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On Relations Between Right and Left Eigenvectors of Nonselfadjoint Matrix Pencils

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The authors consider (from the reactor physicist viewpoint) some useful relations between right and left eigenvectors of nonselfadjoint generalized matrix eigenvalue problems via their equivalence to properly constructed ordinary eigenvalue problems, and their use for the determination of the perturbed dominant eigenvalue.

Авторы занимаются, с точки зрения физики реакторов, некоторыми часто используемыми соотношениями между правыми и левыми собственными векторами в несамосопряженных обобщенных матричных задачах на собственные значения с помощью их эквивалентности с надлежащим образом построеными обыкновенными задачами на собственные значения и их использованием для вычисления возмущенного доминантного собственного значения.

Autoři studují, s hlediska reaktorové fyziky, některé užitečné vztahy mezi pravými a levými vlastními vektory zobecněného nesamoadjungovaného maticového problému vlastních hodnot pomocí vhodně sestrojeného ekvivalentního obyčejného problému vlastních hodnot, a jejich užití při výpočtu porušeného dominantního vlastního čísla.

Introduction

In this note, there are summarized and discussed in detail the problems concerning the equivalence question of the generalized nonselfadjoint eigenvalue problem $Px = \lambda Qx$ for the right eigenvector and also of the corresponding eigenvalue problem $y^* P = \lambda y^* Q$ for the left eigenvector in the E_N space. There is accepted a reactor physicist's notation and point of view, i.e., the matrix Q is assumed to be nonsingular and the matrix $M = Q^{-1}P \in \mathscr{A}(E_N)$ of Perron – Frobenius type i.e., with λ as positive dominant eigenvalue and x, z^* resp. as the corresponding positive neutron flux vector or neutron importance vector resp. [1]).

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There are also discussed the relations between these eigenvectors and their use for the determination of the perturbed dominant eigenvalue, needed in some reactor criticality calculations [2].

I. Equivalence problems for matrix pencils

Let E_N be a real N-dimensional Euclidean space with $\langle x, y \rangle$, $x, y \in E_N$ as scalar product. Let $\mathscr{A}(E_N)$ denote the algebra of linear endomorphisms on E_N with the involution $A \leftrightarrow A^*$, $A, A^* \in \mathscr{A}(E_N)$, A^* transposed. Let A be the neutron absorbtion matrix, P the neutron production matrix (both in the sense of the heterogeneous method of reactor physics) and let I be the identity matrix in E_N. Further, let us suppose that the matrix

(1.1) $$Q = I + A \in \mathscr{A}(E_N)$$

is nonsingular, i.e.

(1.1a) $$Q^{-1} \in \mathscr{A}(E_N)$$

It is well known [2] that the matrices P and A, and therefore also $Q = I + A$ are nonselfadjoint, i.e.

(1.2) $$P \neq P^*, \quad Q \neq Q^*, \quad Q^{-1} \neq (Q^{-1})^* = (Q^*)^{-1}$$

so that we have in general

(1.3) $$M = Q^{-1}P \neq M^*, \quad N = PQ^{-1} \neq N^*$$

and the matrix N is similar to the matrix M

(1.3a) $$QMQ^{-1} = Q(Q^{-1}P)Q^{-1} = PQ^{-1} = N$$

Now, let us make following **Assumption I** for have a reactor criticality mathematical model in E_N (N denoting the number of fuel, safety and control rods in the reactor core):
(I) The matrix $M = Q^{-1}P$ is of Perron – Frobenius type, with λ as dominant simple positive eigenvalue.
Clearly, λ is also a dominant simple positive eigenvalue of N, N being similar to M in virtue of (1.3a).

We shall consider both the following eigenvalue problems for the nonselfadjoint matrix pencil $P - \lambda Q \in \mathscr{A}(E_N)$ and its adjoint pencil $P^* - \lambda Q^* \in \mathscr{A}(E_N)$:

(1.4) $$Px = \lambda Qx$$

(1.5) $$P^*y = \lambda Q^*y$$

Clearly, in virtue of the assumption (1.1a), the generalized eigenvalue problem (1.4) is equivalent to the following non-selfadjoint matrix eigenvalue problem for the matrix $M = Q^{-1}P$

(1.6) $$Mx = \lambda x$$

and the corresponding adjoint matrix eigenvalue problem (1.5) is equivalent to the following nonselfadjoint matrix eigenvalue problem for the matrix

$$N^* = [PQ^{-1}]^* = (Q^{-1})^* P^* = (Q^*)^{-1} P^*$$

(17) $$N^* y = \lambda y$$

Because the matrices $M, N \in \mathscr{A}(E_N)$ are in general nonselfadjoint, i.e., $M \neq M^* = P^*(Q^{-1})^* = P^*(Q^*)^{-1}, N^{**} = N = PQ^{-1} \neq N^*$, we have, for using the perturbation theory, to solve both the adjoint eigenvalue problems for (1.6)

