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SADDLE POINTS CRITERIA VIA A SECOND ORDER η-APPROXIMATION APPROACH FOR NONLINEAR MATHEMATICAL PROGRAMMING INVOLVING SECOND ORDER INVEX FUNCTIONS

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In this paper, by using the second order η-approximation method introduced by Antczak [3], new saddle point results are obtained for a nonlinear mathematical programming problem involving second order invex functions with respect to the same function η. Moreover, a second order η-saddle point and a second order η-Lagrange function are defined for the so-called second order η-approximated optimization problem constructed in this method. Then, the equivalence between an optimal solution in the original mathematical programming problem and a second order η-saddle point of the second order η-Lagrangian in the associated second order η-approximated optimization problem is established. Finally, some example of using this approach to characterize of solvability of some O.R. problem is given.

Keywords: second order η-approximated optimization problem, second order η-saddle point, second order η-Lagrange function, second order invex function with respect to η, second order optimality conditions

Classification: 90C26, 90C46

1. INTRODUCTION

The Lagrange multipliers of a mathematical programming problem and the saddle points of its Lagrange function have been studied by many authors (see, for example, [5, 12, 13], and others). It is well known that, a fundamental result of optimization theory is that a saddle point of the Lagrange function is equivalent to an optimal solution of the associated convex programming problem satisfying a suitable constraint qualification (see, for example, [12, 13]). This result plays an important role in optimization theory and economics.

In the recent years, various classes of generalized convex functions have been proposed for the purpose of weakening the limitation of convexity in this result. Among these, the concept of an invexity function proposed by Hanson [11] has received more attention. Later Craven [10] named a class of functions introduced by Hanson as invex. Hanson's initial results inspired a great deal of subsequent work which has greatly expanded the role of invexity in optimization. Some of the generalization of Hanson's definition of an invex function is a second order invexity notion introduced by Bector and Bector [6, 7, 8] (biinvexity, bonvexity in Bector and Bector terminology).

Recently, new methods for characterizing of solvability of a nonlinear mathematical programming problem via an associated modified optimization problem have been proposed in the literature. One of such methods is the so-called (first order) η-approximation method proposed by Antczak [1]. In this method, the so-called η-approximated optimization problem is constructed by modifying both the objective and constraint functions of the given mathematical programming problem at an arbitrary but fixed feasible point $\overline{x}$. Furthermore, the equivalence between it and the original nonconvex mathematical programming problem has been established.

In [2], Antczak introduced a definition of the so-called (first order) η-saddle point and a definition of the so-called (first order) η-Lagrange function in the (first order) η-approximated optimization problem. Further, he established the equivalence between (first order) η-saddle-points of the η-Lagrange function in the η-approximated optimization problem and an optimal solution in the original mathematical programming problem. To prove this result, he assumed that the functions constituting the original mathematical programming problem are invex with respect to the same function η.

In [3] and [4], Antczak introduced two second order approximation methods, that is, the so-called second order objective function method and the second order η-approximation method, respectively, for obtaining the second order sufficient optimality conditions in nonlinear mathematical programming problem with twice differentiable functions. To prove the equivalence between the original mathematical programming problem and its associated second order modified optimization problem constructed in these methods, he assumed that all functions constituting the original mathematical programming problem are second order invex with respect to the same function η and some extra constraint is imposed on the function η.

The purpose of the present paper is to extend the results obtained by Antczak in [2] to the case when the functions involved in the original mathematical programming problem are twice differentiable. Thus, we study a new aspect of using the second order η-approximation method proposed by Antczak in [4]. To this end, we introduce a definition of a so-called second order η-saddle point and a definition of a so-called second order η-Lagrange function in the second order η-approximated optimization problem constructed in this method. Further, it is shown the equivalence between a second order η-saddle-point of the second order η-Lagrange function in the second order η-approximated optimization problem and an optimal solution in the original mathematical programming problem. To illustrate this result, some examples of nonlinear nonconvex optimization problems with second order invex functions are presented.

2. PRELIMINARIES

Throughout the paper we write $\nabla f(\overline{x})$ and $\nabla^2 f(\overline{x})$ for the gradient of f and for the Hessian of f evaluated at $\overline{x}$, respectively. We recall some definitions that will be used in the present paper.

Definition 2.1. (Hanson [11]) Let $f : X \to R$ be a differentiable function on a nonempty open set $X \subset R^n$ and $u \in X$. If, there exists $\eta : X \times X \to R^n$ such that, for all $x \in X$,

$$f(x) - f(u) \geq [\eta(x,u)]^T \nabla f(u), \tag{2.1}$$

then f is first order invex (or shortly, invex in Hanson terminology) at u on X (with respect to η). If the inequality (2.1) holds for any $u \in X$, then f is invex on X (with respect to η).

Definition 2.2. (Bector and Bector [6], Bector and Chandra [8]) Let $f : X \to R$ be a twice differentiable function defined on a nonempty open set $X \subset R^n$ and $u \in X$. If there exists $\eta : X \times X \to R^n$ such that, the following inequality

$$f(x) - f(u) \geq [\eta(x,u)]^T \left[\nabla f(u) + \nabla^2 f(u) y\right] - \frac{1}{2} y^T \nabla^2 f(u) y \quad (>) \tag{2.2}$$

holds for all $y \in R^n$ and for all $x \in X$ $(x \neq u)$, then f is said to be second order (strictly) invex (bonvex in Bector and Bector terminology) at u on X (with respect to η). If the inequality (2.2) holds for any $u \in X$, then f is second order invex on X (with respect to η).

In the paper, we consider the nonlinear constrained mathematical programming problem

$$\begin{array}{ll} & f(x) \to \min \\ \text{subject to} & g_j(x) \leq 0, \quad j \in J = \{1, \dots, m\}, \end{array} \tag{P}$$

where $f : X \to R$ and $g_j : X \to R$, $j \in J$, are twice differentiable functions on a nonempty open set $X \subset R^n$.

Let

$$D := \{x \in X : g_j(x) \leq 0, \quad j = 1, \dots, m\}$$

denote the set of all feasible solutions in (P). Further, we denote by $J(\overline{x})$ the set

$$J(\overline{x}) := \{j \in J : g_j(\overline{x}) = 0\},$$

that is, the index set of constraints active at the given feasible point $\overline{x}$.

Definition 2.3. The function $L : D \times R^m \to R$ defined by

$$L(x, \xi) := f(x) + \xi^T g(x)$$

is said the Lagrange function or the Lagrangian in the original mathematical programming problem (P).

It is well known that the Karush–Kuhn–Tucker conditions are (first order) necessary conditions for optimality in such optimization problems (see, for example, [5, 12]).

Theorem 2.1. Let $\overline{x}$ be an optimal solution in (P) and a suitable constraint qualification [5] be satisfied at $\overline{x}$. Then, there exists $\overline{\xi} \in R^m$ such that

$$\nabla f(\overline{x}) + \overline{\xi}^T \nabla g(\overline{x}) = 0, \tag{2.3}$$

$$\overline{\xi}_j g_j(\overline{x}) = 0, \ \ j \in J, \tag{2.4}$$

$$\overline{\xi} \geq 0. \tag{2.5}$$

To introduce the second order necessary optimality conditions in the dual form for a mathematical programming problem, Ben-Tal [9] gave a definition of the so-called set of critical directions at $\overline{x}$.

Definition 2.4. The set

$$C(\overline{x}) := \left\{ d \in R^n : d^T \nabla f(\overline{x}) \leq 0 \wedge d^T \nabla g_j(\overline{x}) \leq 0, \, j \in J(\overline{x}) \right\}$$

is said to be the set of critical directions at $\overline{x}$.

It is also known (see, for example, [9]) that the second-order optimality conditions (in the so-called dual form) for a nonlinear mathematical programming problem are necessary for $\overline{x}$ to be an optimal solution in the considered mathematical programming problem.

