Debdas Mishra; Pratima Panigrahi

Some new classes of graceful Lobsters obtained from diameter four trees

Mathematica Bohemica, Vol. 135 (2010), No. 3, 257–278

Persistent URL: `http://dml.cz/dmlcz/140703`

SOME NEW CLASSES OF GRACEFUL LOBSTERS OBTAINED FROM DIAMETER FOUR TREES

Debdas Mishra, Bhubaneswar, Pratima Panigrahi, Kharagpur

(Received January 16, 2009)

Abstract. We observe that a lobster with diameter at least five has a unique path $H = x_0, x_1, \ldots, x_m$ with the property that besides the adjacencies in H both x_0 and x_m are adjacent to the centers of at least one $K_{1,s}$, where $s > 0$, and each x_i, $1 \leqslant i \leqslant m-1$, is adjacent at most to the centers of some $K_{1,s}$, where $s \geqslant 0$. This path H is called the central path of the lobster. We call $K_{1,s}$ an even branch if s is nonzero even, an odd branch if s is odd and a pendant branch if $s = 0$. In the existing literature only some specific classes of lobsters have been found to have graceful labelings. Lobsters to which we give graceful labelings in this paper share one common property with the graceful lobsters (in our earlier works) that each vertex x_i, $0 \leqslant i \leqslant m-1$, is even, the degree of x_m may be odd or even. However, we are able to attach any combination of all three types of branches to a vertex x_i, $1 \leqslant i \leqslant m$, with total number of branches even. Furthermore, in the lobsters here the vertices x_i, $1 \leqslant i \leqslant m$, on the central path are attached up to six different combinations of branches, which is at least one more than what we find in graceful lobsters in the earlier works.

Keywords: graceful labeling, lobster, odd branch, even branch, inverse transformation, component moving transformation

MSC 2010: 05C78

1. Introduction

Recall that a *graceful labeling* of a tree T with q edges is a bijection $f\colon V(T) \to \{0, 1, 2, \ldots, q\}$ such that $\{|f(u) - f(v)|\colon \{u, v\}$ is an edge of $T\} = \{1, 2, \ldots, q\}$. A tree which has a graceful labeling is called a *graceful tree*. A *lobster* is a tree having a path from which every vertex has distance at most two. If L is a lobster with diameter at least five and P is a path of maximum length in L then we obtain the path $H = x_0, x_1, \ldots, x_m$ from P by deleting two vertices from both the ends. H is independent of P, i.e. H is unique, and it is called the *central path* of L. Throughout the paper we use H to denote the central path of a lobster with diameter at least five. It follows directly from the definition of a lobster that besides the adjacencies in H each x_i is adjacent at most to the centers of some stars $K_{1,s}$ where $s \geqslant 0$. For $x_i \in V(H)$, if x_i is adjacent to the center of $K_{1,s}$ where $s \geqslant 0$ then we call $K_{1,s}$ an *even branch* if s is nonzero even, an *odd branch* if s is odd, and a *pendant branch* if $s = 0$. Furthermore, whenever we say x_i, for some $0 \leqslant i \leqslant m$, is attached to an even number of branches we mean a "non zero" even number of branches unless otherwise stated.

In 1979, Bermond [1] conjectured that all lobsters are graceful, which is a special case of the famous and unsolved "graceful tree conjecture" of Ringel and Kotzig (1964) [11], [12], which states that all trees are graceful. Bermond's conjecture is also open and very few classes of lobsters are known to be graceful. Ng [9], Wang et al. [13], Chen et al. [2], Morgan [8] (see [3]), and Mishra and Panigrahi [5], [6], [7], [10] have given graceful labeling to some classes of lobsters. In the graceful lobsters due to Ng [9] and Chen et al. [2], the vertices of the central path are attached to the isomorphic copies of at most two non isomorphic branches. Morgan [8] has proved that all lobsters with perfect matching are graceful. The graceful lobsters of this paper share one common feature with the graceful lobsters in [5], [6], [7], [10], [13] that the degree of each x_i, $0 \leqslant i \leqslant m-1$, is even and the degree of x_m is odd. However, the graceful lobsters of this paper possess simultaneously the following features, which we do not find in the graceful lobsters appearing in the earlier works mentioned above.

1. The vertices x_i, $1 \leqslant i \leqslant m$, on the central path are attached up to six different combinations of branches, which is at least one more than what we find in graceful lobsters in the earlier works [5], [6], [7], [10], [13].

2. The central path contains a vertex that may be attached to only one type, any combination of two types, or any combination of all three types, of branches with total number of branches even.

3. In this paper we find graceful lobsters with vertices on the central path attached to combination(s) containing all three types of branches preceded by the vertices attached to combination(s) containing two types of branches. Only in [10], some lobsters satisfy this property with some restrictions on the number of odd, even, and pendant branches. The graceful lobsters appearing in [7], [10], [13] are particular cases of the graceful lobsters of this paper in which one or more combinations are absent.

The lobsters of this paper have one of the following properties.

1. The vertex x_0 is attached to $(e, 0, o)$. For some t_1, $0 \leqslant t_1 < m$, if $t_1 \geqslant 1$ then each x_i, $1 \leqslant i \leqslant t_1$, is attached to $(o, 0, o)$. For integers t_2, t_3, t_4, and t_5 with

$1 \leqslant t_1 < t_2 < t_3 < t_4 \leqslant t_5 \leqslant m$, each x_i, $t_1 + 1 \leqslant i \leqslant t_2$, is attached to (o, e, o) and we have either (I) or (II) below.

(I) Each x_i, $t_2 + 1 \leqslant i \leqslant t_3$, is attached to (e, o, o) and we have (a) or (b) below.

(a) Each x_i, $t_3 + 1 \leqslant i \leqslant t_4$, is attached to $(0, o, o)$, each x_i, $t_4 + 1 \leqslant i \leqslant t_5$, is attached to $(0, e, e)$, and each of the rest of the x_i is attached to $(0, e, 0)$.

(b) Each x_i, $t_3 + 1 \leqslant i \leqslant t_4$, is attached to (e, e, e) and we have either (i) or (ii) below.

(i) Each x_i, $t_3 + 1 \leqslant i \leqslant t_4$, is attached to $(e, e, 0)$ and each of the rest of the x_i is attached to $(e, 0, 0)$ (or $(0, e, 0)$).

(ii) Each x_i, $t_4 + 1 \leqslant i \leqslant t_5$, is attached to $(e, 0, e)$ ($(0, e, e)$) and each of the rest of the x_i is attached to $(e, 0, 0)$ (respectively, $(0, e, 0)$).

(II) Each x_i, $t_2 + 1 \leqslant i \leqslant t_3$, is attached to (o, o, e) and we have one of the following.

(a) Each x_i, $t_3 + 1 \leqslant i \leqslant t_4$, is attached to $(o, o, 0)$, each x_i, $t_4 + 1 \leqslant i \leqslant t_5$, is attached to $(e, e, 0)$, and each of the rest of the x_i is attached to $(e, 0, 0)$ or $(0, e, 0)$.

(b) Same as (I)(b).

2. Lobsters obtained from those in (1) above by putting $t_1 = 0$.

3. The vertex x_0 is attached to one of the combinations of $(0, o, 0)$, $(e, o, 0)$, $(0, o, e)$, $(0, e, o)$, (e, o, e), and (e, e, o). For integers t_1, t_2 with $1 \leqslant t_1 < t_2 \leqslant m$, each x_i, $1 \leqslant i \leqslant t_1$, is attached to $(o, o, 0)$, each x_i, $t_1 + 1 \leqslant i \leqslant t_2$, is attached to $(e, e, 0)$, and each of the rest of the x_i, if any, is attached to $(e, 0, 0)$ or $(0, e, 0)$.

4. The vertex x_0 is attached to one of the combinations of $(0, e, o)$, (e, e, o), and (o, o, o). For integers t_1, t_2 with $1 \leqslant t_1 < t_2 \leqslant m$, each x_i, $1 \leqslant i \leqslant t_1$, is attached to $(0, o, o)$, each x_i, $t_1 + 1 \leqslant i \leqslant t_2$, is attached to $(0, e, e)$, and each of the rest of the x_i, if any, is attached to $(0, e, 0)$.

2. Preliminaries

To prove our results we need some definitions, terminology and existing results which are described below.

Lemma 2.1 [4], [13]. *If f is a graceful labeling of a tree T with n edges then the inverse transformation of f, defined as $f_n(v) = n - f(v)$ for all $v \in V(T)$, is also a graceful labeling of T.*

Definition 2.2. For an edge $e = \{u, v\}$ of a tree T, we define $u(T)$ as that connected component of $T - e$ which contains the vertex u. Here we say $u(T)$ is a *component incident on* the vertex v. If a and b are vertices of a tree T, $u(T)$ is a component incident on a and the component $u(T)$ does not contain the vertex b, then deleting the edge $\{a, u\}$ from T and making b and u adjacent is called the *component moving transformation.* Here we say the component $u(T)$ has been moved from a to b.

Throughout the paper we write "the component u" instead of writing "the component $u(T)$". Therefore, whenever we wish to refer to u as a vertex, we write "the vertex u". By the label of the component "$u(T)$" we mean the label of the vertex u. Moreover, we will not distinguish between a vertex and its label.

