let F be a filter of M. Let us suppose that F is positive implicative. Let $\nu_F(x \to ((y \to z) \to y)) \wedge \nu_F(x) = \alpha$. Then $\nu_F(x \to ((y \to z) \to y)) = \alpha = \nu_F(x)$, thus $x \to ((y \to z) \to y)$, $x \in F$, and hence $y \in F$, that means $\nu_F(y) = \alpha$. Therefore we get that ν_F is a positive implicative fuzzy filter of M.

Conversely, let ν_F be a positive implicative fuzzy filter of M. Let $x \to ((y \to z) \to y) \in F$ and $x \in F$. Then $\nu_F(x \to ((y \to z) \to y)) = \alpha = \nu_F(x)$, hence $\nu_F(y) = \alpha$ and so $y \in F$. That means F is a positive implicative filter of M. $\square$

Theorem 5.3 ([20, Theorem 3.8])**.** *Let F be a filter of an* Rl*-monoid M. Then the following conditions are equivalent.*

(a) *F is a positive implicative filter of M.*
(b) *$(x \to y) \to x \in F$ implies $x \in F$ for any $x, y \in M$.*
(c) *$(x^- \to x) \to x \in F$ for any $x \in M$.*

Theorem 5.4. *Let F be a filter of an* Rl*-monoid M. Then the following conditions are equivalent.*

(a) *ν_F is a positive implicative fuzzy filter of M.*
(b) *$\nu_F((x \to y) \to x) \leqslant \nu_F(x)$ for any $x, y \in M$.*
(c) *$\nu_F((x^- \to x) \to x) = \nu_F(1)$.*

Proof. Analogous to that for Theorem 4.4. $\square$

Theorem 5.5. *Let ν be a fuzzy filter of an* Rl*-monoid M. Then ν is a positive implicative fuzzy filter of M if and only if $U(\nu;\alpha)$ is a positive implicative filter of M for every $\alpha \in [0,1]$ such that $U(\nu;\alpha) \neq \emptyset$.*

Proof. Let us suppose that ν is a fuzzy filter of M. Let ν be positive implicative, $\alpha \in [0,1]$, $U(\nu;\alpha) \neq \emptyset$, $x, y, z \in M$ and $x \to ((y \to z) \to y) \in U(\nu;\alpha)$, $x \in U(\nu;\alpha)$. Then $\nu(x \to ((y \to z) \to y))$, $\nu(x) \geqslant \alpha$, hence $\nu(x \to ((y \to z) \to y)) \wedge \nu(x) \geqslant \alpha$. Since $\nu(y) \geqslant \nu(x \to ((y \to z) \to y)) \wedge \nu(x)$, we get $y \in U(\nu;\alpha)$. Therefore the filter $U(\nu;\alpha)$ is positive implicative.

Conversely, let ν be such that $U(\nu;\alpha)$ is a positive implicative filter for any $\alpha \in [0,1]$ such that $U(\nu;\alpha) \neq \emptyset$. If $x, y, z \in M$, then $x \to ((y \to z) \to y)$, $x \in U(\nu; (x \to$

$((y \to z) \to y)) \wedge x)$, thus also $y \in U(\nu; (x \to ((y \to z) \to y)) \wedge x)$, hence $\nu(y) \geqslant \nu((x \to ((y \to z) \to y)) \wedge x) = \nu(x \to ((y \to z) \to y))) \wedge \nu(x)$. That means ν is a positive implicative fuzzy filter. ☐

Theorem 5.6. *Every positive implicative fuzzy filter of an* Rl-*monoid M is implicative.*

P r o o f. Let ν be a positive implicative fuzzy filter of M. Then by Theorem 5.5, if $\alpha \in [0,1]$ is such that $U(\nu;\alpha) \neq \emptyset$ then $U(\nu;\alpha)$ is a positive implicative filter of M. Hence by [20, Proposition 3.7], $U(\nu;\alpha)$ is also an implicative filter of M. Therefore by Theorem 4.5, ν is an implicative fuzzy filter of M. ☐

Theorem 5.7 ([20, Proposition 3.11])**.** *Let F be an implicative filter of an* Rl-*monoid M. Then the following conditions are equivalent.*

(a) *F is a positive implicative filter of M.*
(b) *$(x \to y) \to y \in F$ implies $(y \to x) \to x \in F$ for any $x, y \in M$.*

Theorem 5.8. *Let F be an implicative filter of an* Rl-*monoid M. Then the following conditions are equivalent.*

(a) *ν_F is a positive implicative fuzzy filter of M.*
(b) *$\nu_F((x \to y) \to y) = \nu_F((y \to x) \to x)$ for any $x, y \in M$.*

P r o o f. (a)⇒(b): Let ν_F be a positive implicative fuzzy filter of M. Then by Theorem 5.2, F is a positive implicative filter of M hence $(x \to y) \to y \in F$ implies $(y \to x) \to x \in F$ for any $x, y \in M$. Let $\nu_F((x \to y) \to y) = \alpha$. Then also $\nu_F((y \to x) \to x) = \alpha$, and thus ν_F satisfies (b).