Theorem 2.2. Let $\overline{x}$ be an optimal solution in (P) and a suitable constraint qualification (CQ) be satisfied at $\overline{x}$ (see [9]). Then, for every $d \in C(\overline{x})$, there exists $\overline{\xi} \in R^m$ such that

$$\nabla L(\overline{x}, \overline{\xi}) = 0, \tag{2.6}$$

$$d^T \nabla^2 L(\overline{x}, \overline{\xi})\, d \geq 0, \tag{2.7}$$

$$\overline{\xi}_j g_j(\overline{x}) = 0, \quad j \in J, \tag{2.8}$$

$$\nabla f(\overline{x})\, d = 0, \tag{2.9}$$

$$\overline{\xi}_j \nabla g_j(\overline{x})\, d = 0, \quad j \in J(\overline{x}), \tag{2.10}$$

$$\overline{\xi} \geq 0. \tag{2.11}$$

Remark 2.1. Whenever we assume that a suitable constraint qualification (CQ) is satisfied for the considered optimization problem (P) we shall mean that some of the constraint qualifications considered in [9] is satisfied.

Now, we give a definition of a classical saddle point of the Lagrangian in the original mathematical programming problem (P).

Definition 2.5. (Bazaraa et al. [5], Mangasarian [12]) A point $(\overline{x}, \overline{\xi}) \in X \times R^m_+$ is said to be a saddle point in the original problem (P), if

i) $L(\overline{x}, \xi) \leq L(\overline{x}, \overline{\xi}), \quad \forall \xi \in R^m_+,$

ii) $L(\overline{x}, \overline{\xi}) \leq L(x, \overline{\xi}), \quad \forall x \in X.$

3. SECOND ORDER η-SADDLE POINT OPTIMALITY CRITERIA FOR THE SECOND ORDER η-APPROXIMATED OPTIMIZATION PROBLEM

In the second order η-approximation approach introduced by Antczak [4] a so-called second order η-approximated optimization problem $(P^2_\eta(\overline{x}))$ equivalent to the original mathematical programming problem (P) is constructed.

Now, for the benefit of the reader, we give a definition of the second order η-approximated optimization problem $(\mathrm{P}^2_\eta(\overline{x}))$ associated with the mathematical programming problem (P) considered in the paper.

Let $\overline{x}$ be the given feasible solution in (P). We consider the following optimization problem $(\mathrm{P}^2_\eta(\overline{x}))$ defined as follows:

$$\begin{array}{c} f(\overline{x}) + [\eta(x,\overline{x})]^T \nabla f(\overline{x}) + \frac{1}{2}[\eta(x,\overline{x})]^T \nabla^2 f(\overline{x})\eta(x,\overline{x}) \to \min \\ g_j(\overline{x}) + [\eta(x,\overline{x})]^T \nabla g_j(\overline{x}) + \frac{1}{2}[\eta(x,\overline{x})]^T \nabla^2 g_j(\overline{x})\eta(x,\overline{x}) \leq 0,\ j \in J, \end{array} \qquad (P^2_\eta(\overline{x}))$$

where f, g_j, $j = 1, \ldots, m$, X are defined as in problem (P) and, moreover, the function η is defined by $\eta : X \times X \to R^n$. We call $(P^2_\eta(\overline{x}))$ the associated second order η-approximated optimization problem.

Let

$$D(\overline{x}) := \left\{ x \in R^n : g_j(\overline{x}) + [\eta(x,\overline{x})]^T \nabla g_j(\overline{x}) + \frac{1}{2}\left[\eta(x,\overline{x})\right]^T \nabla^2 g_j(\overline{x})\eta(x,\overline{x}) \leq 0,\ j \in J \right\}$$

denote the set of all feasible solutions in $(\mathrm{P}^2_\eta(\overline{x}))$.

Definition 3.1. A function $L^2_\eta : X \times R^m_+ \to R$ defined by

$$\begin{aligned} L^2_\eta(x,\xi) \; : \; &= f(\overline{x}) + \xi^T g(\overline{x}) + [\eta(x,\overline{x})]^T \left[\nabla f(\overline{x}) + \xi^T \nabla g(\overline{x})\right] \\ &\quad + \frac{1}{2}[\eta(x,\overline{x})]^T \left[\nabla^2 f(\overline{x}) + \xi^T \nabla^2 g(\overline{x})\right] \eta(x,\overline{x}) \end{aligned}$$

is called the second order η-approximated Lagrange function, or shortly, the second order η-approximated Lagrangian in the second order η-approximated optimization problem $(\mathrm{P}^2_\eta(\overline{x}))$.

Remark 3.1. By Definition 2.3, it follows that

$$L^2_\eta(x,\xi) := L(\overline{x},\xi) + [\eta(x,\overline{x})]^T \nabla L(\overline{x},\xi) + \frac{1}{2}[\eta(x,\overline{x})]^T \nabla^2 L(\overline{x},\xi)\eta(x,\overline{x}).$$

In the analogous manner as for the original mathematical programming problem (P), we define a so-called second order η-saddle point for the second order η-approximated optimization problem $(\mathrm{P}^2_\eta(\overline{x}))$.

Definition 3.2. A point $\left(\overline{x},\overline{\xi}\right) \in D \times R^m_+$ is said to be a second order η-saddle point in the second order η-approximated optimization problem $(\mathrm{P}^2_\eta(\overline{x}))$, if

i) $L^2_\eta(\overline{x},\xi) \leq L^2_\eta\left(\overline{x},\overline{\xi}\right), \quad \forall\, \xi \in R^m_+$,

ii) $L^2_\eta\left(\overline{x},\overline{\xi}\right) \leq L^2_\eta\left(x,\overline{\xi}\right), \quad \forall\, x \in D(\overline{x})$.

Using Definition 3.1, the second order η-saddle point criteria given above can be rewritten in the following form:

i) $f(\overline{x}) + \xi^T g(\overline{x}) + [\eta(\overline{x},\overline{x})]^T \left[\nabla f(\overline{x}) + \xi^T \nabla g(\overline{x})\right] + \frac{1}{2}[\eta(\overline{x},\overline{x})]^T \left[\nabla^2 f(\overline{x})\right.$

$\left. + \xi^T \nabla^2 g(\overline{x})\right] \eta(\overline{x},\overline{x}) \leq f(\overline{x}) + \overline{\xi}^T g(\overline{x}) + [\eta(\overline{x},\overline{x})]^T \left[\nabla f(\overline{x}) + \overline{\xi}^T \nabla g(\overline{x})\right]$

$+\frac{1}{2}[\eta(\overline{x},\overline{x})]^T \left[\nabla^2 f(\overline{x}) + \overline{\xi}^T \nabla^2 g(\overline{x})\right] \eta(\overline{x},\overline{x}), \qquad \forall \xi \in R^m_+,$

ii) $f(\overline{x}) + \overline{\xi}^T g(\overline{x}) + [\eta(\overline{x},\overline{x})]^T \left[\nabla f(\overline{x}) + \overline{\xi}^T \nabla g(\overline{x})\right] + \frac{1}{2}[\eta(\overline{x},\overline{x})]^T \left[\nabla^2 f(\overline{x})\right.$

$\left. + \overline{\xi}^T \nabla^2 g(\overline{x})\right] \eta(\overline{x},\overline{x}) \leq f(\overline{x}) + \overline{\xi}^T g(\overline{x}) + [\eta(x,\overline{x})]^T \left[\nabla f(\overline{x}) + \overline{\xi}^T \nabla g(\overline{x})\right]$

$+\frac{1}{2}[\eta(x,\overline{x})]^T \left[\nabla^2 f(\overline{x}) + \overline{\xi}^T \nabla^2 g(\overline{x})\right] \eta(x,\overline{x}), \qquad \forall x \in D(\overline{x}).$

Now, we prove the equivalence between a second order η-saddle point of the second order η-Lagrangian and an optimal solution in the second order η-approximated optimization problem $(\mathrm{P}^2_\eta(\overline{x}))$.

Proposition 3.1. If $(\overline{x},\overline{\xi}) \in D \times R^m_+$ is a second order η-saddle point in problem $(\mathrm{P}^2_\eta(\overline{x}))$, then $\overline{x}$ is optimal in problem $(\mathrm{P}^2_\eta(\overline{x}))$.