Lemma 2.3 [4]**.** *Let f be a graceful labeling of a tree T; let a and b be two vertices of T; let $u(T)$ and $v(T)$ be two components incident on a where $u(T) \cup v(T) \not\ni b$. Then the following assertions hold:*

(i) *if $f(u) + f(v) = f(a) + f(b)$ then the tree T^* obtained from T by moving the components $u(T)$ and $v(T)$ from a to b is also graceful.*

(ii) *if $2f(u) = f(a) + f(b)$ then the tree T^{**} obtained from T by moving the component $u(T)$ from a to b is also graceful.*

Lemma 2.4 [4]**.** *Let T be a diameter four tree with q edges. If a_0 is the center vertex and the degree of a_0 is $2k+1$ then there exists a graceful labeling f of T such that*

(a) *$f(a_0) = 0$ and the labelings of the neighbours of a_0 are $1, 2, \ldots, k, q, q-1, \ldots, q-k$;*

(b) *from the sequence $S = (q, 1, q-1, 2, q-2, 3, \ldots, q-k+1, k, q-k)$ of vertex labels, the centers of the odd branches get labels consecutively from the beginning, then the centers of the even branches get labels consecutively and finally the centers of the pendant branches get labels.*

3. Results

We begin this section with a theorem (Theorem 3.3) which describes a technique by which one can generate graceful trees from a given graceful tree of a certain type. Subsequently, we apply this technique to a diameter four tree whose center has odd degree to construct graceful lobsters. The lemma given below is used in proving Theorem 3.3.

Lemma 3.1. *Let $S_0 = (t_1, t_2, \ldots, t_{2p})$ be a finite sequence of natural numbers in which the sums of consecutive terms are alternately $l+1$ and l, beginning (and ending) with the sum $l+1$. For an integer $r \geqslant 1$, let $S_r = \varphi_{l+r}(S_r')$, where S_r' is the sequence obtained from S_{r-1} by deleting any odd number of terms from the* *beginning as well as from the end and* $\varphi_{l+r}(S'_r) = (l+r-x)_{x\in S'_r}$. *Then the sums of consecutive terms in the sequence* S_r *are alternately* $l+r+1$ *and* $l+r$, *beginning (and ending) with the sum* $l+r+1$.

P r o o f. We first consider the case when $r=1$. Let the sequence S'_1 be obtained from S_0 by deleting $2k+1$ terms from the beginning and $2k_1+1$ terms from the end. For $2k+2 \leqslant i \leqslant 2p-2k_1-1$ we have

$$\begin{aligned}\varphi_{l+1}(t_i)+\varphi_{l+1}(t_{i+1}) &= 2(l+1)-(t_i+t_{i+1})\\ &= \begin{cases} 2(l+1)-(l+1) & \text{if } (t_i+t_{i+1}) = l+1\\ 2(l+1)-l & \text{if } (t_i+t_{i+1}) = l\end{cases}\\ &= \begin{cases} l+1 & \text{if } (t_i+t_{i+1}) = l+1.\\ l+2 & \text{if } (t_i+t_{i+1}) = l.\end{cases}\end{aligned}$$

Therefore, the sums of consecutive terms of the sequence S_1 are $l+1$ and $l+2$ alternately. Moreover, the sum of the first two terms, i.e. $\varphi_{l+1}(t_{2k+2})+\varphi_{l+1}(t_{2k+3})$, is $l+2$ as $t_{2k+2}+t_{2k+3}=l$. Since the total number of terms in S_1 is even the sum of the last two terms is $l+2$. Thus, the lemma holds if we take $r=1$. For $r>1$ the proof follows if we repeat the above procedure r times. □

C o n s t r u c t i o n 3.2. Let T be a graceful tree with q edges. Let a_0 be a non pendant vertex of T with degree $2k+1$. Suppose there exists a graceful labeling f of T in which a_0 gets the label 0 and the labels of the neighbours of a_0 are $1,2,\ldots,k,q,q-1,q-2,\ldots,q-k$ (see Figure 1).

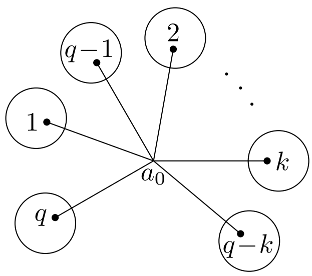

Figure 1. The tree T with vertex a_0 and its neighbours. The circles around the neighbouring vertices of represent the respective components incident on a_0.

We consider the sequence $A=(q,1,q-1,2,q-2,3,\ldots,k,q-k)$ of vertices adjacent to a_0 (as we do not distinguish between a vertex and its label). We construct a tree T_1 (see Figure 2) from T by identifying the vertex y_0 of a path $H'=y_0,y_1,\ldots,y_m$ with a_0 and moving the components (incident on the vertex a_0) in A to y_i in the following way:

(1) At y_0 we retain $2\lambda_0+1$ components, where $\lambda_0 \geqslant 0$. In particular, we retain $2p_0$ components, $0 \leqslant p_0 \leqslant \lambda_0$, whose labels are from the beginning of A, namely $q, 1, q-1, 2, q-2, 3, \ldots, q-p_0+1, p_0$, and $2\lambda_0+1-2p_0$ components whose labels are from the end of A, namely $q-k, k, q-k+1, k-1, \ldots, k-\lambda_0+p_0+1, q-k+\lambda_0-p_0$. Then we delete the components from A retained at y_0 and denote the sequence of the remaining terms of A by $A^{(1)}$.

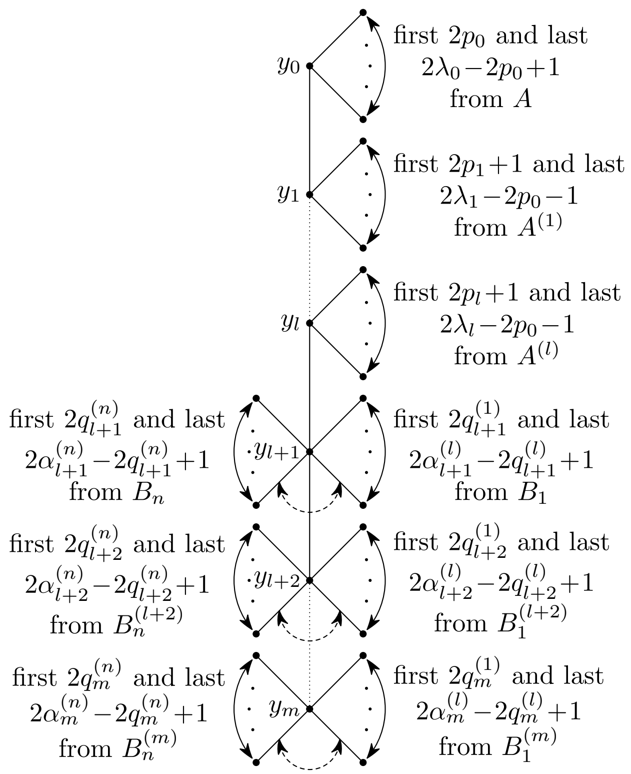

Figure 2. The tree T_1 obtained from T. Here we take $s_1 = s_2 = m$.

(2) Let l, $1 \leqslant l < m$, be a fixed integer. For $i = 1, 2, \ldots, l$, we move $2\lambda_i$ components from $A^{(i)}$ to y_i, where $\lambda_i \geqslant 1$. In particular, we move $2p_i+1$, $0 \leqslant p_i < \lambda_i$, components whose labels are from the beginning of $A^{(i)}$ and $2\lambda_i-2p_i-1$ components whose labels are from the end of $A^{(i)}$, where, for $i \geqslant 2$, $A^{(i)}$ is obtained from $A^{(i-1)}$ by deleting the components which are moved to y_{i-1}.

(3) Let $2p_0+\sum\limits_{i=1}^{l}(2p_i+1) = k_1$ and $2(\lambda_0-p_0)+1+\sum\limits_{i=1}^{l}(2\lambda_i-2p_i-1) = k_2$. Here we notice that if l is odd (even) then k_1 is odd (even) and k_2 is even (respectively, odd). Let $A^{(l+1)}$ be the sequence obtained from $A^{(l)}$ by deleting the components which are moved to y_l. Then one finds that $A^{(l+1)} = (\frac{1}{2}(k_1-1)+1, q-\frac{1}{2}(k_1-1)-1, \ldots,$

$k - \frac{1}{2}k_2, q - k + \frac{1}{2}k_2)$ if l is odd and $A^{(l+1)} = (q - \frac{1}{2}k_1, \frac{1}{2}k_1, \ldots, q - k + \frac{1}{2}(k_2 - 1) + 2, k - \frac{1}{2}(k_2 - 1) - 1)$ if l is even.

(4) For any $n \in \mathbb{N}$, if possible, partition $A^{(l+1)}$ into n parts, say $A^{(l+1)} = B_1 \cup B_2 \cup \ldots \cup B_n$, where $|B_i| = 2r_i$ (say), in such a way that the first $2r_1$ terms of $A^{(l+1)}$ are in B_1, the next $2r_2$ terms of $A^{(l+1)}$ are in B_2 and so on.

Now the components in B_j, $1 \leqslant j \leqslant n$, are distributed to the vertices y_i, $l+1 \leqslant i \leqslant s_j$, $l+1 \leqslant s_j \leqslant m$, of H' in the following way:

For $l+1 \leqslant i \leqslant s_j$, we move $2\alpha_i^{(j)}$ components from $B_j^{(i)}$ to y_i, where $\alpha_i^{(j)} \geqslant 1$. In particular, we move $2q_i^{(j)} + 1$, $0 \leqslant q_i^{(j)} < \alpha_i^{(j)}$, components whose labels are from the beginning of $B_j^{(i)}$, and $2\alpha_i^{(j)} - 2q_i^{(j)} - 1$ components whose labels are from the end of $B_j^{(i)}$, where for $i \geqslant l+2$ (if $s_j \geqslant l+2$), $B_j^{(i)}$ is obtained from $B_j^{(i-1)}$ by deleting the components which are moved to y_{i-1} and $B_j^{(l+1)} = B_j$.