(b)⇒(a): Let ν_F satisfy (b) and let $\nu_F(x \to ((y \to z) \to y)) = \alpha = \nu_F(x)$. Then $x \to ((y \to z) \to y)$, $x \in F$, and hence also $(y \to z) \to y \in F$. Further, $(y \to z) \to y \leqslant (y \to z) \to ((y \to z) \to z)$, therefore $(y \to z) \to ((y \to z) \to z) \in F$. Since F is an implicative filter of M, by Theorem 4.3 we get $(y \to z) \to z \in F$, and consequently by Theorem 5.7, F is a positive implicative filter of M. Therefore by Theorem 5.2, ν_F is a positive implicative fuzzy filter of M. ☐

Theorem 5.9 ([20, Theorem 3.12])**.** *Let M be an* Rl-*monoid. Then the following conditions are equivalent.*

(a) *$\{1\}$ is a positive implicative filter.*
(b) *Every filter of M is positive implicative.*
(c) *M is a Boolean algebra.*

Theorem 5.10. *Let M be an* Rl-*monoid. Then the following conditions are equivalent.*

(a) *M is a Boolean algebra.*

(b) *Every fuzzy filter of M is positive implicative.*

(c) *Every fuzzy filter ν of M such that $\nu(1) = 1$ is positive implicative.*

(d) *$\chi_{\{1\}}$ is a positive implicative fuzzy filter of M.*

P r o o f. (a)$\Rightarrow$(b): Let M be a Boolean algebra and ν a fuzzy filter of M. Let $\alpha \in [0,1]$ be such that $U(\nu;\alpha) \neq \emptyset$. Then by Theorem 5.9, $U(\nu;\alpha)$ is a positive implicative filter of M. Hence by Theorem 5.5 we obtain that ν is a positive implicative fuzzy filter of M.

(b)$\Rightarrow$(c), (c)$\Rightarrow$(d): Obvious.

(d)$\Rightarrow$(a): Let $\chi_{\{1\}} = \nu_{\{1\}}(1;0)$ be a positive implicative fuzzy filter of M. Then by Theorem 5.2 we get that $\{1\}$ is a positive implicative filter of M, and therefore by Theorem 5.9, M is a Boolean algebra. $\square$

A filter F of an Rl-monoid M is called a *Boolean filter* of M, if for any $x \in M$, $x \vee x^- \in F$.

A fuzzy filter ν of an Rl-monoid M is called a *Boolean fuzzy filter* of M, if for any $x \in M$, $\nu(x \vee x^-) = \nu(1)$.

Theorem 5.11. *A filter F of an* Rl-*monoid M is Boolean if and only if ν_F is a Boolean fuzzy filter of M.*

P r o o f. Let F be a Boolean filter of M and let $x \in M$. Then $\nu_F(x \vee x^-) = \alpha = \nu_F(1)$, hence ν_F is a Boolean fuzzy filter of M.

Conversely, let ν_F be a Boolean fuzzy filter of M and let $x \in F$. Then $\nu_F(x \vee x^-) = \nu_F(1) = \alpha$, then $x \vee x^- \in F$, that means F is a Boolean filter of M. $\square$

Theorem 5.12. *Let ν be a fuzzy filter of an* Rl-*monoid M. Then the following conditions are equivalent.*

(a) *ν is a Boolean fuzzy filter of M.*

(b) *If $\alpha \in [0,1]$ is such that $U(\nu;\alpha) \neq \emptyset$, then $U(\nu;\alpha)$ is a Boolean filter of M.*

(c) *$M_\nu = U(\nu;\nu(1))$ is a Boolean filter of M.*

P r o o f. It follows from (TP). $\square$

Theorem 5.13. *Let ν be a fuzzy filter of an* Rl*-monoid M. Then ν is Boolean if and only if the quotient* Rl*-monoid $M/U(\nu;\alpha)$ is a Boolean algebra for any $\alpha \in [0,1]$ such that $U(\nu;\alpha) \neq \emptyset$.*