Proof. We proceed by contradiction. Suppose that $\overline{x}$ is not optimal in $(\mathrm{P}^2_\eta(\overline{x}))$. Then, there exists $\widetilde{x} \in D(\overline{x})$ such that

$$\begin{aligned} f(\overline{x}) + [\eta(\widetilde{x},\overline{x})]^T \nabla f(\overline{x}) + \tfrac{1}{2}[\eta(\widetilde{x},\overline{x})]^T \nabla^2 f(\overline{x})\eta(\widetilde{x},\overline{x}) \\ < f(\overline{x}) + [\eta(\overline{x},\overline{x})]^T \nabla f(\overline{x}) + \tfrac{1}{2}[\eta(\overline{x},\overline{x})]^T \nabla^2 f(\overline{x})\eta(\overline{x},\overline{x}), \end{aligned} \tag{3.12}$$

and

$$g(\overline{x}) + \nabla g(\overline{x})\eta(\widetilde{x},\overline{x}) + \frac{1}{2}[\eta(\widetilde{x},\overline{x})]^T \nabla^2 g(\overline{x})\eta(\widetilde{x},\overline{x}) \leq 0. \tag{3.13}$$

Therefore, $\overline{\xi} \in R^m_+$ gives

$$\overline{\xi}^T \left(g(\overline{x}) + \nabla g(\overline{x})\eta(\widetilde{x},\overline{x}) + \frac{1}{2}[\eta(\widetilde{x},\overline{x})]^T \nabla^2 g(\overline{x})\eta(\widetilde{x},\overline{x})\right) \leq 0. \tag{3.14}$$

In the similar manner, using the feasibility of $\overline{x}$ in $(\mathrm{P}^2_\eta(\overline{x}))$ together with $\overline{\xi} \in R^m_+$, we get

$$\overline{\xi}^T \left(g(\overline{x}) + \nabla g(\overline{x})\eta(\overline{x},\overline{x}) + \frac{1}{2}[\eta(\overline{x},\overline{x})]^T \nabla^2 g(\overline{x})\eta(\overline{x},\overline{x})\right) \leq 0. \tag{3.15}$$

Since $\left(\overline{x},\overline{\xi}\right)$ is a second order η-saddle point in $(\mathrm{P}^2_\eta(\overline{x}))$, then, by Definition 3.2 i), the following inequality

$$\begin{aligned} & f(\overline{x})+\xi^T g(\overline{x})+\left[\eta\left(\overline{x},\overline{x}\right)\right]^T\left[\nabla f(\overline{x})+\xi^T\nabla g\left(\overline{x}\right)\right] \\ & \quad +\frac{1}{2}\left[\eta\left(\overline{x},\overline{x}\right)\right]^T\left[\nabla^2 f(\overline{x})+\xi^T\nabla^2 g\left(\overline{x}\right)\right]\eta\left(\overline{x},\overline{x}\right) \\ \leq \; & f(\overline{x})+\overline{\xi}^T g(\overline{x})+\left[\eta\left(\overline{x},\overline{x}\right)\right]^T\left[\nabla f(\overline{x})+\overline{\xi}^T\nabla g\left(\overline{x}\right)\right] \\ & \quad +\frac{1}{2}\left[\eta\left(\overline{x},\overline{x}\right)\right]^T\left[\nabla^2 f(\overline{x})+\overline{\xi}^T\nabla^2 g\left(\overline{x}\right)\right]\eta\left(\overline{x},\overline{x}\right) \end{aligned}$$

holds for all $\xi\in R^m_+$. Thus, for $\xi=0$,

$$\overline{\xi}^T\left(g(\overline{x})+\nabla g\left(\overline{x}\right)\eta\left(\overline{x},\overline{x}\right)+\frac{1}{2}\left[\eta\left(\overline{x},\overline{x}\right)\right]^T\nabla^2 g\left(\overline{x}\right)\eta\left(\overline{x},\overline{x}\right)\right)\geq 0. \tag{3.16}$$

Then, by (3.15) and (3.16), it follows that

$$\overline{\xi}^T\left(g(\overline{x})+\nabla g\left(\overline{x}\right)\eta\left(\overline{x},\overline{x}\right)+\frac{1}{2}\left[\eta\left(\overline{x},\overline{x}\right)\right]^T\nabla^2 g\left(\overline{x}\right)\eta\left(\overline{x},\overline{x}\right)\right)=0. \tag{3.17}$$

Using (3.12) and (3.14) together with (3.17), we get

$$\begin{aligned} & f(\overline{x})+\overline{\xi}^T g(\overline{x})+\left[\eta\left(\widetilde{x},\overline{x}\right)\right]^T\left[\nabla f(\overline{x})+\overline{\xi}^T\nabla g\left(\overline{x}\right)\right] \\ & \quad +\frac{1}{2}\left[\eta\left(\widetilde{x},\overline{x}\right)\right]^T\left[\nabla^2 f(\overline{x})+\overline{\xi}^T\nabla^2 g\left(\overline{x}\right)\right]\eta\left(\widetilde{x},\overline{x}\right) \\ < \; & f(\overline{x})+\overline{\xi}^T g(\overline{x})+\left[\eta\left(\overline{x},\overline{x}\right)\right]^T\left[\nabla f(\overline{x})+\overline{\xi}^T\nabla g\left(\overline{x}\right)\right] \\ & \quad +\frac{1}{2}\left[\eta\left(\overline{x},\overline{x}\right)\right]^T\left[\nabla^2 f(\overline{x})+\overline{\xi}^T\nabla^2 g\left(\overline{x}\right)\right]\eta\left(\overline{x},\overline{x}\right). \end{aligned}$$

Hence, by Definition 3.1, we obtain the following inequality

$$L^2_\eta\left(\widetilde{x},\overline{\xi}\right)<L^2_\eta\left(\overline{x},\overline{\xi}\right),$$

which is a contradiction to the inequality ii) in Definition 3.2. □

Remark 3.2. Note that we establish Proposition 3.1 without any assumption to which a class of functions the functions involving in problem $(\mathrm{P}^2_\eta(\overline{x}))$ belong. Moreover, no constraints are imposed on the function η.

Proposition 3.2. Let the objective function and the constraint function involving in the second order η-approximated optimization problem $(\mathrm{P}^2_\eta(\overline{x}))$ be second order invex at $\overline{x}$ on $D\left(\overline{x}\right)$ with respect to the same function $\vartheta : D\left(\overline{x}\right)\times D\left(\overline{x}\right)\to R^n$, not necessarily equal to the function η, and the suitable constraint qualification (CQ) [5] be satisfied at $\overline{x}$. Moreover, we assume that ϑ satisfies the condition $\vartheta\left(\overline{x},\overline{x}\right)=0$. If $\overline{x}$ is optimal in problem $(\mathrm{P}^2_\eta(\overline{x}))$, then there exists $\overline{\xi}\in R^m_+$ such that $\left(\overline{x},\overline{\xi}\right)$ is a second order η-saddle point in $(\mathrm{P}^2_\eta(\overline{x}))$.

P r o o f. By assumption, $\overline{x}$ is optimal in problem $(\mathrm{P}^2_\eta(\overline{x}))$ and the suitable constraint qualification (CQ) is satisfied at $\overline{x}$. Then, the second order necessary optimality conditions (2.6) − (2.11) are fulfilled at $\overline{x}$.