The positive integers $\alpha_i^{(j)}$, $i = l+1, l+2, \ldots, s_j$, are chosen in such a way that $\sum\limits_{i=l+1}^{s_j} \alpha_i^{(j)} = r_j$. Therefore, $2k+1 = k_1 + k_2 + 2\sum\limits_{j=1}^{n} r_j$. □

In the following theorem, for a graceful tree R with n edges and a graceful labeling g of R we use the notation "$g(R)$" to denote the tree R with the graceful labeling g. Also, for any sequence $F = (a_1, a_2, \ldots, a_r)$, $g_n(F)$ is the sequence $(n - a_1, n - a_2, \ldots, n - a_r)$.

Theorem 3.3. *The tree T_1 in Construction 3.2 is graceful.*

Proof. Recall that we denote an edge with end points x and y by $\{x, y\}$. We first consider the tree $T \cup \{y_0, y_1\}$, where the vertices a_0 and y_0 are identified. We give the label $q+1$ to y_1. Clearly $T \cup \{y_0, y_1\}$ is graceful with the graceful labeling $f^{(1)}$, where $f^{(1)}$ is the same as f on T and gives the label $q+1$ to y_1. Then we move all the components in $A^{(1)}$ to y_1 and let the resulting tree be $T^{(1)}$. One can notice that $A^{(1)}$ can be partitioned into pairs of labels whose sum is $q+1$ (consecutive terms). By Lemma 2.3(i), $T^{(1)}$ is a graceful tree with the graceful labeling $f^{(1)}$.

Next, we consider the inverse transformation $f_{q+1}^{(1)}$ of $f^{(1)}$ of $T^{(1)}$. By Lemma 2.1, $f_{q+1}^{(1)}$ is a graceful labeling of $T^{(1)}$ and the label of y_1 in $f_{q+1}^{(1)}(T^{(1)})$ is 0. Next, we make y_2 adjacent to y_1 and give the label $q+2$ to y_2. Obviously, the tree $T^{(1)} \cup \{y_1, y_2\}$ is graceful with the graceful labeling $f^{(2)}$, where $f^{(2)}$ is the same as $f_{q+1}^{(1)}$ on $T^{(1)}$ and gives the label $q+2$ to y_2. We move all the components in $f_{q+1}^{(1)}(A^{(2)})$ from y_1 to y_2 and let the resulting tree be $T^{(2)}$. Observe that the sums of consecutive terms in $A^{(1)}$ are alternately $q+1$ and q beginning and ending with $q+1$ so by Lemma 3.1 the sums of consecutive terms in $f_{q+1}^{(1)}(A^{(2)})$ are alternately $q+2$ and $q+1$ beginning and ending with the sum $q+2$, i.e. $f_{q+1}^{(1)}(A^{(2)})$ can be partitioned into pairs of labels whose sum is $q+2$. Therefore, by Lemma 2.3(i), $T^{(2)}$ is graceful.

Repeating the above procedure for $l+1$ times we find that the tree $T^{(l+1)}$ with the vertex set $V(T) \cup \{y_0, y_1, \ldots, y_l, y_{l+1}\}$, is graceful with the graceful labeling $f^{(l+1)}$ in which the vertex y_{l+1} gets the label $q+l+1$ and the components of $f^{(l)}_{q+l} f^{(l-1)}_{q+l-1} \ldots f^{(1)}_{q+1}(A^{(l+1)})$ are incident on x_{l+1}. By Lemma 3.1, we find that the sums of consecutive terms in $f^{(l)}_{q+l} f^{(l-1)}_{q+l-1} \ldots f^{(1)}_{q+1}(A^{(l+1)})$ are $q+l+1$ and $q+l$ beginning and ending with the sum $q+l+1$. Since B_1 contains the first $2r_1$ terms of S and for $2 \leqslant j \leqslant n$, B_j contains the first $2r_j$ terms of $S^{(l+1)} \setminus B_1 \cup B_2 \cup \ldots \cup B_{j-1}$, the sums of consecutive terms in $f^{(l)}_{q+l} f^{(l-1)}_{q+l-1} \ldots f^{(1)}_{q+1}(B^{(l+1)}_j)$, $1 \leqslant j \leqslant n$, are $q+l+1$ and $q+l$ beginning and ending with the sum $q+l+1$.

Next, we take the inverse transformation $f^{(1+1)}_{q+l+1}$ of $f^{(l+1)}$ of $T^{(l+1)}$. By Lemma 2.1, $f^{(l+1)}_{q+l+1}$ is a graceful labeling of $T^{(l+1)}$ and the label of y_{l+1} in $f^{(l+1)}_{q+l+1}(T^{(l+1)})$ is 0. Next, we make y_{l+2} adjacent to y_{l+1} and give the label $q+l+2$ to y_{l+2}. Obviously, $T^{(l+1)} \cup \{y_{l+1}, y_{l+2}\}$ is graceful with the graceful labeling $f^{(l+2)}$, where $f^{(l+2)}$ is the same as $f^{(l+1)}_{q+l+1}$ on $T^{(l+1)}$ and gives the label $q+l+2$ to y_{l+2}.

For those j with $s_j \geqslant l+2$, $1 \leqslant j \leqslant n$, we move all the components in $f^{(l+1)}_{q+l+1} f^{(l)}_{q+l} f^{(l-1)}_{q+l-1} \ldots f^{(1)}_{q+1}(B^{(l+2)}_j)$ from y_{l+1} to y_{l+2} and let the resulting tree be $T^{(l+2)}$. By Lemma 3.1, the sums of consecutive terms in $f^{(l+1)}_{q+l+1} f^{(l)}_{q+l} f^{(l-1)}_{q+l-1} \ldots f^{(1)}_{q+1}$ $(B^{(l+2)}_j)$ are alternately $q+l+2$ and $q+l+1$ beginning and ending with $q+l+2$. One sees that each $f^{(l+1)}_{q+l+1} f^{(l)}_{q+l} f^{(l-1)}_{q+l-1} \ldots f^{(1)}_{q+1}(B^{(l+2)}_j)$ can be partitioned into pairs of labels whose sum is $q+l+2$. By Lemma 2.3(i), $T^{(l+2)}$ is graceful.

Let $s^\star = \max\{s_1, s_2, \ldots s_n\}$. Repeating the above procedure $s^\star - l - 1$ times we get the graceful tree $T^{(s^\star)}$ with vertex set $V(T) \cup \{y_1, \ldots, y_{s^\star}\}$ in which the vertex $y_{s^\star}$ gets the label $q+s^\star$. If $s^\star = m$ then we stop, otherwise we proceed as follows.

We apply inverse transformation to the graceful tree $T^{(s^\star)}$ so that the vertex $y_{s^\star}$ gets the label 0. Then make the vertex $y_{s^\star+1}$ adjacent to $y_{s^\star}$ and give the label $q+s^\star+1$ to $y_{s^\star+1}$. If $s^\star+1 = m$ then we stop, otherwise we repeat this procedure until the vertex y_m gets a label. The graceful tree that is obtained on the vertex set $V(T) \cup V(H')$ is easily seen to be the tree T_1. □

In the following theorem we demonstrate how we give graceful labeling to certain classes of lobsters by applying Theorem 3.3 to a graceful diameter four tree.

Theorem 3.4. *The lobsters in Tables* 3.1 *and* 3.2 *below are graceful.*

Description of Tables. In the column headings, the triple (x, y, z) represents the number of odd, even and pendant branches, respectively, where e means any even number of branches (nonzero, unless otherwise stated), o means any odd number of branches and 0 means no branch. For example, $(e, 0, o)$ means an even number of odd branches, no even branch and an odd number of pendant branches. If in a triple e or o appear more than once then it does not mean that the corresponding branches are equal in number. For example, (e, e, o) does not mean that the number of odd branches is equal to the number of even branches.