(1.6a) $$M^* z = \lambda z$$

and for (1.7)

(1.7a) $$Nw = \lambda w$$

Using the well known relations [3]

(1.8) $$(Q^{-1})^* = (Q^*)^{-1}, QQ^{-1} = Q^{-1}Q = I, Q^*(Q^*)^{-1} = (Q^*)^{-1}Q^* = I$$

we obtain from (1.6a) and (1.5) the following useful relation between the left eigenvectors z, y resp. of $M, N \in \mathscr{A}(E_N)$ resp.:

(1.9) $$\lambda Q^*[(Q^*)^{-1}z] = P^*[(Q^*)^{-1}z] \Rightarrow \alpha z = Q^* y$$

Similarly, from (1.7a) and (1.4) using (1.8), there follows the following useful relation between the right eigenvectors x, w resp. of $M, N \in \mathscr{A}(E_N)$ resp.:

(1.10) $$\lambda Q[Q^{-1}w] = P[Q^{-1}w] \Rightarrow \beta w = Qx$$

Remark 1.1: $\alpha, \beta \neq 0$ resp. in (1.9), (1.10) resp. are suitable real numbers (norming constants).

2. Biorthogonality relations, perturbed dominant eigenvalue

It is well known [3], that, if assuming both the matrices $M, N^* \in \mathscr{A}(E_N)$, with eigenvalues $\mu_i \in \sigma(M) \subset \mathbf{C}, \nu_i^* \in \sigma(N^*) \subset \mathbf{C}$ resp., $i = 1, 2, \ldots, N$ have the complete (i.e. forming a base in E_N) system of normalized eigenvectors $x_i = x(\mu_i) \neq 0, y_i = y(\nu_i^*) \neq 0, \|x_i\| = \|y_i\| = 1$, then the vectors z_i, w_i of the corresponding biorthogonal (with respect to the usual scalar product $\langle x, y \rangle$ in E_N) bases

(2.1) $$z_i = z(\mu_i^*) \in E_N, \quad \langle z_i, x_j \rangle = \delta_{ij} = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$$

(ν_i^*, μ_i^* resp. denotes the complex conjugate to

$$\mu_i \in \sigma(M) \subset \mathbf{C}, \quad \nu_i \in \sigma(N) \subset \mathbf{C} \text{ resp., } i, j = 1, 2, \ldots, N)$$

(2.2) $$w_i = w(\nu_i) \in E_N, \quad \langle y_i, w_j \rangle = \delta_{ij} = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$$

are just the normalized eigenvectors of the adjoint matrices $M^*, N \in \mathscr{A}(E_N)$ resp., i.e., they fulfil the relations (1.6a), (1.7a) resp.

Using this fact, we obtain by help of the usual perturbation theory [4] the following explicite expressions for the perturbations $\Delta_M\lambda, \Delta_N\lambda$ of the simple positive dominant eigenvalue λ of M, N, dues to the perturbation ΔM, $\Delta N \in \mathscr{A}(E_N)$ resp. of $M, N \in \mathscr{A}(E_N)$ resp.:

$$\Delta_M\lambda = \frac{\langle \Delta Mx, z\rangle}{\langle x, z\rangle} \left(= \langle \Delta Mx, z\rangle \quad \text{if} \quad \|x\| = \|z\| = 1\right) \tag{2.3}$$

$$\Delta_N\lambda = \frac{\langle \Delta Nw, y\rangle}{\langle w, y\rangle} \left(= \langle \Delta Nw, y\rangle \quad \text{if} \quad \|w\| = \|y\| = 1\right) \tag{2.4}$$

Theorem 2.1: Let $P, P_0 \in \mathscr{A}(E_N)$.

Let $Q = (I + A) \in \mathscr{A}(E_N)$ and $Q_0 = (I + A_0) \in \mathscr{A}(E_N)$ be nonsingular.

Let $\lambda > 0$ be a simple dominant eigenvalue of $M_0 = Q_0^{-1}P_0, N_0 = P_0Q_0^{-1} \in \mathscr{A}(E_N)$, with the corresponding right eigenvectors x, $M_0x = \lambda x, w$, $N_0w = \lambda w$ and left eigenvectors $z, M_0^*z = \lambda z, y, N_0^*y = \lambda y$ resp.

Let $M = Q^{-1}P, N = PQ^{-1}$.

Then for the first order perturbations $\Delta_M\lambda, \Delta_N\lambda$ resp. of λ, dues to the perturbations $\Delta M = M - M_0, \Delta N = N - N_0$ resp. of M, N resp. the following relation is valid:

$$\Delta_N\lambda = \Delta_M\lambda + \frac{\langle (M_0 - Q^{-1}N_0Q)\, x, z\rangle}{\langle x, z\rangle} \tag{2.5}$$

where $\Delta_M\lambda, \Delta_N\lambda$ resp. are given by (2.3), (2.4) resp.