Now, we prove the inequality i) of Definition 3.2. Since $\overline{x}$ is feasible in problem $(\mathrm{P}^2_\eta(\overline{x}))$, then

$$g(\overline{x}) + \nabla g(\overline{x})\, \eta(\overline{x},\overline{x}) + \frac{1}{2}\left[\eta(\overline{x},\overline{x})\right]^T \nabla^2 g(\overline{x})\, \eta(\overline{x},\overline{x}) \leq 0.$$

Thus, for all $\xi \in R^m_+$,

$$\xi^T \left(g(\overline{x}) + \nabla g(\overline{x})\, \eta(\overline{x},\overline{x}) + \frac{1}{2}\left[\eta(\overline{x},\overline{x})\right]^T \nabla^2 g(\overline{x})\, \eta(\overline{x},\overline{x}) \right) \leq 0. \tag{3.18}$$

Hence, using the second order optimality condition (2.8), we have

$$\overline{\xi}^T \left(g(\overline{x}) + \nabla g(\overline{x})\, \eta(\overline{x},\overline{x}) + \frac{1}{2}\left[\eta(\overline{x},\overline{x})\right]^T \nabla^2 g(\overline{x})\, \eta(\overline{x},\overline{x}) \right) = 0. \tag{3.19}$$

By (3.18) and (3.19), the inequality

$$\begin{aligned} & f(\overline{x}) + \xi^T g(\overline{x}) + \left[\eta(\overline{x},\overline{x})\right]^T \left[\nabla f(\overline{x}) + \xi^T \nabla g(\overline{x})\right] \\ & \qquad + \frac{1}{2}\left[\eta(\overline{x},\overline{x})\right]^T \left[\nabla^2 f(\overline{x}) + \xi^T \nabla^2 g(\overline{x})\right] \eta(\overline{x},\overline{x}) \\ \leq \; & f(\overline{x}) + \overline{\xi}^T g(\overline{x}) + \left[\eta(\overline{x},\overline{x})\right]^T \left[\nabla f(\overline{x}) + \overline{\xi}^T \nabla g(\overline{x})\right] \\ & \qquad + \frac{1}{2}\left[\eta(\overline{x},\overline{x})\right]^T \left[\nabla^2 f(\overline{x}) + \overline{\xi}^T \nabla^2 g(\overline{x})\right] \eta(\overline{x},\overline{x}) \end{aligned}$$

is satisfied for all $\xi \in R^m_+$. By Definition 3.1, the inequality

$$L^2_\eta(\overline{x},\xi) \leq L^2_\eta\left(\overline{x},\overline{\xi}\right) \tag{3.20}$$

holds for all $\xi \in R^m_+$.

To prove the inequality ii) in Definition 3.2, we denote

$$\begin{aligned} F(x) &:= f(\overline{x}) + \left[\eta(x,\overline{x})\right]^T \nabla f(\overline{x}) + \frac{1}{2}\left[\eta(x,\overline{x})\right]^T \nabla^2 f(\overline{x}) \eta(x,\overline{x}), \\ G(x) &:= g(\overline{x}) + \nabla g(\overline{x})\, \eta(x,\overline{x}) + \frac{1}{2}\left[\eta(x,\overline{x})\right]^T \nabla^2 g(\overline{x})\, \eta(x,\overline{x}), \end{aligned}$$

that is, we introduce the denotations for the objective function and the constraint function in the second order η-approximated optimization problem $(\mathrm{P}^2_\eta(\overline{x}))$, respectively. By assumption, F and G are second order invex at $\overline{x}$ on $D(\overline{x})$ with respect to the same function ϑ, not necessarily equal to the function η, satisfying the condition: $\vartheta(\overline{x},\overline{x}) = 0$. Then, by Definition 2.2, we have, respectively, that the inequalities

$$F(x) - F(\overline{x}) \geq \left[\vartheta(x,\overline{x})\right]^T \left[\nabla F(\overline{x}) + \nabla^2 F(\overline{x}) y\right] - \frac{1}{2} y^T \nabla^2 F(\overline{x}) y,$$

$$G(x)-G(\overline{x}) \geq [\vartheta(x,\overline{x})]^T \left[\nabla G(\overline{x})+\nabla^2 G(\overline{x})y\right]-\frac{1}{2}y^T\nabla^2 G(\overline{x})y$$

hold for any $x \in D(\overline{x})$ and for all $y \in R^n$. Hence, they are also satisfied for $y=\vartheta(x,\overline{x})$. Thus,

$$F(x)-F(\overline{x}) \geq [\vartheta(x,\overline{x})]^T \nabla F(\overline{x})+\frac{1}{2}[\vartheta(x,\overline{x})]^T \nabla^2 F(\overline{x})\vartheta(x,\overline{x}), \quad (3.21)$$

$$G(x)-G(\overline{x}) \geq [\vartheta(x,\overline{x})]^T \nabla G(\overline{x})+\frac{1}{2}[\vartheta(x,\overline{x})]^T \nabla^2 G(\overline{x})\vartheta(x,\overline{x}). \quad (3.22)$$

Using (3.22) together with the second order necessary optimality condition (2.11), we obtain

$$\overline{\xi}^T G(x)-\overline{\xi}^T G(\overline{x}) \geq \overline{\xi}^T \left([\vartheta(x,\overline{x})]^T \nabla G(\overline{x})+\frac{1}{2}[\vartheta(x,\overline{x})]^T \nabla^2 G(\overline{x})\vartheta(x,\overline{x})\right). \quad (3.23)$$

Adding both sides of the inequalities (3.21) and (3.23), we get

$$F(x)+\overline{\xi}^T G(x)-F(\overline{x})-\overline{\xi}^T G(\overline{x}) \geq [\vartheta(x,\overline{x})]^T \left[\nabla F(\overline{x})+\overline{\xi}^T \nabla G(\overline{x})\right]$$
$$+\tfrac{1}{2}[\vartheta(x,\overline{x})]^T \left[\nabla^2 F(\overline{x})+\overline{\xi}^T \nabla^2 G(\overline{x})\right]\vartheta(x,\overline{x}).$$

Then, by the second order necessary optimality condition (2.6), it follows that the following inequality

$$F(x)+\overline{\xi}^T G(x)-F(\overline{x})-\overline{\xi}^T G(\overline{x}) \geq \frac{1}{2}[\vartheta(x,\overline{x})]^T \left[\nabla^2 F(\overline{x})+\overline{\xi}^T \nabla^2 G(\overline{x})\right]\vartheta(x,\overline{x}). \quad (3.24)$$

holds for all $x \in D(\overline{x})$. Using the second order necessary optimality condition (2.7), we obtain that the inequality

$$F(x)+\overline{\xi}^T G(x) \geq F(\overline{x})+\overline{\xi}^T G(\overline{x})$$

holds for all $x \in D(\overline{x})$. Thus, by the definitions of F and G, the inequality

$$f(\overline{x})+\overline{\xi}^T g(\overline{x})+[\eta(x,\overline{x})]^T\left[\nabla f(\overline{x})+\overline{\xi}^T\nabla g(\overline{x})\right]+\frac{1}{2}[\eta(x,\overline{x})]^T\left[\nabla^2 f(\overline{x})\right.$$
$$\left.+\overline{\xi}^T\nabla^2 g(\overline{x})\right]\eta(x,\overline{x}) \geq f(\overline{x})+\overline{\xi}^T g(\overline{x})+[\eta(\overline{x},\overline{x})]^T\left[\nabla f(\overline{x})+\overline{\xi}^T\nabla g(\overline{x})\right]$$
$$+\frac{1}{2}[\eta(\overline{x},\overline{x})]^T\left[\nabla^2 f(\overline{x})+\overline{\xi}^T\nabla^2 g(\overline{x})\right]\eta(\overline{x},\overline{x})$$

holds for all $x \in D(\overline{x})$. Hence, by Definition 3.1, the inequality

$$L_\eta^2(\overline{x},\overline{\xi}) \leq L_\eta^2(x,\overline{\xi}) \quad (3.25)$$

holds for any $x \in D(\overline{x})$. By (3.20) and (3.25), we get the conclusion of theorem. □

4. SECOND ORDER η-SADDLE POINT OPTIMALITY CRITERIA FOR THE ORIGINAL MATHEMATICAL PROGRAMMING PROBLEM

In [4], by using the so-called second order η-approximation method, Antczak established the equivalence between the original mathematical programming problem (P) and its second order η-approximated optimization problem $(\mathrm{P}^2_\eta(\overline{x}))$ constructed in this method. Indeed, under assumption that all functions constituting the original mathematical programming (P) are second order invex at $\overline{x}$ on D with respect to the same function η satisfying $\eta(\overline{x},\overline{x}) = 0$, he showed that if $\overline{x}$ is an optimal solution in (P), then it is also optimal in $(\mathrm{P}^2_\eta(\overline{x}))$, and conversely, if $\overline{x}$ is an optimal solution in $(\mathrm{P}^2_\eta(\overline{x}))$, then it is also optimal in (P). Now, we show a new aspect of using this approach. We prove the equivalence between an optimal solution $\overline{x}$ in the original mathematical programming problem (P) and a second order η-saddle point in its associated second order η-approximated optimization problem $(\mathrm{P}^2_\eta(\overline{x}))$.