Lobsters ↓	$(e,0,o)$	$(o,0,o)$	(o,e,o)	(e,o,o)	$(o,o,0)$	$(0,o,o)$	$(e,e,0)$	$(e,0,0)^1$ or $(0,e,0)^2$	$(0,0,e)$
a	0	$1\to t_1$, $t_1 < m-2$	$t_1+1\to t_2$, $t_2 < m-1$	$t_2+1\to t_3$, $t_3 < m$	—	$t_3+1\to t^\star$, $t^\star \leqslant m$	—	$t^\star+1\to m(2)$, if $t^\star < m$	—
b	0	$1\to t_1$, $t_1 < m-2$	$t_1+1\to t_2$, $t_2 < m-1$	$t_2+1\to t_3$, $t_3 < m$	—	—	$t_3+1\to t'$, $t' \leqslant m$	$t'+1\to m$, if $t' < m$	—
c	0	$1\to t_1$, $t_1 < m-2$	$t_1+1\to t_2$, $t_2 < m-1$	$t_2+1\to t_3$, $t_3 < m$	—	—	—	$t_3+1\to m(1)$	$t_3+1\to s$, $s \leqslant m$
d	0	$1\to t_1$, $t_1 < m-1$	$t_1+1\to t_2$, $t_2 < m$	$t_2+1\to t'$, $t' \leqslant m$	—	—	—	$t'+1\to m$ if $t' < m$	—
e	0	$1\to t_1$, $t_1 < m-1$	$t_1+1\to t_2$, $t_2 < m$	—	$t_2+1\to t'$, $t' \leqslant m$	—	—	$t'+1\to m$, if $t' < m$	$t_2+1\to s$, $s \leqslant m$
f	0	$1\to t_1$, $t_1 < m-1$	$t_1+1\to t_2$, $t_2 < m$	—	—	—	$t_2+1\to t'$, $t' \leqslant m$	$t'+1\to m$ if $t' < m$	—
g	0	$1\to t_1$, $t_1 < m-1$	$t_1+1\to t_2$, $t_2 < m$	—	—	—	—	$t_2+1\to m$	$t_2+1\to s$, $s \leqslant m$
h	0	$1\to t_1$, $t_1 < m-1$	$t_1+1\to t_2$, $t_2 < m$	—	$t_2+1\to t'$, $t' \leqslant m$	—	—	$t'+1\to m$, if $t' < m$	—
i	0	$1\to t_1$, $t_1 < m-1$	$t_1+1\to t_2$, $t_2 < m$	—	—	$t_2+1\to t'$, $t' \leqslant m$	—	$t'+1\to m$ (2) if $t' < m$	—
j	0	$1\to t$, $t < m-1$	$t+1\to t'$, $t' \leqslant m$,	—	—	—	—	$t'+1\to m$ if $t' < m$	—

Table 3.1

1st column: 0 means that x_0 is attached to any one of the mentioned combinations of branches. The notation $0(r)$, $r = 1, 2$ (or $r = 1, 2, 3, 4, 5, 6, 7$), means that x_0 is attached to the combination of branches mentioned in the column heading in which r is the superscript.

Other columns: $i \to j$ (or $i \to j(r)$, $r = 1, 2$) means that each x_l, $i \leqslant l \leqslant j$, is attached to the mentioned combination or any one of the combinations of branches (respectively, the branches mentioned in the triple with superscript r).

Further, when some vertex x_i on the central path is attached to two combinations $(x, y, 0)$ and $(0, 0, e)$, we mean that x_i is attached to the combination (x, y, e). For example, in Table 3.1(c), x_{t_3+1} is attached to the combinations $(e, 0, 0)$ and $(0, 0, e)$, which means that x_{t_3+1} is attached to the combination $(e, 0, e)$.

Lob-sters ↓	$(e,o,0)^1$ or $(e,o,e)^2$ or $(0,o,0)^3$ or $(0,o,e)^4$ or $(e,e,o)^5$ or $(0,e,o)^6$ or $(o,o,o)^7$ or $(e,0,o)^8$	$(o,o,0)^1$ or $(0,o,o)^2$	$(e,e,0)$	$(e,0,0)^1$ or $(0,e,0)^2$	$(0,0,e)$
a	0 (any one of the combinations from 1 to 6)	$1\to t(1)$, $t<m$	$t+1\to t'$, $t'\leqslant m$	$t'+1\to m$ if $t'<m$	—
b	0 (any one of the combinations from 5 to 8)	$1\to t(2)$, $t<m$	—	$t+1\to m(2)$	$t+1\to s$, $s\leqslant m$

Table 3.2

Proof. For every lobster L we first construct a diameter four tree, say $T(L)$, by successively merging the vertices x_i, $i = 1, 2, \ldots, m$ of H with x_0. It is clear that x_0 is the center of $T(L)$ and its degree is odd. Let $|E(T(L))| = q$ and $\deg(x_0) = 2k + 1$. We give the label 0 to x_0. We consider the sequence A in Construction 3.2. We use the notation l, n, k_1, k_2, $A^{(i)}$, $i \geqslant 1$, and B_j, $j = 1, 2, \ldots, n$, of Construction 3.2 and determine them for each lobster of this theorem.

Let L be a lobster of type (a) in Table 3.1. We follow the steps given below.

1. For $i = 0, 1, \ldots, t_1$, the centers of the odd (pendant) branches incident on x_i in L get labels consecutively from the beginning (end) of the sequence $A^{(i)}$, where $A^{(0)} = A$. We take k_1 (k_2) as the sum total of the number of odd (respectively, pendant) branches incident on x_i, $0 \leqslant i \leqslant t_1$.

2. (i) Take $l = t_1$, $n = 2$ and determine B_1 and B_2. For $i = t_1 + 1, \ldots, t_2$, let the number of odd branches incident on x_i be $2\lambda_i + 1$, where $\lambda_i \geqslant 0$. The centers of these branches will get labels from B_1. For $i = t_1 + 1, \ldots, t_2$, let the number of even branches incident on x_i be $2\alpha_i$, $\alpha_i > 0$, among which the centers of $2\beta_i + 1$ branches for arbitrary integers β_i, $0 \leqslant \beta_i < \alpha_i$, will get labels from B_1, and the centers of the rest of these branches will get labels from B_2. Let $\sum\limits_{i=t_1+1}^{t_2} (2\lambda_i + 1) + \sum\limits_{i=t_1+1}^{t_2} (2\beta_i + 1) = 2p_1$.

(ii) Let the number of odd branches incident on x_i, $i = t_2+1, \ldots, t_3$, be $2\lambda_i$, $\lambda_i \geqslant 1$. The centers of these branches get labels from the sequence B_1. Let $\sum\limits_{i=t_2+1}^{t_3} 2\lambda_i = 2p_2$. Let $|B_1| = 2r_1 = 2(p_1 + p_2)$.

3. We give labelings to the centers of the branches incident on x_i, $t_1 + 1 \leqslant i \leqslant m$, in the following manner.

(i) For $i = t_1+1, \ldots, t_2$, the centers of $2\lambda_i+1$ odd branches incident on x_i get labels consecutively from the beginning of $B_1^{(i)}$, the centers of $2\alpha_i$ even branches incident on x_i get $2\beta_i + 1$ labels consecutively from the end of $B_1^{(i)}$ and $2\alpha_i - 2\beta_i - 1$ labels consecutively from the beginning of $B_2^{(i)}$, and the centers of the pendant branches incident on x_i get labels consecutively from the end of $B_2^{(i)}$.

(ii) For $i = t_2 + 1, \ldots, t_3$, among the odd branches incident on x_i, the centers of any odd number of branches get labels consecutively from the beginning of $B_1^{(i)}$ and the centers of the rest of these branches get labels consecutively from the end of $B_1^{(i)}$.

(iii) For $i = t_2 + 1, \ldots, t^\star$, the centers of the even (pendant) branches incident on x_i get labels consecutively from the beginning (respectively, end) of $B_2^{(i)}$.

If $t^\star < m$ we do the following additional step.

(iv) For $i = t^\star + 1, \ldots, m$, among the even branches incident on x_i, the centers of any odd number of branches get labels consecutively from the beginning of $B_2^{(i)}$ and the centers of the rest of these branches get labels consecutively from the end of $B_2^{(i)}$.

Here we notice that the above labeling of the centers of the branches incident on the center x_0 of $T(L)$ follows Lemma 2.4. Therefore, by Lemma 2.4 there exists a graceful labeling of $T(L)$ with the above labels of the center x_0 and the centers of the branches incident on x_0 (i.e. we give labeling to the remaining vertices of $T(L)$ using the techniques of [4]). Finally, we apply Theorem 3.3 for $n = 2$ to $T(L)$ and the path $H = x_0, x_1, \ldots, x_m$, so as to get a graceful labeling of L (see example below). This approach will be the same for all the remaining cases of this theorem and hence we will just indicate the modifications we do in steps 1 to 3.

Example 1. Consider the lobster L presented in Figure 3 which is of the type (a) in Table 3.1. We construct the diameter four tree $T(L)$ shown in Figure 4. $|E(T(L))| = q = 73$ and $k = 13$. Therefore, $A = (73, 1, 72, 2, \ldots, 61, 13, 60)$. Here

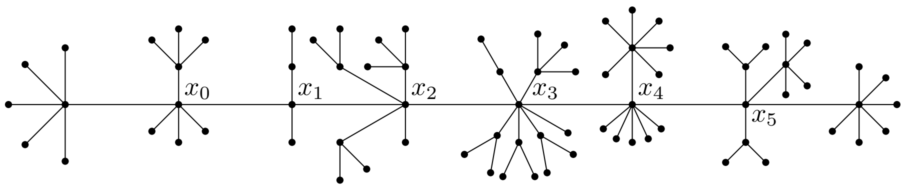

Figure 3. A lobster L of type (a) in Table 3.1. Here $m = 4$, $t_1 = 1$, $t_2 = 2$ and $t^\star = 3$.

$m = 5$, $t_1 = 1$, $t_2 = 2$, $t_3 = 3$, $t^\star = 4$, $k_1 = 3$, $k_2 = 4$. Therefore, $A^{(t_1+1)} = A^{(2)} = (2, 71, 3, \ldots, 61, 11, 62)$. Here $\lambda_2 = 0$, $\lambda_3 = 1$, $\alpha_2 = 1$, $\beta_2 = 0$, so $|B_1| = 4$, i.e. $B_1 = (2, 71, 3, 70)$ and $B_2 = (4, 69, 5 \ldots, 61, 11, 62)$. We give the label 0 to the vertex x_0 and give labelings to the centers of the branches incident on x_0 as per the steps 1 and 3. Using the techniques of [4] (Theorem 1 of [4]) we obtain a graceful labeling of $T(L)$ given in Figure 4. Then in Figure 5 we make x_1 adjacent to x_0, give the label 74 to x_1 and move all the components in $A^{(1)}$ to x_1. The tree in Figure 6 is obtained by applying inverse transformation to the lobster found in Figure 5, making x_2 adjacent to x_1, giving the label 75 to x_2 and moving all the components in $f_{74}^{(1)}(A^{(2)})$ to x_2. Then we proceed as per the technique described in Theorem 3.3 and get a graceful labeling of L. Figure 8 represents L with a graceful labeling.