P r o o f. In [18] it is proved that a filter F of an Rl-monoid M is Boolean if and only if M/F is a Boolean algebra. Hence the assertion is a corollary of the preceding theorem. $\square$

Theorem 5.14 ([21])**.** *A filter F of an* Rl*-monoid M is positive implicative if and only if F is a Boolean filter.*

As a consequence of Theorems 5.2, 5.11 and 5.14 we get

Theorem 5.15. *If F is a filter of an* Rl*-monoid M then for the fuzzy filter ν_F the following conditions are equivalent.*
(a) *ν_F is a positive implicative fuzzy filter of M.*
(b) *ν_F is a Boolean fuzzy filter of M.*

Analogously, from Theorems 5.5, 5.12 and 5.14 we obtain

Theorem 5.16. *Let ν be a fuzzy filter of an* Rl*-monoid M. Then the following conditions are equivalent.*
(a) *If $\alpha \in [0,1]$ is such that $U(\nu;\alpha) \neq \emptyset$, then $U(\nu;\alpha)$ is a positive implicative filter of M.*
(b) *If $\alpha \in [0,1]$ is such that $U(\nu;\alpha) \neq \emptyset$, then $U(\nu;\alpha)$ is a Boolean filter of M.*

R e m a r k 5.17. Theorems 4.8 and 5.16 give an alternative proof of Theorem 5.6.

6. Fantastic fuzzy filters

Let M be an Rl-monoid and F a subset of M. Then F is called a *fantastic filter* of M if
(I) $1 \in F$;
(IV) $z \to (y \to x) \in F$ and $z \in F$ imply $((x \to y) \to y) \to x \in F$ for any $x, y, z \in M$.

By [20], every fantastic filter is a filter of M.

A fuzzy subset ν in an Rl-monoid M is called a *fantastic fuzzy filter* of M if for any $x, y, z \in M$,
(1) $\nu(1) \geqslant \nu(x)$;
(4) $\nu(z \to (y \to x)) \wedge \nu(z) \leqslant \nu(((x \to y) \to y) \to x)$.

Proposition 6.1. *Every fantastic fuzzy filter of an* Rl-*monoid M is a fuzzy filter of M.*

P r o o f. Let ν be a fantastic fuzzy filter of M. Let $\alpha \in [0,1]$ and $U(\nu;\alpha) \neq \emptyset$. Then $1 \in U(\nu;\alpha)$. Let x, $x \to y \in U(\nu;\alpha)$, i.e., $\nu(x)$, $\nu(x \to y) \geqslant \alpha$. Then $\nu(x \to (1 \to y)) = \nu(x \to y) \geqslant \alpha$, hence $\nu(x \to (1 \to y)) \wedge \nu(x) \geqslant \alpha$, thus by (4), $\nu(y) = \nu(1 \to y) = \nu(((y \to 1) \to 1) \to y) \geqslant \nu(x \to (1 \to y)) \wedge \nu(x) \geqslant \alpha$, and so $y \in U(\nu;\alpha)$. Therefore by Theorem 3.6, ν is a fuzzy filter of M. $\square$

Theorem 6.2. *A filter F of an* Rl-*monoid M is fantastic if and only if ν_F is a fantastic fuzzy filter of M.*

P r o o f. Let F be a filter of M. Let us suppose that F is fantastic. Let $\nu_F(z \to (y \to x)) \wedge \nu_F(z) = \alpha$. Then $\nu_F(z \to (y \to x)) = \alpha = \nu_F(z)$, thus $z \to (y \to x) \in F$, $z \in F$, and hence $((x \to y) \to y) \to x \in F$, that means $\nu(((x \to y) \to y) \to x) = \alpha$. Therefore we get that ν_F is a fantastic fuzzy filter of M.

Conversely, let ν_F be a fantastic fuzzy filter of M. Let $z \to (y \to x) \in F$ and $z \in F$. Then $\nu_F(z \to (y \to x)) = \alpha = \nu_F(z)$, hence $\nu_F(((x \to y) \to y) \to x) = \alpha$, and therefore $((x \to y) \to y) \to x \in F$. That means, F is a fantastic filter of M. $\square$

Theorem 6.3 ([20, Theorems 4.2, 4.4])**.** *Let F be a filter of an* Rl-*monoid M. Then the following conditions are equivalent:*

(a) *F is a fantastic filter of M.*
(b) *$y \to x \in F$ implies $((x \to y) \to y) \to x \in F$ for every $x, y \in M$.*
(c) *$x^{--} \to x \in F$ for every $x \in M$.*
(d) *$x \to z \in F$ and $y \to z \in F$ imply $((x \to y) \to y) \to z \in F$ for every $x, y, z \in M$.*