Before proving the main result of this section, we give some useful lemma the simple proof of which is omitted in the paper.

Lemma 4.1. If g is second order invex at $\overline{x}$ on D with respect to η, then any feasible solution in the original mathematical programming problem (P) is also feasible in the second order η-approximated optimization problem $(\mathrm{P}^2_\eta(\overline{x}))$, that is, $D \subset D(\overline{x})$.

Theorem 4.1. Let $\overline{x}$ be a feasible solution in (P). We assume that f and g are second order invex at $\overline{x}$ on D with respect to the same function η satisfying the condition $\eta(\overline{x},\overline{x}) = 0$. If $(\overline{x},\overline{\xi}) \in D \times R^m_+$ is a second order η-saddle point in the second order η-approximated optimization problem $(\mathrm{P}^2_\eta(\overline{x}))$, then $\overline{x}$ is optimal in the original mathematical programming problem (P).

P r o o f. By assumption, $(\overline{x},\overline{\xi}) \in D \times R^m_+$ is a second order η-saddle point in the second order η-approximated optimization problem $(\mathrm{P}^2_\eta(\overline{x}))$. Then, by the inequality i) in Definition 3.2) we have, for all $\xi \in R^m_+$,

$$L^2_\eta(\overline{x},\xi) \leq L^2_\eta(\overline{x},\overline{\xi}).$$

From the definition of the second order η-Lagrange function in $(\mathrm{P}^2_\eta(\overline{x}))$, we obtain, for $\xi = 0$,

$$\overline{\xi}^T\left(g(\overline{x}) + [\eta(\overline{x},\overline{x})]^T\nabla g(\overline{x}) + \frac{1}{2}[\eta(\overline{x},\overline{x})]^T\nabla^2 g(\overline{x})\,\eta(\overline{x},\overline{x})\right) \geq 0.$$

By hypothesis, $\eta(\overline{x},\overline{x}) = 0$. Thus,

$$\overline{\xi}^T g(\overline{x}) \geq 0. \tag{4.26}$$

By assumption, g is second order invex at $\overline{x}$ on D with respect to η. Hence, by Lemma 4.1, it follows that $D \subset D(\overline{x})$. Thus, $\overline{x} \in D$ implies $\overline{x} \in D(\overline{x})$. Since $\overline{x} \in D(\overline{x})$ and $\overline{\xi} \in R^m_+$, then

$$\overline{\xi}^T\left(g(\overline{x}) + [\eta(\overline{x},\overline{x})]^T\nabla g(\overline{x}) + \frac{1}{2}[\eta(\overline{x},\overline{x})]^T\nabla^2 g(\overline{x})\,\eta(\overline{x},\overline{x})\right) \leq 0.$$

By hypothesis, $\eta(\overline{x}, \overline{x}) = 0$. Thus,

$$\overline{\xi}^T g(\overline{x}) \leq 0. \tag{4.27}$$

Combining (4.26) and (4.27), we get

$$\overline{\xi}^T g(\overline{x}) = 0. \tag{4.28}$$

Hence, by the definition of a second order η-saddle point (the inequality ii) in Definition 3.2) we have, for all $x \in D(\overline{x})$,

$$L_\eta^2\left(x, \overline{\xi}\right) \geq L_\eta^2\left(\overline{x}, \overline{\xi}\right).$$

Since $D \subset D(\overline{x})$, then the above inequality is fulfilled for all $x \in D$. Now, using the definition of the second order η-Lagrange function in $(\mathrm{P}_\eta^2(\overline{x}))$, we get, for all $x \in D$,

$$\begin{gathered}[\eta(x, \overline{x})]^T \left[\nabla f(\overline{x}) + \overline{\xi}^T \nabla g(\overline{x})\right] + \tfrac{1}{2}[\eta(x, \overline{x})]^T \left[\nabla^2 f(\overline{x}) + \overline{\xi}^T \nabla^2 g(\overline{x})\right] \eta(x, \overline{x}) \\ \geq [\eta(\overline{x}, \overline{x})]^T \left[\nabla f(\overline{x}) + \overline{\xi}^T \nabla g(\overline{x})\right] + \tfrac{1}{2}[\eta(\overline{x}, \overline{x})]^T \left[\nabla^2 f(\overline{x}) + \overline{\xi}^T \nabla^2 g(\overline{x})\right] \eta(\overline{x}, \overline{x})\end{gathered} \tag{4.29}$$

By assumption, g is second order invex at $\overline{x}$ on D with respect to η. Hence, by Definition 2.2, the following inequality

$$g(x) \geq g(\overline{x}) + [\eta(x, \overline{x})]^T \left[\nabla g(\overline{x}) + \nabla^2 g(\overline{x}) y\right] - \frac{1}{2} y^T \nabla^2 g(\overline{x}) y$$

holds for any $x \in D$ and all $y \in R^n$. Thus, it is satisfied for $y = \eta(x, \overline{x})$. Therefore, by $\overline{\xi} \in R_+^m$, it follows that

$$\overline{\xi}^T g(x) \geq \overline{\xi}^T \left(g(\overline{x}) + \nabla g(\overline{x}) \eta(x, \overline{x}) + \tfrac{1}{2}[\eta(x, \overline{x})]^T \nabla^2 g(\overline{x}) \eta(x, \overline{x})\right). \tag{4.30}$$

Since $x \in D$, then $D \subset D(\overline{x})$ gives $x \in D(\overline{x})$. Hence, $\overline{\xi}^T g(x) \leq 0$, and, therefore, the inequality

$$\overline{\xi}^T \left(g(\overline{x}) + \nabla g(\overline{x}) \eta(x, \overline{x}) + \tfrac{1}{2}[\eta(x, \overline{x})]^T \nabla^2 g(\overline{x}) \eta(x, \overline{x})\right) \leq 0 \tag{4.31}$$

holds for all $x \in D$. By (4.28) and (4.31),

$$\overline{\xi}^T \left(\nabla g(\overline{x}) \eta(x, \overline{x}) + \tfrac{1}{2}[\eta(x, \overline{x})]^T \nabla^2 g(\overline{x}) \eta(x, \overline{x})\right) \leq 0. \tag{4.32}$$

By assumption, f is second order invex at $\overline{x}$ on D with respect to η. Therefore, by Definition 2.2, the inequality

$$f(x) - f(\overline{x}) \geq [\eta(x, \overline{x})]^T \left[\nabla f(\overline{x}) + \nabla^2 f(\overline{x}) y\right] - \frac{1}{2} y^T \nabla^2 f(\overline{x}) y \tag{4.33}$$

holds for any $x \in D$ and all $y \in R^n$. Hence, it is also satisfied for $y = \eta(x, \overline{x})$. Then, (4.33) gives

$$f(x) - f(\overline{x}) \geq [\eta(x,\overline{x})]^T \nabla f(\overline{x}) + \frac{1}{2}[\eta(x,\overline{x})]^T \nabla^2 f(\overline{x})\, \eta(x,\overline{x}). \tag{4.34}$$

Therefore, by (4.32),

$$\begin{aligned} f(x) - f(\overline{x}) \geq\ & [\eta(x,\overline{x})]^T \left[\nabla f(\overline{x}) + \overline{\xi}^T \nabla g(\overline{x})\right] \\ & + \tfrac{1}{2}[\eta(x,\overline{x})]^T \left[\nabla^2 f(\overline{x}) + \overline{\xi}^T \nabla^2 g(\overline{x})\right] \eta(x,\overline{x}). \end{aligned}$$

Hence, by (4.29), it follows that

$$\begin{aligned} f(x) - f(\overline{x}) \geq\ & [\eta(\overline{x},\overline{x})]^T \left[\nabla f(\overline{x}) + \overline{\xi}^T \nabla g(\overline{x})\right] \\ & + \tfrac{1}{2}[\eta(\overline{x},\overline{x})]^T \left[\nabla^2 f(\overline{x}) + \overline{\xi}^T \nabla^2 g(\overline{x})\right] \eta(\overline{x},\overline{x}). \end{aligned}$$

Thus, $\eta(\overline{x},\overline{x}) = 0$ implies that the inequality

$$f(x) \geq f(\overline{x})$$

holds for all $x \in D$. This means that $\overline{x}$ is optimal in the original mathematical programming problem (P). □

Now, we show that if, $\overline{x}$ is an optimal solution in the original mathematical programming problem (P) and a suitable constraint qualification is fulfilled at $\overline{x}$, then there exists $\overline{\xi} \in R^m_+$ such that $(\overline{x}, \overline{\xi})$ is a second order η-saddle point in the second order η-approximated optimization problem ($\mathrm{P}^2_\eta(\overline{x})$).