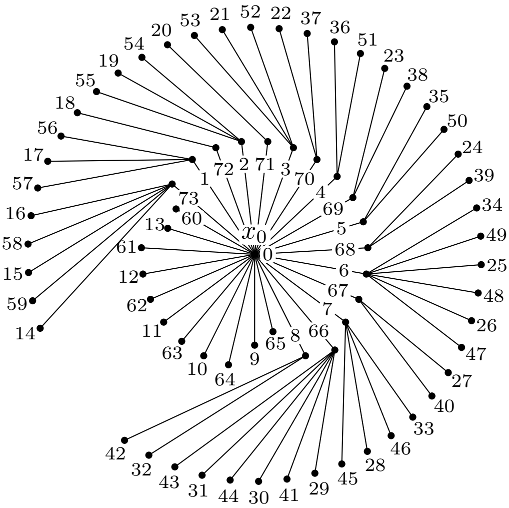

Figure 4. The diameter four tree $T(L)$ corresponding to L with a graceful labeling.

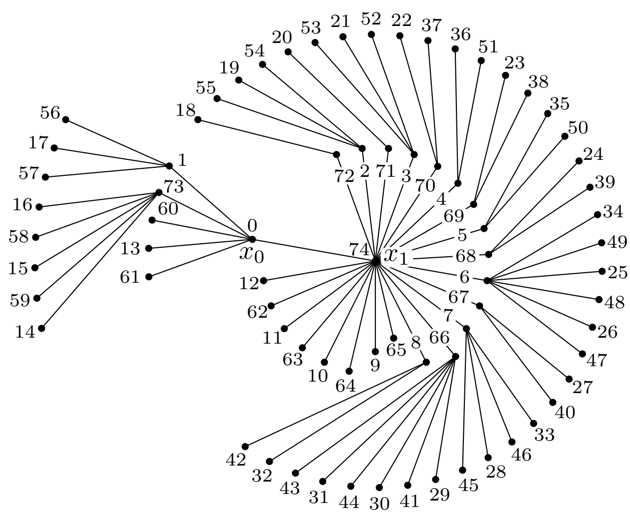

Figure 5. The graceful lobster with the graceful labeling $f^{(1)}$ obtained by making x_1 adjacent to x_0, giving the label 74 to x_1, and moving all the branches in $A^{(1)}$ to x_1.

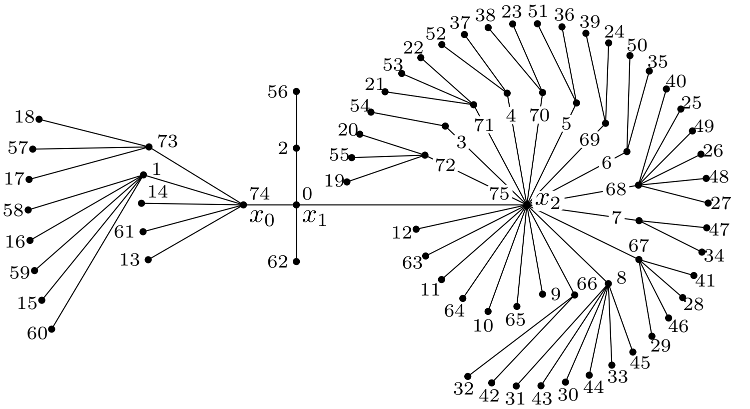

Figure 6. The graceful lobster with the graceful labeling $f^{(2)}$ obtained by applying inverse transformation to the lobster in Figure 5, making x_2 adjacent to x_1, giving the label 75 to x_2, and moving all the components in $f^{(1)}_{74}(A^{(2)})$ to x_2.

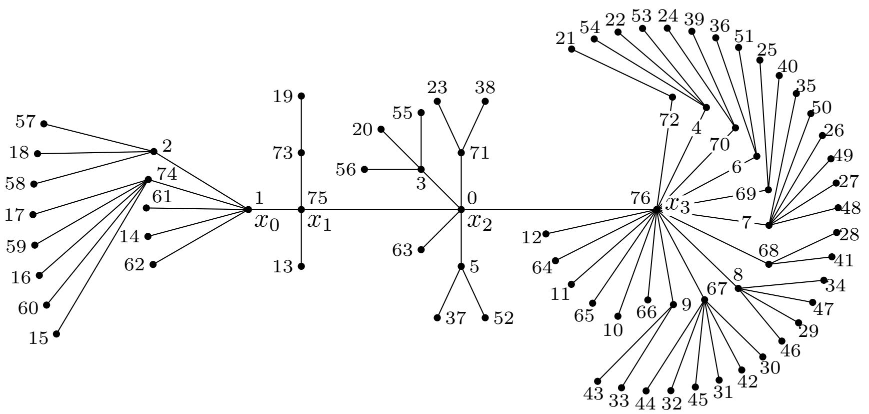

Figure 7. The graceful lobster with the graceful labeling $f^{(3)}$ obtained by applying inverse transformation to the lobster in Figure 6, making x_3 adjacent to x_2, giving the label 76 to x_3, and moving all the components in $f^{(2)}_{75}f^{(1)}_{74}(A^{(2)})$ to x_3.

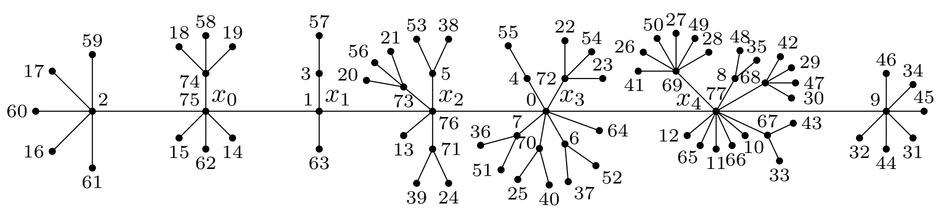

Figure 8. The graceful lobster with the graceful labeling $f^{(4)}$ obtained by applying inverse transformation to the lobster in Figure 7, making x_4 adjacent to x_3, giving the label 77 to x_4, and moving all the components in $f^{(2)}_{76}f^{(2)}_{75}f^{(1)}_{74}(A^{(3)})$ to x_4.

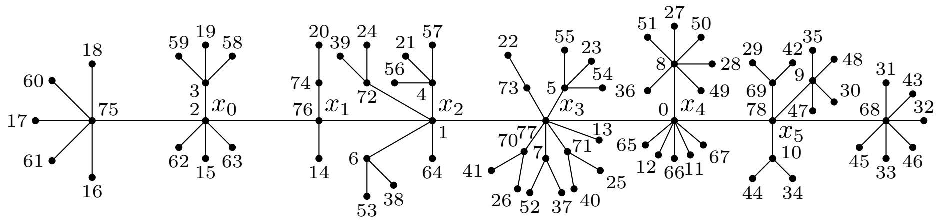

Figure 9. The lobster L with a graceful labeling.

For lobsters of type (x), $x = b, \ldots, j$, in Table 3.1, the proof follows if we proceed as in the proof involving the lobsters of type (a) in Table 3.1 by modifying steps 1, 2 and 3. For lobsters of type (b) we first define an integer p as $p = m$ if either $t' = m$ or $t' < m$ with each x_i, $i = t' + 1, \ldots, m$, being attached to an even number of odd branches and $p = t'$ if $t' < m$ with each x_i, $i = t' + 1, \ldots, m$, being attached to an even number of even branches; and this definition of p will hold henceforth in the text. Next, we set $t_3 = p$, $t^\star = t_3$ and $m = m + t' - p$ in steps 1, 2, and 3. For lobsters of type (c), we set $t_3 = m$, $t^\star = t_3$ and $m = s$ in steps 1, 2 and 3, and replace even branches by pendant branches in step 3(iv). For lobsters of type (d), we set $t_3 = p$ and $t^\star = t'$ in steps 1, 2, 3(i), 3(ii), and 3(iii), and furthermore, if $p = t'$ then we set $t^\star = t'$ in step 3(iv).

For lobsters L of type (e), we repeat steps 1 and 2(i), set $t_3 = m$, and replace the number of odd branches by the sum total of the number of odd and even branches in step 2(ii), repeat step 3(i), and modify steps 3(ii) to 3(iv) in the following manner.

3(ii) For $i = t_2 + 1, \ldots, t'$, the centers of the odd (even) branches incident on x_i get labels consecutively from the beginning (respectively, end) of $B_1^{(i)}$.

3(iii) Set $t^\star = t_2$ and $m = s$, and replace B_1 by B_2 and even branches by pendant branches in step 3(iv).

If $t' < m$ then we do the following additional step.

3(iv) For $i = t' + 1, \ldots, m$, among the odd (or even) branches incident on x_i, the centers of any odd number of these branches get labels consecutively from the beginning of $B_1^{(i)}$ and the centers of the rest of these branches get labels consecutively from the end of $B_1^{(i)}$.