Proof:

Clearly, we have, in virtue of the relations (1.9), (1.10)

$$\beta\, \langle w, y\rangle = \langle Qx, y\rangle = \langle x, Q^*y\rangle = \alpha\langle x, z\rangle \tag{2.6}$$

so that the denominators in both the expressions (2.3), (2.4) resp. for $\Delta_M\lambda, \Delta_N\lambda$ resp. are proportional. For the numerator in (2.4), we have obviously

$$\begin{aligned} \frac{\beta}{\alpha}\langle \Delta Nw, y\rangle &= \langle (N - N_0)\, w, (Q^{-1})^*z\rangle = \\ &= \langle Q^{-1}(PQ^{-1} - P_0Q_0^{-1})\, Qx, z\rangle \\ &= \langle (M - Q^{-1}N_0Q)\, x, z\rangle \\ &= \langle \Delta Mx, z\rangle + \langle M_0 - Q^{-1}N_0Q)\, x, z\rangle \end{aligned} \tag{2.7}$$

Q.E.D.

Corollary 2.1:

If $Q = Q_0$, i.e., $A = A_0$, (no change in neutron absorbtion during the reactor perturbation), then we have $\Delta_N\lambda = \Delta_M\lambda$.

Proof: For $Q = Q_0$ we have $Q^{-1}N_0Q = Q_0^{-1}(P_0Q_0^{-1})\, Q_0 = Q_0^{-1}P_0 = M_0$,

Q.E.D.

Remark 2.1: Physical interpretation of the relations between the eigenvector pairs x, z and w, y is the following: Because the matrix $M = Q^{-1}P$ is assumed to be of Frobenius type with $\lambda \in \sigma(M)$ as dominant simple positive eigenvalue, M^* is also of Frobenius type and both the corresponding left and right eigenvectors $x = x(\lambda)$ and $z = z(\lambda)$ of M are positive. Therefore, x has the physical meaning of neutron flux vector (giving the neutron fluxes on the rods in the reactor core) and (2.3) implies, that the positive left eigenvector z of M is the weighting vector for the neutron flux vector x, giving the importance of the individual components of x with respect to the perturbation $\Delta_M\lambda$ of λ. Therefore, x is called the neutron importance vector. Clearly, the physical interpretation of the equivalence between the equations (1.4) and (1.6) is the following: The neutron absorbtion A is so little, i.e. $\|A\| \ll 1$, that $P' = Q^{-1}P \approx (I-A)\,P$ can again be interpreted as production matrix, to which there corresponds in (1.6) a zero absorbtion matrix $A' = 0$.

The matrix N, being similar to M, need not be of Perron – Frobenius type, so that the right and left eigenvectors $w = w(\lambda)$ and $y = y(\lambda)$, corresponding to the dominant simple positive eigenvalue $\lambda \in \sigma(N)$ of N have in general no direct physical interpretation in reactor physics. But if N is also of the Perron – Frobenius type, the same physical interpretation of w, y can be applied.

If Q is a matrix with positive entries, then Q^* has the same property, so that both Q and Q^* in this case leave invariant the cone of positive vectors in E_N. But, Q being nonsingular, $(Q^*)^{-1} = (Q^{-1})^*$ needs not leave the cone of the positive vectors in E_N invariant. Therefore, if $M = Q^{-1}P$ is of Perron – Frobenius type, and both the matrices $Q, (Q^{-1})^*P(Q^{-1})^*$ leave invariant the cone of positive vectors in E_N, then the right and left eigenvectors (with $\alpha > 0, \beta > 0$)

$w(\lambda) = \beta^{-1}Qx(\lambda)$ and $y(\lambda) = \frac{\alpha}{\lambda}\,(Q^*)^{-1}P^*(Q^*)^{-1}z(\lambda)$ of N, corresponding to its dominant simple positive eigenvalue $\lambda \in \sigma(N)$, are both positive and therefore can again be interpreted as the neutron flux w and neutron importance y in a reactor with production matrix $P'' = PQ^{-1} \approx P(I-A)$ and absorbtion matrix $A'' = 0$.

Now, we shall generalize the biorthogonality property [3] to the nonselfadjoint matrix pencil $P - \lambda Q \in \mathscr{A}(E_N)$:

Theorem 2.2

Let $P - \lambda Q \in \mathscr{A}(E_N)$ be a given matrix pencil, with $Q \in \mathscr{A}(E_N)$ nonsingular, $Q^{-1} \in \mathscr{A}(E_N)$. Let $\langle x, y\rangle$ be the (complex type) scalar product of $x, y \in E_N \subset U_N$. Let the set $\{x_k\}, (P - \lambda_k Q)\,x_k = 0,\ x_k \neq 0,\ k = 1, 2, \ldots, N$ of right eigenvectors $x_k = x(\lambda_k)$ of the matrix pencil $P - \lambda Q$, corresponding to its eigenvalues λ_k, span E_N.