Theorem 4.2. Let $\overline{x}$ be an optimal solution in the original mathematical programming problem (P) and a suitable constraint qualification [9] be fulfilled at $\overline{x}$. Then there exists $\overline{\xi} \in R^m_+$ such that $(\overline{x}, \overline{\xi})$ is a second order η-saddle point in its second order η-approximated optimization problem ($\mathrm{P}^2_\eta(\overline{x})$).

P r o o f. By assumption, $\overline{x}$ is an optimal solution in the original mathematical programming problem (P) and, moreover, a suitable constraint qualification [9] is fulfilled at $\overline{x}$. Then, there exists $\overline{\xi} \in R^m_+$ such that the Karush–Kuhn–Tucker optimality conditions (2.3) – (2.5) for the original mathematical programming problem (P) are satisfied at $\overline{x}$. Then, by the Karush–Kuhn–Tucker condition (2.3), it follows that

$$\begin{aligned} & f(\overline{x}) + \overline{\xi}^T g(\overline{x}) + [\eta(\overline{x},\overline{x})]^T \left[\nabla f(\overline{x}) + \overline{\xi}^T \nabla g(\overline{x})\right] \\ & \qquad = f(\overline{x}) + \overline{\xi}^T g(\overline{x}) + [\eta(x,\overline{x})]^T \left[\nabla f(\overline{x}) + \overline{\xi}^T \nabla g(\overline{x})\right]. \end{aligned} \tag{4.35}$$

holds for any $x \in D(\overline{x})$. By assumption, $\eta(\overline{x},\overline{x}) = 0$. Then, by (4.35) and (2.7)

$$\begin{aligned} & f(\overline{x}) + \overline{\xi}^T g(\overline{x}) + [\eta(\overline{x},\overline{x})]^T \left[\nabla f(\overline{x}) + \overline{\xi}^T \nabla g(\overline{x})\right] + \frac{1}{2}[\eta(\overline{x},\overline{x})]^T \left[\nabla^2 f(\overline{x}) + \right. \\ & \quad \left. + \overline{\xi}^T \nabla^2 g(\overline{x})\right] \eta(\overline{x},\overline{x}) \leq f(\overline{x}) + \overline{\xi}^T g(\overline{x}) + [\eta(x,\overline{x})]^T \left[\nabla f(\overline{x}) + \overline{\xi}^T \nabla g(\overline{x})\right] \\ & \quad + \frac{1}{2}[\eta(x,\overline{x})]^T \left[\nabla^2 f(\overline{x}) + \overline{\xi}^T \nabla^2 g(\overline{x})\right] \eta(x,\overline{x}). \end{aligned}$$

Thus, by Definition 3.1, we get that the inequality

$$L_{\eta}^{2}\left(\overline{x},\overline{\xi}\right) \leq L_{\eta}^{2}\left(x,\overline{\xi}\right) \tag{4.36}$$

holds for any $x \in D\left(\overline{x}\right)$. Hence, we conclude that the inequality ii) in Definition 3.2 is fulfilled.

Now, we prove the inequality i) in Definition 3.2. Using the Karush–Kuhn–Tucker optimality conditions (2.3) − (2.5), it follows that

$$\overline{\xi}^{T} g\left(\overline{x}\right) + \left[\eta\left(\overline{x},\overline{x}\right)\right]^{T}\left(\nabla f\left(\overline{x}\right) + \overline{\xi}^{T}\nabla g(\overline{x})\right) = 0. \tag{4.37}$$

By assumption, $\eta\left(\overline{x},\overline{x}\right) = 0$. Hence, (4.37) gives

$$\overline{\xi}^{T} g\left(\overline{x}\right) + \left[\eta\left(\overline{x},\overline{x}\right)\right]^{T}\left(\nabla f\left(\overline{x}\right) + \overline{\xi}^{T}\nabla g(\overline{x})\right) + \frac{1}{2}\left[\eta\left(\overline{x},\overline{x}\right)\right]^{T}\overline{\xi}^{T}\nabla^{2} g\left(\overline{x}\right)\eta\left(\overline{x},\overline{x}\right) = 0. \tag{4.38}$$

Since $\overline{x}$ is optimal in the original mathematical programming problem (P), then it is feasible in (P). Therefore, for any $\xi \in R_{+}^{m}$,

$$\xi^{T} g\left(\overline{x}\right) \leq 0. \tag{4.39}$$

By assumption, $\eta\left(\overline{x},\overline{x}\right) = 0$. Hence, by (4.39),

$$\xi^{T}\left(g\left(\overline{x}\right) + \nabla g\left(\overline{x}\right)\eta\left(\overline{x},\overline{x}\right) + \frac{1}{2}\left[\eta\left(\overline{x},\overline{x}\right)\right]^{T}\nabla^{2} g\left(\overline{x}\right)\eta\left(\overline{x},\overline{x}\right)\right) \leq 0. \tag{4.40}$$

Using $\eta\left(\overline{x},\overline{x}\right) = 0$ together with (4.40), we get

$$\begin{aligned} &\xi^{T} g\left(\overline{x}\right) + \left[\eta\left(\overline{x},\overline{x}\right)\right]^{T}\left[\nabla f\left(\overline{x}\right) + \xi^{T}\nabla g\left(\overline{x}\right)\right] \\ &\qquad + \tfrac{1}{2}\left[\eta\left(\overline{x},\overline{x}\right)\right]^{T}\left[\nabla^{2} f\left(\overline{x}\right) + \xi^{T}\nabla^{2} g\left(\overline{x}\right)\right]\eta\left(\overline{x},\overline{x}\right) \leq 0. \end{aligned} \tag{4.41}$$

By (4.38) and (4.41), it follows that the inequality

$$\begin{aligned} f\left(\overline{x}\right) &+ \xi^{T} g\left(\overline{x}\right) + \left[\eta\left(\overline{x},\overline{x}\right)\right]^{T}\left[\nabla f\left(\overline{x}\right) + \xi^{T}\nabla g\left(\overline{x}\right)\right] + \frac{1}{2}\left[\eta\left(\overline{x},\overline{x}\right)\right]^{T}\left[\nabla^{2} f\left(\overline{x}\right)\right. \\ &\left. + \xi^{T}\nabla^{2} g\left(\overline{x}\right)\right]\eta\left(\overline{x},\overline{x}\right) \leq f\left(\overline{x}\right) + \overline{\xi}^{T} g\left(\overline{x}\right) + \left[\eta\left(\overline{x},\overline{x}\right)\right]^{T}\left[\nabla f\left(\overline{x}\right) + \overline{\xi}\nabla g\left(\overline{x}\right)\right] \\ &+ \frac{1}{2}\left[\eta\left(\overline{x},\overline{x}\right)\right]^{T}\left[\nabla^{2} f\left(\overline{x}\right) + \overline{\xi}^{T}\nabla^{2} g\left(\overline{x}\right)\right]\eta\left(\overline{x},\overline{x}\right), \end{aligned}$$

holds for all $\xi \in R_{+}^{m}$. By the definition of the second order η-Lagrange function the following inequality

$$L_{\eta}\left(\overline{x},\xi\right) \leq L_{\eta}\left(\overline{x},\overline{\xi}\right) \tag{4.42}$$

holds for all $\xi \in R_{+}^{m}$. Then, by (4.36) and (4.42), we get the conclusion of theorem. □

Remark 4.1. Note that to prove Theorem 4.2 we don't need to assume that the objective function f and the constraint function g are second order invex at $\overline{x}$ on D with respect to the same function η.