For lobsters L of type (f), we repeat steps 1 and 2(i), set $t_3 = p$ in step 2(ii), repeat step 3(i), set $t_3 = p$ in step 3(ii), and set $t^\star = t_2$ and $m = m + t' - p$ in step 3(iv). For lobsters L of type (g), we repeat steps 1 and 2(i), set $t_3 = m$ and replace odd branches by odd (or even) branches in step 2(ii), repeat step 3(i), set $t_3 = m$ and replace odd branches by odd (or even) branches in step 3(ii), and set $t^\star = t_2$ and $m = s$ and replace even branches with pendant branches in step 3(iv). For lobsters L of type (h), we repeat steps 1, 2, and 3 excluding step 3(iii) in the proof involving the lobsters of type (e). For lobsters L of type (i), we repeat steps 1 and 2(i) and let

$|B_1| = 2r_1 = 2p_1$, repeat step 3(i), set $t_3 = t_2$ and $t^\star = t'$ in step 3(iii). Furthermore, if $t' < m$ then we set $t^\star = t'$ in step 3(iv). For lobsters L of type (j), we set $t_1 = t$ and $t_2 = t'$ in steps 1 and 2(i). If $t' < m$ then we set $t_2 = t'$ and $t_3 = m$ and replace odd branches with odd (or even) branches in step 2(ii). Here $|B_1| = 2r_1 = 2p_1$ if $t' = m$ and $2p_1 + 2p_2$ if $t' < m$. Set $t_1 = t$ and $t_2 = t'$ in step 3(i). Furthermore, if $t' < m$ then we set $t_2 = t'$ and $t_3 = m$ and replace odd branches with odd (or even) branches in step 3(ii).

For lobsters L of type (a) in Table 3.2, the proof follows if we proceed as in the proof involving the lobsters of type (a) in Table 3.1 with the changes in steps 1 to 3 as per the following.

1. The centers of odd (pendant branches followed by even branches) incident on x_0 get labels from the beginning (respectively, end) of A. For $i = 1, 2, \dots, t$, the centers of the odd (even) branches incident on x_i get labels consecutively from the beginning (respectively, end) of the sequence $A^{(i)}$. We take k_1 (k_2) as the sum total of the number of odd (respectively, sum total of even and pendant) branches incident on x_i, $0 \leqslant i \leqslant t$.

2. Take $n = 2$ and $l = t$ and determine B_1 and B_2. Take $|B_1| = 2r_1$ as the total number of odd branches incident on the vertices x_i, $i = t+1, t+2, \dots, p$.

3. Omit step 3(i). Set $t_1 = t$ and $t_2 = p$ in step 3(ii). Omit step 3(iii). Set $t^\star = t$ and $m = m + t' - p$ in step 3(iv).

For lobsters L of type (b) in Table 3.2, the proof follows if we proceed as in the proof involving the lobsters of type (a) in Table 3.1 with the changes in steps 1 to 3 as per the following.

1. The centers of odd branches followed by even branches (pendant branches) incident on x_0, get labels from the beginning (end) of A. For $i = 1, 2, \dots, t$, the centers of the even (pendant) branches incident on x_i get labels consecutively from the beginning (end) of the sequence $A^{(i)}$. We take k_1 (k_2) as the sum of the total number of odd and even branches (respectively, number of pendent) branches incident on x_i, $0 \leqslant i \leqslant t$.

2. Take $n = 2$ and $l = t$ and determine B_1 and B_2. Take $|B_1| = 2r_1$ as the sum total of number of even branches incident on the vertices x_i, $i = t+1, t+2, \dots, m$.

3. Omit step 3(i). Set $t_1 = t$ and $t_2 = m$ and replace odd branches with even branches in step 3(ii). Omit step 3(iii). Set $t^\star = t$ and $m = s$ and replace even branches by pendant branches in step 3(iv). □

Next, we show that for the case $n = 2$ in Construction 3.2 by distributing the branches in B_j, $j = 1, 2$, to the vertices $y_i, 0 \leqslant i \leqslant m$ of H' in a slightly different manner we get a graceful tree T_2 (may be different from T_1). By applying this result to diameter four trees we obtain some more graceful lobsters.

C o n s t r u c t i o n 3.5. Let the tree T, the path H', the graceful labeling f (of T) and the sequence A be the same as in Construction 3.2 (see Figure 1). We construct a tree T_2 (see Figure 10) from T by identifying the vertex y_0 of H' with a_0 and distributing the components (incident on the vertex a_0) in A to y_i, $i = 0, 1, 2, \ldots, m$, in the following manner.

(1) The components in A are distributed to the vertices $y_0, y_1, \ldots, y_l$ in the same manner as described in Construction 3.2. The integers k_1 and k_2 are defined as in Construction 3.2.

(2) We take $n = 2$ in Construction 3.2, i.e. we partition the sequence $A^{(l+1)}$ into two parts: $A^{(l+1)} = B_1 \cup B_2$. Let $\{l_1, l_2\} = \{1, 2\}$. For $l + 1 \leqslant i \leqslant s_{l_j}$, we move $2\alpha_i^{(l_j)}$ components from $B_{l_j}^{(i)}$ to y_i, where $\alpha_i^{(l_j)} \geqslant 1$.

The components in B_{l_1} are distributed to the vertices $y_{l+1}, y_{l+2}, \ldots, y_{s_{l_1}}$, in the same manner as described in Construction 3.2. The components in B_{l_2} are distributed to the vertices $y_{l+1}, y_{l+2}, \ldots, y_{s_{l_2}}$, in the following way:

(i) For some integer s', $l + 1 \leqslant s' < s_{l_2}$, the components of B_{l_2} are distributed to y_i, $l + 1 \leqslant i \leqslant s'$, in the same manner as described in Construction 3.2.

(ii) For $i = s' + 1, \ldots, s_{l_2}$, the components of B_{l_2} are distributed to y_i in the following manner. Suppose $|B_{l_2}^{(s'+1)}| = 2k_3$. We first partition $B_{l_2}^{(s'+1)}$ as $B_{l_2}^{(s'+1)} = C_1 \cup C_2$, where for some integer k_4, $1 \leqslant k_4 < k_3$, C_1 consists of $2k_4$ terms from the beginning of $B_{l_2}^{(s'+1)}$ and $C_2 = B_{l_2}^{(s'+1)} \setminus C_1$. Let $s_{l_2}^{(1)}$ and $s_{l_2}^{(2)}$ be integers, where $s_{l_2}^{(1)}, s_{l_2}^{(2)} \geqslant s' + 1$ and $\max(s_{l_2}^{(1)}, s_{l_2}^{(2)}) = s_{l_2}$. For $l' = 1, 2$ and $i = s' + 1, \ldots, s_{l_2}^{(l')}$, we move $2\beta_i^{(l')}$, $\beta_i^{(l')} \geqslant 1$, components from $C_{l'}$ to y_i. In particular, we move $2\gamma_i^{(l')} + 1$ components for arbitrary integers $\gamma_i^{(l')}$, $0 \leqslant \gamma_i^{(l')} < \beta_{l'}$, whose labels appear consecutively from the beginning of $C_{l'}^{(i)}$ and $2\beta_i^{(l')} - 2\gamma_i^{(l')} - 1$ components whose labels appear consecutively from the end of $C_{l'}^{(i)}$, where $C_{l'}^{(s'+1)} = C_{l'}$ and for $i \geqslant s' + 2$, $C_{l'}^{(i)}$ is obtained from $C_{l'}^{(i-1)}$ by deleting the components which are retained at y_{i-1}. The numbers $\beta_i^{(l')}$, $i = s' + 1, \ldots, s_{l_2}^{(l')}$, $l' = 1, 2$, are chosen in such a way that

$$\sum_{i=s'+1}^{s_{l_2}^{(l)}} \beta_i^{(l)} = k_4 \quad \text{and} \quad \sum_{i=s'+1}^{s_{l_2}^{(2)}} \beta_i^{(2)} = k_3 - k_4.$$

The numbers $\alpha_i^{(l_1)}$, $i = l + 1, \ldots, s_{l_1}$, and $\alpha_i^{(l_2)}$, $i = l + 1, \ldots, s'$, $j = 1, 2$, are chosen in such a way that

$$\sum_{i=l+1}^{s_{l_1}} \alpha_i^{(l_1)} = r_{l_1} \quad \text{and} \quad \sum_{i=l+1}^{s'} \alpha_i^{(l_2)} = r_{l_2} - k_3.$$