Theorem 6.4. *Let F be a filter of an* Rl-*monoid M. Then the following conditions are equivalent.*

(a) *ν_F is a fantastic fuzzy filter of M.*
(b) *$\nu_F(y \to x) \leqslant \nu_F(((x \to y) \to y) \to x)$ for any $x, y \in M$.*
(c) *$\nu_F(x^{--} \to x) = \nu_F(1)$ for any $x \in M$.*
(c) *$\nu_F(x \to z) \wedge \nu_F(y \to z) \leqslant \nu_F(((x \to y) \to y) \to z)$ for any $x, y, z \in M$.*

P r o o f. Analogous to that for Theorem 4.4. $\square$

Theorem 6.5. *Let ν be a fuzzy filter of an* Rl-*monoid M. Then ν is a fantastic fuzzy filter of M if and only if $U(\nu;\alpha)$ is a fantastic filter of M for every $\alpha \in [0,1]$ such that $U(\nu;\alpha) \neq \emptyset$.*

P r o o f. It follows from (TP). $\square$

As a consequence we obtain the following theorem.

Theorem 6.6. *Let ν be a fuzzy filter of an* Rl-*monoid M. Then ν is a fantastic fuzzy filter of M if and only if $U(\nu;\alpha)$ satisfies each of conditions* (b)–(d) *of Theorem* 6.3 *for every $\alpha \in [0,1]$ such that $U(\nu;\alpha) \neq \emptyset$.*

Theorem 6.7. *Every positive implicative fuzzy filter of an* Rl-*monoid M is fantastic.*

P r o o f. If ν is a positive implicative fuzzy filter of M, then by Theorem 5.5, $U(\nu;\alpha)$ is a positive implicative filter of M for every $U(\nu;\alpha) \neq \emptyset$. Hence by [20, Theorem 4.3], $U(\nu;\alpha)$ is also a fantastic filter of M, and hence, by the preceding theorem, ν is a fantastic filter of M. □

Theorem 6.8 ([20, Theorem 4.6]). *A filter F of an* Rl-*monoid M is fantastic if and only if M/F is an* MV-*algebra.*

Theorem 6.9. *If ν is a fuzzy filter of an* Rl-*monoid M, then ν is a fantastic fuzzy filter of M if and only if the quotient* Rl-*monoid $M/U(\nu;\alpha)$ is an* MV-*algebra for every $\alpha \in [0,1]$ such that $U(\nu;\alpha) \neq \emptyset$.*

P r o o f. Follows from Theorems 6.5 and 6.8. □

Theorem 6.10 ([20, Proposition 4.10]). *Let M be an* Rl-*monoid. Then the following conditions are equivalent:*

(1) *M is an* MV-*algebra.*
(2) *Every filter of M is fantastic.*
(3) *$\{1\}$ is a fantastic filter of M.*

Theorem 6.11. *Let M be an* Rl-*monoid. Then the following conditions are equivalent:*

(a) *M is an* MV-*algebra.*
(b) *Every fuzzy filter of M is fantastic.*
(c) *Every fuzzy filter ν of M such that $\nu(1) = 1$ is fantastic.*
(d) *$\chi_{\{1\}}$ is a fantastic fuzzy filter of M.*

P r o o f. (a)⇒(b): Let M be an MV-algebra and let ν be a fuzzy filter of M. Let $\alpha \in [0,1]$ be such that $U(\nu;\alpha) \neq \emptyset$. Then by Theorem 6.10, $U(\nu;\alpha)$ is a fantastic filter of M, hence by Theorem 6.5 we get that ν is a fantastic fuzzy filter of M.

(b)⇒(c), (c)⇒(d): Obvious.

(d)⇒(a): Let $\chi_{\{1\}} = \nu_{\{1\}}(1;0)$ be a fantastic fuzzy filter of M. Then by Theorem 6.2 we have that $\{1\}$ is a fantastic filter of M, and therefore by Theorem 6.8, M is an MV-algebra. □

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Authors' addresses: *J. Rachůnek*, Department of Algebra and Geometry, Faculty of Sciences, Palacký University, 17. listopadu 12, 771 46 Olomouc, Czech Republic, e-mail: rachunek@inf.upol.cz; *D. Šalounová*, Department of Mathematical Methods in Economy, Faculty of Economics, VŠB-Technical University Ostrava, Sokolská 33, 701 21 Ostrava, Czech Republic, e-mail: dana.salounova@vsb.cz.