In view of Theorem 4.1, if we assume that both the objective and constraint functions involved in problem (P) are second order invex with respect to the same function η at $\overline{x}$ on the set of feasible solutions D, and, moreover, η satisfies the condition $\eta(\overline{x},\overline{x})=0$, then it can be found optimal solutions in the original mathematical programming problem (P) by the help of a second order η-saddle point of the η-Lagrange function in its associated second order η-approximated optimization problem $(\mathrm{P}^2_\eta(\overline{x}))$.

Now, we give an example of a mathematical programming problem (P) involving second order invex functions with respect to the same function η satisfying the condition $\eta(\overline{x},\overline{x})=0$. To characterize its solvability, we use the approach discussed in this paper.

Example 4.1. Consider the following nonlinear mathematical programming problem

$$\begin{aligned} f(x) = x_1^6 \ln^2\left(x_1^2+x_1+1\right) + x_1^4 + \exp(x_1^2) + \ln(x_1^2+1) \\ + x_2^4 \ln^2\left(x_2^2+\tfrac{1}{2}\right) + e^{x_2}\arctan x_2 \to \min \\ g_1(x) = x_2^2 - x_1 \le 0, \\ g_2(x) = \tfrac{1}{2}x_2^2 - \arctan x_2 \le 0, \\ X = R^2. \end{aligned} \qquad (\mathrm{P})$$

Note that $D=\left\{(x_1,x_2)\in X : x_2^2-x_1\le 0 \ \wedge\ \frac{1}{2}x_2^2-\arctan x_2\le 0\right\}$ and $\overline{x}=(0,0)$ is optimal in the considered nonlinear optimization problem (P). Moreover, f and g are second order invex at $\overline{x}$ on D, for example, with respect to η defined by

$$\eta(x,\overline{x}) = \begin{bmatrix} \eta_1(x,\overline{x}) \\ \eta_2(x,\overline{x}) \end{bmatrix} = \begin{bmatrix} 2(x_1-\overline{x}_1) \\ x_2-\overline{x}_2. \end{bmatrix}$$

Note that the function η satisfies $\eta(\overline{x},\overline{x})=0$. Now, using the approach discussed in the paper, we construct problem $(\mathrm{P}^2_\eta(\overline{x}))$ by the second order η-approximation both the objective function f and the constraint function g near $\overline{x}$. Thus, we obtain the following quadratic optimization problem

$$\begin{aligned} 4x_1^2 + x_2^2 + x_2 \to \min \\ x_2^2 - 2x_1 \le 0, \\ \tfrac{1}{4}x_2^2 - x_2 \le 0 \end{aligned} \qquad (\mathrm{P}^2_\eta(\overline{x})).$$

Hence, $D(\overline{x})=\left\{(x_1,x_2)\in X : x_2^2-2x_1\le 0 \ \wedge\ 0\le x_2\le 4\right\}$. The second order η-approximated Lagrangian L^2_η in problem $(\mathrm{P}^2_\eta(\overline{x}))$ is

$$L^2_\eta(x,\xi) = 4x_1^2 + (1+\xi_1+\xi_2)\,x_2^2 - 2\xi_1 x_1 + (1-\xi_2)\,x_2.$$

It is not difficult to show, by Definition 3.2, that $\left(\overline{x},\overline{\xi}\right)=((0,0),(0,1))$ is a second order η-saddle point in the second order η-approximated Lagrangian L^2_η of problem $(\mathrm{P}^2_\eta(0))$. Then, by Theorem 4.1, we conclude that $\overline{x}=(0,0)$ is optimal in the considered mathematical programming problem (P).

Remark 4.2. Note that a quadratic convex optimization problem $(\mathrm{P}^2_\eta(\overline{x}))$ is obtained in Example 4.1 by the help of the second order η-approximation approach.

As it was shown in [4], if a function η is linear with respect to the first component, then the original mathematical programming problem (P) is transformed to the quadratic optimization problem $(\mathrm{P}^2_\eta(\overline{x}))$.

This property of the considered second order η-approximation approach is important from the practical point of view, because it can be exploited to find an optimal solution in a complicated nonlinear constrained optimization problem by solving saddle points conditions of a single less complicated optimization problem, that is, its associated quadratic (second order η-approximated) optimization problem $(\mathrm{P}^2_\eta(\overline{x}))$.

Now, we consider the example of an optimization problem involving second order invex functions with respect to the same function η, which is not linear with respect to the first component.

Example 4.2. Consider the following nonlinear mathematical programming problem

$$\begin{array}{c} f(x) = \sum_{i=1}^{k} (\arctan x)^k \to \min \\ g(x) = \left(1+x^4\right)\left[(\arctan x)^2 - \arctan x\right] \leq 0, \end{array} \qquad (\mathrm{P})$$

where k is a finite integer number. Note that $D = \left\{x \in R : 0 \leq x \leq \frac{\pi}{4}\right\}$ and $\overline{x} = 0$ is optimal in the considered nonlinear optimization problem (P). Moreover, f and g are second order invex at $\overline{x}$ on D, for example, with respect to $\eta(x, \overline{x}) = \frac{1}{2}(\arctan x - \arctan \overline{x})$. Note that the function η satisfies $\eta(\overline{x}, \overline{x}) = 0$. Now, using the approach discussed in the paper, we construct problem $(\mathrm{P}^2_\eta(\overline{x}))$ by the second order η-approximation both the objective function f and constraint function g near $\overline{x}$. Thus, we obtain the following optimization problem

$$\begin{array}{c} \frac{1}{4}(\arctan x)^2 + \frac{1}{2}\arctan x \to \min \\ \frac{1}{2}(\arctan x)^2 - \arctan x \leq 0. \end{array} \qquad (\mathrm{P}^2_\eta(0))$$

Hence, $D(\overline{x}) = \left\{x \in R : \frac{1}{2}(\arctan x)^2 - \arctan x \leq 0\right\}$. The second order η-approximated Lagrangian L^2_η in problem $(\mathrm{P}^2_\eta(0))$ is

$$\begin{aligned} L^2_\eta(x, \xi) &= (1-\xi)\,\eta(x, \overline{x}) + \frac{1}{2}(2+2\xi)\left[\eta(x, \overline{x})\right]^2 \\ &= \frac{1}{2}(1-\xi)\arctan x + \frac{1}{4}(1+\xi)(\arctan x)^2 . \end{aligned}$$

It is not difficult to show by Definition 3.2 that $\left(\overline{x}, \overline{\xi}\right) = (0, 1)$ is a second order η-saddle point of the second order η-approximated Lagrangian L^2_η in problem $(\mathrm{P}^2_\eta(0))$. Then, by Theorem 4.1, we conclude that $\overline{x} = 0$ is optimal in the considered mathematical programming problem (P). It is not difficult to see that also in this case when the function η with respect to which all functions constituting the original mathematical programming problem is not linear with respect to the first component, the second order η-saddle point criteria for the second order η-Lagrange function are easier to solve than the saddle point criteria for the original Lagrange function.

Remark 4.3. Note that there exists, in general, no a unique function η satisfying the condition $\eta(\overline{x}, \overline{x}) = 0$, and, moreover, with respect to which all functions constituting the original mathematical programming problem (P) are second order invex at the point $\overline{x}$ on D (Theorem 4.1). This fact is important from the practical point of view, since it is possible in the easier way to find such functions η, which satisfy all hypotheses of Theorem 4.1. In other words, there exists, for the considered mathematical programming problem (P) involving second order invex functions at $\overline{x}$ with respect to the same function η, in general, more than one associated second order η-approximated optimization problem $(\mathrm{P}^2_\eta(\overline{x}))$, which satisfies all hypotheses of Theorem 4.1.

This property has a place for the optimization problem considered in Example 4.1. Indeed, it is not difficult to prove that all functions involving constituting the original mathematical programming problem (P) are second order invex at $\overline{x}$ with respect to each functions η defined by

$$\eta(x, \overline{x}) = \begin{bmatrix} \eta_1(x, \overline{x}) \\ \eta_2(x, \overline{x}) \end{bmatrix} = \begin{bmatrix} \alpha(x_1 - \overline{x}_1) \\ x_2 - \overline{x}_2 \end{bmatrix},$$

where α is a real number satisfying $1 \leq \alpha \leq 2$. In this way, by using the second order η-approximation approach, we obtain, for the original optimization problem (P), more than one second order η-approximated optimization problem $(\mathrm{P}^2_\eta(\overline{x}))$. What is more, each of these second order η-approximated optimization problems $(\mathrm{P}^2_\eta(\overline{x}))$ is equivalent to the original mathematical programming problem (P) in the sense discussed in the paper (this follows from the fact that each of functions η defined above satisfies the condition $\eta(\overline{x}, \overline{x}) = 0$).