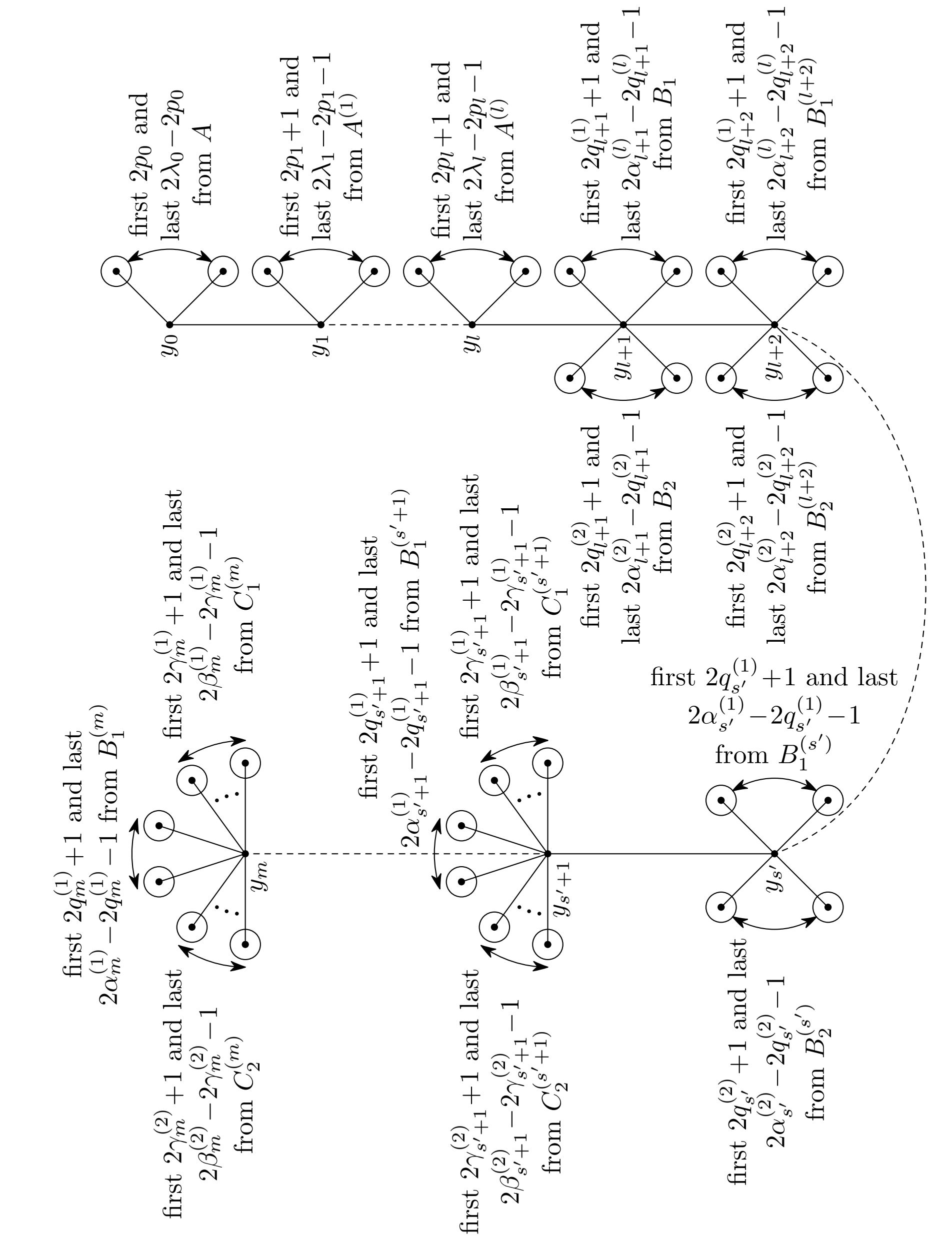

Figure 10. The tree T_2 obtained from T. Here we take $l_1 = 1$, $l_2 = 2$, $s_1 = s_2^{(1)} = s_2^{(2)} = m$.

Theorem 3.6. *The tree T_2 in Construction 3.5 is graceful.*

P r o o f. We first proceed as in the proof of Theorem 3.3 for $n = 2$ by setting $\max(s_1, s_2, \ldots, s_n) = s' + 1$. Next, we continue the above process for $n = 3$, i.e. the parts are B_{l_1}, C_1, C_2 (or C_1, C_2, B_{l_1}) if $s_{l_1} > s'$ ($n = 2$, i.e. the parts are $C_1^{(i)}$ and $C_2^{(i)}$ if $s_{l_1} \leqslant s'$). □

Theorem 3.7. *The lobsters in Table* 3.3 *below are graceful.*

Lobsters ↓	$(e,0,o)$	$(o,0,o)$	(o,e,o)	(e,o,o)	$(o,o,0)$	$(0,o,o)$	$(e,e,0)$	$(e,0,0)^1$ or $(0,e,0)^2$	$(0,0,e)$
a	0	$1\to t_1$, $t_1 < m-3$	$t_1+1\to t_2$, $t_2 < m-2$	$t_2+1\to t_3$, $t_3 < m-1$	—	$t_3+1\to t_4$, $t_4 < m$	—	$t_4+1\to m(2)$	$t_4+1\to s, s\leqslant m$
b	0	$1\to t_1$, $t_1 < m-2$	$t_1+1\to t_2$, $t_2 < m-1$	$t_2+1\to t_3$, $t_3 < m$	—	—	$t_3+1\to t'$ $t'\leqslant m$	$t'+1\to m$, if $t' < m$	$t_3+1\to s$, $s\leqslant m$
c	0	$1\to t_1$, $t_1 < m-2$	$t_1+1\to t_2$, $t_2 < m-1$	$t_2+1\to t_3$, $t_3 < m$	—	—	—	$t_3+1\to m$ (2)	$t_3+1\to s$, $s\leqslant m$
d	0	$1\to t_1$, $t_1 < m-2$	$t_1+1\to t_2$, $t_2 < m-1$	—	—	$t_2+1\to t_3$, $t_3 < m$	—	$t_3+1\to m$ (2)	$t_3+1\to s$, $s\leqslant m$
e	0	$1\to t_1$, $t_1 < m-2$	$t_1+1\to t_2$, $t_2 < m-1$	—	$t_2+1\to t_3$, $t_3 < m$	—	$t_3+1\to t'$, $t'\leqslant m$	$t'+1\to m$, if $t' < m$	$t_2+1\to s$, $s\leqslant m$
f	0	$1\to t_1$, $t_1 < m-2$	$t_1+1\to t_2$, $t_2 < m-1$	—	$t_2+1\to t_3$, $t_2 < m$	—	$t_3+1\to t'$, $t'\leqslant m$	$t'+1\to m$, if $t' < m$	—
g	0	$1\to t_1$, ; $t_1 < m-1$	$t_1+1\to t_2$, $t_2 < m$	—	—	—	$t_2+1\to t'$, $t'\leqslant m$	$t'+1\to m$ if $t' < m$	$t_2+1\to s$, $s\leqslant m$

Table 3.3

Description of Table 3.3. Same as Table 3.1.

Proof. As in the proof of Theorem 3.4, for every lobster L of this theorem we first construct the diameter four tree $T(L)$. Let $|E(T(L))| = q$ and $\deg(x_0) = 2k+1$. We give the label 0 to the center x_0. Here we use the notation A, l, k_1, k_2, k_3, k_4, l_1, l_2, $A^{(i)}$, $i \geqslant 1$, B_j, $j = 1,2$, and C_j, $j = 1,2$, s_{l_1}, $s_{l_2}^{(1)}$, and $s_{l_1}^{(2)}$ of Construction 3.5 and determine them for each lobster of this theorem.

Let L be a lobster of type (a) in Table 3.3. We proceed as per the steps given below.

Set $t^\star = t_4$ and repeat steps 1, 2, 3(i), 3(ii), and 3(iii) in the proof involving the lobsters of type (a) in Table 3.1. Set $l_1 = 1$, $l_2 = 2$, and $s_{l_1} = s_1 = t_3$.

4. Set $s_{l_2}^{(1)} = s_2^{(1)} = m$ and $s_{l_2}^{(2)} = s_2^{(2)} = s$. Take $s' = t_4$; hence k_3 is determined, i.e. $2k_3 = |B_{l_2}^{(s'+1)}| = |B_2^{(t_4+1)}|$. Next, we determine k_4 and hence C_1 and C_2. The terms of C_1 (C_2) will be the labels given to the centers of the even (pendant) branches incident on each x_i, $i = t_4+1, t_4+2, \ldots, m$ ($i = t_4+1, t_4+2, \ldots, s$),

i.e. $|C_1|$ ($|C_2|$) is the sum of the number of even (pendant) branches incident on x_i, $i = t_4+1, t_4+2, \dots, m$ (respectively, $i = t_4+1, t_4+2, \dots, s$).

5. For $i = t_4+1, t_4+2, \dots, m$ ($i = t_4+1, t_4+2, \dots, s$), among the even (pendant) branches incident on x_i, the centers of any odd number of branches get labels consecutively from the beginning of $C_1^{(i)}$ ($C_2^{(i)}$) and the centers of rest of these branches get labels consecutively from the end of $C_1^{(i)}$ (respectively, $C_2^{(i)}$).

Here we notice that the above labeling of the centers of the branches incident on the center x_0 of $T(L)$ follows Lemma 2.4. Therefore, by Lemma 2.4 there exists a graceful labeling of $T(L)$ with the above labels of the center x_0 and the centers of the branches incident on x_0 (i.e. we can give labeling to the remaining vertices of $T(L)$ using the techniques of [4]). Finally, we apply Theorem 3.3 for $n = 2$ to $T(L)$ and the path $H = x_0, x_1, \dots, x_m$, so as to get a graceful labeling of L (see example below). This approach will be the same for all the remaining cases of this theorem and hence we will just indicate the modification we do in steps 1 to 5. □

Example 2. Consider the lobster L presented in Figure 11 which is of the type (a) in Table 3.3. We construct the diameter four tree $T(L)$ shown in Figure 12. $|E(T(L))| = q = 83$ and $k = 14$. Therefore, $A = (83, 1, 87, 2, 81, 3, \dots, 13, 70, 14, 69)$. Here $m = 6$, $t_1 = 1$, $t_2 = 2$, $t_3 = 3$, $t_4 = 4$, $s = 5$, $l = t_1 = 1$, $k_1 = 3$, $k_2 = 2$. Therefore, $A^{(t_1+1)} = A^{(2)} = (2, 81, 3, 80, \dots, 12, 71, 13, 70)$. Here $\lambda_2 = 0$, $\alpha_1 = 1$, $\beta_1 = 0$, $2p_1 = (2\lambda_2 + 1) + (2\beta_1 + 1) = 2$, $\lambda_3 = 1$, $2p_2 = 2\lambda_3 = 2$. So $|B_1| = 2(p_1 + p_2) = 4$, $B_1 = (2, 81, 3, 80)$ and $B_2 = (4, 79, 5, \dots, 12, 71, 13, 70)$. Here $l_1 = 1$, $l_2 = 2$, $s' = t_4 = 4$, $s_{l_1} = s_1 = t_3 = 3$, $s_{l_2}^{(1)} = s_2^{(1)} = m = 6$, and $s_{l_2}^{(2)} = s_2^{(2)} = s = 5$. $|B_{l_2}^{(s'+1)}| = |B_2^{(5)}| = 2k_3 = 8$. $|C_1| = 2k_4 = 4$ is the sum of the number of even branches incident on x_i, $i = 5, 6$; and $|C_2| = 2(k_3 - k_4) = 4$ is the number of pendant branches incident on x_5. We give the label 0 to the vertex x_0 and give labelings to the centers of the branches incident on x_0 as per steps 1 to 5. Using the techniques of [4] (Theorem 1 of [4]) we obtain a graceful labeling of $T(L)$ given in Figure 12. Then we proceed as per the technique described in Theorem 3.6 and get a graceful labeling of L. Figure 13 represents the lobster L with a graceful labeling.

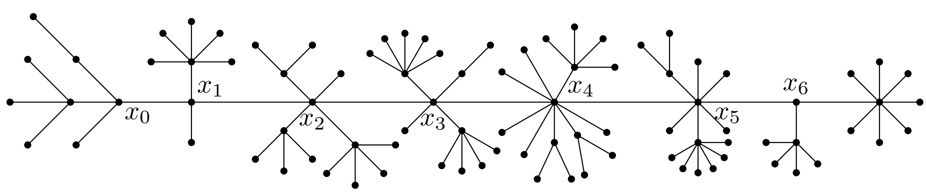

Figure 11. A lobster L of type (a) in Table 3.1. Here $m = 6$, $t_1 = 1$, $t_2 = 2$, $t_3 = 3$, $t_4 = 4$, $s = 5$.

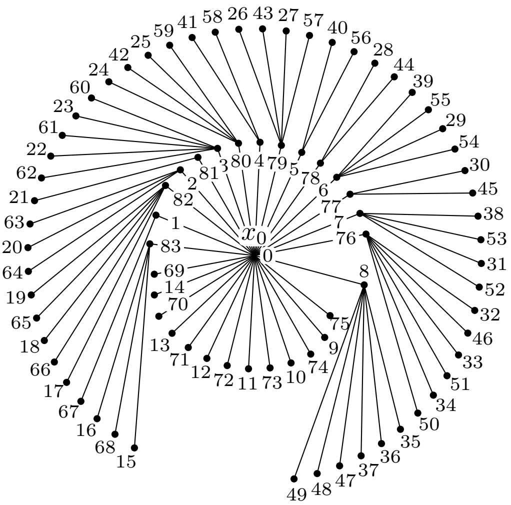

Figure 12. The diameter four tree $T(L)$ corresponding to L with a graceful labeling.

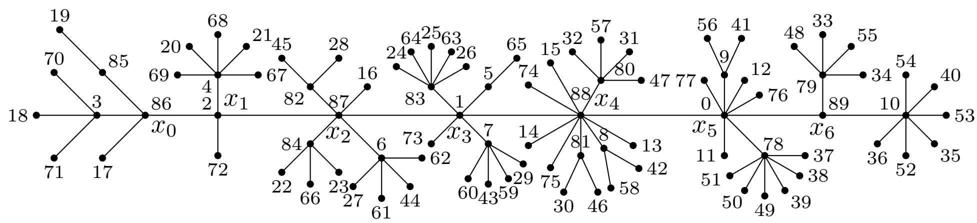

Figure 13. The lobster L with a graceful labeling.

For lobsters of type (x), $x = b, c, d, e$, in Table 3.3, the proof follows if we proceed as in the proof involving the lobsters of type (a) in Table 3.36 by modifying steps 1 to 5.

For lobsters of type (b) we do the following.

1. Repeat steps 1, 2, 3(i), 3(ii), and 3(iii) in the proof involving the lobsters of type (b) in Table 3.1.

2. Set $t_4 = t_3$ and $m = m + t' - p$ in steps 4 and 5, where p is an integer defined as in the proof for the lobsters of type (b) in Table 3.1.

For lobsters of type (c), we repeat steps 1, 2, and 3(i), (ii) and set $t_4 = t_3$ in steps 3(iii), 4, and 5. For lobsters of type (d), we repeat steps 1, 2, 3(i), omit step 3(ii), set $t_4 = t_3$ in step 3(iii), and set $t_4 = t_3$ in steps 4 and 5.

For lobsters L of type (e) and (f) we do the following.

Steps 1–3: If L is of type (e) (respectively, (f)), then set $t' = t_3$ and repeat steps 1, 2, 3(i), 3(ii), and 3(iii) (steps 1, 2, 3(i), and 3(ii)) in the proof involving the lobsters of type (e) in Table 3.1. Set $l_1 = 2$, $l_2 = 1$, and $s_{l_1} = s_2 = s$.

S t e p s 4–5: Set $s_{l_2}^{(1)} = s_1^{(1)} = p$ and $s_{l_2}^{(2)} = s_1^{(2)} = m + t' - p$. Set $t_4 = t_3$, $m = p$, and $s = m + t' - p$, and replace even branches by odd branches and pendant branches by even branches in steps 4 and 5 in the proof for the lobsters of type (a).

For lobsters L of type (g), the proof follows if we set $t' = t_2$ in steps 1, 2, 3(i), and 3(iii); and set $t_3 = t_2$ in steps 4 and 5 in the proof for the lobsters of type (e) in Table 3.3.

R e m a r k 3.8. With some changes in steps 1 to 5, one can show that the lobsters obtained from the lobsters in Theorems 3.4 and 3.7 by eliminating one or more combinations of branches incident on the central path, are also graceful.

R e m a r k 3.9. In all the lobsters to which we give graceful labelings in this paper, the vertex x_m gets the largest label and x_{m-1} gets the label 0. Therefore we get some more graceful lobsters by attaching a caterpillar to the vertex x_m or by attaching a suitable caterpillar (any number of pendant branches or an odd (or even) branch or the combination of both) to the vertex x_{m-1} in any of the lobsters discussed in Theorem 3.4 and 3.7.

References

[1] *J. C. Bermond*: Graceful graphs, radio antennae and French windmills. Graph Theory and Combinatorics, Proc. Conf. Notes in Maths. *34* (1979), 18–37.

[2] *W. C. Chen, H. I. Lu, Y. N. Yeh*: Operations of interlaced irees and graceful trees. Southeast Asian Bull. Math. *21* (1997), 337–348.

[3] *J. A. Gallian*: A dynamic survey of graph labeling. Electronic Journal of Combinatorics, DS6, Eleventh edition, February 29, 2008.
url: `http://www.combinatorics.org/Surveys/`.

[4] *P. Hrnčiar, A. Haviar*: All trees of diameter five are graceful. Discrete Math. *233* (2001), 133–150.

[5] *D. Mishra, P. Panigrahi*: Some new classes of graceful lobsters obtained from diameter four trees. To appear in Utilitus Mathematica.

[6] *D. Mishra, P. Panigrahi*: Graceful lobsters obtained by component moving of diameter four trees. Ars Combinatoria *81* (October, 2006), 129–146.

[7] *D. Mishra, P. Panigrahi*: Graceful lobsters obtained by partitioning and component moving of diameter four trees. Computers and Mathematics with Applications *50* (August 2005), 367–380.

[8] *D. Morgan*: All lobsters with perfect matchings are graceful. Technical Report, University of Alberta, TR05-01, Jan 2005.
url: `http://www.cs.ualberta.ca/research/techreports/2005.php`.

[9] *H. K. Ng*: Gracefulness of a class of lobsters. Notices AMS *7* (1986), abstract no. 825-05-294.

[10] *P. Panigrahi, D. Mishra*: Graceful lobsters obtained from diameter four trees using partitioning technique. Ars Combinatoria *87* (April, 2008), 291–320.

[11] *G. Ringel*: Problem 25 in theory of graphs and applications. Proceedings of Symposium Smolenice 1963, Academia, 1964, pp. 162.

[12] *A. Rosa*: On certain valuations of the vertices of a graph. Theórie des Graphes, (ed. P. Rosenstiehl), Dunod, Paris, 1968, pp. 349–355.
[13] *J. G. Wang, D. J. Jin, X. G. Lu, D. Zhang*: The gracefulness of a class of lobster trees. Math. Comput. Modelling *20* (1994), 105–110.

Authors' addresses: *Debdas Mishra*, Department of Mathematics, C. V. Raman College of Engineering, Bidya Nagar, Mahura, Janala, Bhubaneswar — 752054, Dist: Khurda, Orissa, India, e-mail: `debdasmishra@gmail.com`; *Pratima Panigrahi*, Department of Mathematics, Indian Institute of Technology, Kharagpur 721302, e-mail: `pratima@maths.iitkgp.ernet.in`.