Of course, not all obtained associated problems $(\mathrm{P}^2_\eta(\overline{x}))$ have to be quadratic (such a case is illustrated by Example 4.2).

5. APPLICATIONS TO O.R. PROBLEMS

In this section, we discuss the potential applications of the second order η-approximation method to solve certain O.R. problems. Nonlinear optimization problems with second order differentiable functions arise in various O.R. applications, for example, in economics. Now, we present an example of an economic optimization problem whose solvability is characterized by using an approach presented in the paper.

A common problem in organizations is determining how much of a needed item should be kept on hand. For retailers, the problem may relate to how many units of each raw material should be kept available. This problem is identified with an area called inventory control, or inventory management. Concerning the question of how much "inventory" to keep on hand, there may be costs associated with having too little or too much inventory on hand.

Now, we consider more precisely an example of such an economic problem. The factory produces some product. The total cost (stated in thousands dollars) of producing x thousand of units of this product is described for this factory by the following function

$$f(x) = 64e^{6.25 - 0.125x} + \ln^4\left(0.0005\left(x^2 - 30x + 1000\right)\right) + x^2 - 91x + 2500.$$

But the plan of production makes that a total amount of producing x should satisfy the condition $g(x) = \ln\left(0,001\left(x^2 - 60x + 1500\right)\right) \leq 0$ (this condition follows, for example, from the possibilities of used machines to a production of this product). We want to determine the number x, which minimizes the production cost under the constraint $g(x) \leq 0$?

Thus, we obtain the following nonlinear optimization problem:

$$f(x) = 64e^{6.25-0.125x} + \ln^4\left(0.0005\left(x^2 - 30x + 1000\right)\right) + x^2 - 92x + 2500 \to \min$$
$$g(x) = \ln\left(0.001\left(x^2 - 60x + 1500\right)\right) \leq 0$$
$$x \in X = \{x \in R : x > 0\}.$$

Note that the set of all feasible solutions $D = \{x \in R : 10 \leq x \leq 50\}$ and the feasible solution $\overline{x} = 50$ satisfies the Karush–Kuhn–Tucker necessary optimality conditions. Further, it can be proved that all functions constituting the considered problem are second order invex at $\overline{x}$ on D with respect to, for example, $\eta(x, \overline{x}) = \frac{5}{4}(x - \overline{x})$ (of course, that is not a unique function η with respect to which the functions involved in the considered problem are second order invex at $\overline{x}$ on D). Now, we construct, for the considered mathematical programming problem, its second order η-approximated optimization problem

$$\begin{gathered} \tfrac{25}{32}(x-50)^2 + 464 \to \min \\ 0.0004(\tfrac{5}{4}(x-50))^2 + 0.04(\tfrac{5}{4}(x-50)) \leq 0 \\ x \in X = \{x \in R : x > 0\}, \end{gathered}$$

or, equivalently,

$$\begin{gathered} 0.78125(x-50)^2 + 464 \to \min \\ 0.025(x-50)(x+50) \leq 0 \\ x \in X = \{x \in R : x > 0\}. \end{gathered}$$

Thus, we obtain a quadratic convex optimization problem which is simpler to solve than the original one. Note that $D(\overline{x}) = \{x \in R : 0 < x \leq 50\}$. Further, by Definition, 3.1, the second order η-Lagrange function is

$$L_\eta^2(x, \xi) = 464 + 0.05(x-50)\xi + 0.390625(1 + 0.0004\xi)(x-50)^2 \tag{5.43}$$

Whereas the Lagrange function, by Definition 2.3, for the original optimization problem is more complex form

$$\begin{aligned} L(x, \xi) = 64e^{6.25-0.125x} &+ \ln^4\left(0.0005\left(x^2 - 30x + 1000\right)\right) + x^2 - 91x + 2500 \\ &+ \xi \ln\left(0.001\left(x^2 - 60x + 1500\right)\right), \end{aligned}$$

and, therefore, the standard saddle points criteria are more complicated to analyze. However, it is not difficult to show, by Definition 3.2 and (5.43), that $\left(\overline{x}, \overline{\xi}\right) = (50, 0)$ is a second order η-saddle point for the second order η-Lagrange function L_η^2 in the constructed second order η-approximated optimization problem $(\mathrm{P}_\eta^2(\overline{x}))$. Since all hypotheses of Theorem 4.1 are fulfilled, then $\overline{x} = 50$ is optimal in the original optimization problem. This means that the factory should produce 50 thousands of this product units to minimize the total cost of producing. Further, the optimal value in both optimization problems is the same and it is equal to 464 thousands dollars.

Note that the second order η-saddle point conditions used to solve the second order η-approximated optimization problem are more convenient and less computational complexity than the saddle point conditions for the original O.R. problem.

6. CONCLUSION

In the paper, a new approach for characterizing of solvability of nonlinear optimization problems with twice differentiable functions has been presented. The so-called second order η-approximation method was used for obtaining new saddle point results for a nonlinear constrained mathematical programming problem involving second order invex functions with respect to the same function η. In this method, the second order η-approximated optimization problem is constructed. Further, for such an optimization problem, the so-called second order η-Lagrange function and second η-saddle points are defined. Thus, the second order η-saddle point optimality criteria for the original mathematical programming problem have been established in the paper.

The results established in this paper are important because it can be exploited to find solutions to complicated twice differentiable nonlinear constrained optimization problems by solving modified second order modified saddle points conditions of a single less complicated optimization problem. In other words, to solve complicated nonlinear constrained optimization problem we are obliged to find a second order η-saddle point of the second order η-Lagrange in its associated second order η-approximated optimization problem.

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REFERENCES

[1] T. Antczak: An η-approximation approach to nonlinear mathematical programming involving invex functions. Numer. Funct. Anal. Optim. *25* (2004), 5–6, 423–438.

[2] T. Antczak: Saddle points criteria in an η-approximation approach for nonlinear mathematical programming involving invex functions. J. Optim. Theory Appl. *132* (2007), 1, 71–87.

[3] T. Antczak: A modified objective function method in mathematical programming with second order invexity. Numer. Funct. Anal. Optim. *28* (2007), 1–2, 1–13.

[4] T. Antczak: A second order η-approximation method for constrained optimization problems involving second order invex functions. Appl. Math. *54* (2009), 433–445.

[5] M. S. Bazaraa, H. D. Sherali, and C. M. Shetty: Nonlinear Programming: Theory and Algorithms. John Wiley and Sons, New York 1991.

[6] C. R. Bector and B. K. Bector: (Generalized)-bonvex functions and second order duality for a nonlinear programming problem. Congr. Numer. *52* (1985), 37–52.

[7] C. R. Bector and B. K. Bector: On various duality theorems for second order duality in nonlinear programming. Cahiers Centre Études Rech. Opér. *28* (1986), 283–292.

[8] C. R. Bector and S. Chandra: Generalized Bonvex Functions and Second Order Duality in Mathematical Programming. Research Report No. 85-2, Department of Actuarial and Management Sciences, University of Manitoba, Winnepeg, Manitoba 1985.

[9] A. Ben-Tal: Second-order and related extremality conditions in nonlinear programming. J. Optim. Theory Appl. *31* (1980), 2, 143–165.

[10] B. D. Craven: Invex functions and constrained local minima. Bull. Austral. Math. Soc. *24* (1981), 357–366.

[11] M. A. Hanson: On sufficiency of the Kuhn–Tucker conditions. J. Math. Anal. Appl. *80* (1981) 545–550.

[12] O. L. Mangasarian: Nonlinear Programming. McGraw-Hill, New York 1969.

[13] R. T. Rockafellar: Convex Analysis. Princeton University Press, 1970.

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