Let $\{z_i\}, i = 1, 2, \ldots, N$ be the basis of E_N biorthonormal to $\{x_k\}$ with respect to the scalar product $\langle z, x\rangle, z, x \in E_N \subset U_N$, [3], i.e.,

(2.8) $$\langle z_i, x_k\rangle = \delta_{ik} = \begin{cases} 0 & \text{if } i \neq k \\ 1 & \text{if } i = k \end{cases}$$

Then

(2.9) $$y_i = y(\lambda_i^*) = (Q^{-1})^* z_i$$

are left eigenvectors of the matrix pencil $A - \lambda I$, corresponding to its eigenvalues λ_i^*, i.e., we have for $\forall\ i = 1, 2, \ldots, N$

(2.9a) $$(P^* - \lambda_i^* Q^*)\, y_i = 0, \quad y_i \neq 0$$

Proof: We have clearly, using the definition (2.9) of y_i, and the relation $(Q^{-1})^* = (Q^*)^{-1}$

(2.9b) $$z_i = Q^* y_i$$

Let us compute, using the definition (2.8) of z_i and the relation (2.9b) between z_i and y_i, the scalar product

$$\begin{aligned}\langle P^* y_i, x_k\rangle &= \langle y_i, Px_k\rangle = \langle y_i, \lambda_k Q x_k\rangle = \lambda_k^* \langle Q^* y_i, x_k\rangle = \\ &= \lambda_k^* \langle z_i, x_k\rangle = \lambda_k^* \delta_{ik} = \lambda_i^* \delta_{ik} = \\ &= \lambda_i^* \langle Q^* y_i, x_k\rangle = \langle \lambda_i^* Q^* y_i, x_k\rangle\end{aligned}$$

Thus, we have, for $\forall\ x_k \in \{x_k\}$,

(2.10) $$\langle (P^* - \lambda_i^* Q^*)\, y_i, x_k\rangle = 0$$

anf therefore, in virtue of the assumption span $\{x_k\} = E_N$, we have

(2.10a) $$P^* y_i = \lambda_i^* Q^* y_i$$

where

(2.10b) $$y_i = (Q^*)^{-1} z_i = (Q^{-1})^* z_i \neq 0 \quad \text{for } \forall\ i = 1, 2, \ldots, N$$

because we have, by (2.8),

(.10c) $$\langle z_i, x_k\rangle = \delta_{ik} \Rightarrow z_i \neq 0 \quad \text{for } \forall\ i = 1, 2, \ldots, N \quad \text{Q.E.D.}$$

Corollary 2.2

Let the adjoint matrix pencil $P^* - \lambda^* Q^* \in \mathscr{A}(E_N)$ with nonsingular $Q^* \in \mathscr{A}(N_N, (Q^*)^{-1} = (Q^{-1})^* \in \mathscr{A}(E_N)$ have a complete system of right eigenvectors $y_k = y(\lambda_k^*) \neq 0$, $(P^* - \lambda_k^* Q^*)\, y_k = 0$, so that we have

(2.11) $$\text{span } \{y_k\}_{k=1}^N = E_N$$

Let $\{w_i\}$, $i = 1, 2. \ldots,$ be the basis of E_N, biorthonormal to $\{y_k\}$ with respect to the scalar product $\langle w, y\rangle$, $w, y \in E_N$, i.e.

(2.12) $$\langle w_i, y_k\rangle = \delta_{ik} = \begin{cases} 0 & \text{if } i \neq k \\ 1 & \text{if } i = k \end{cases}$$

Then

(2.13) $$x_i = x(\lambda_i) = Q^{-1} w_i$$

are the left eigenvectors of the adjoint matrix pencil $P^* - \lambda^* Q^*$, corresponding to its eigenvalues λ_i^*, i.e., we have for $\forall\ i = 1, 2, \ldots, N$

(2.13a) $$(P - \lambda_i Q)\, x_i = 0\,, \quad x_i \neq 0$$

Proof. Because the adjoint matrix pencil $P^* - \lambda^* Q^*$ fulfills in this case all assumptions of Theorem 2.2., we obtain both the assertions (2.13), (2.13a) by applying this theorem and the involutory properties of the involution operation * in $\mathscr{A}(E_N)$. Q.E.D.

There is another possible way of reducing the generalized eigenvalue problems (1.4), (1.5) resp. to equivalent ordinary eigenvalue problems: via inverse eigenvalue problems instead inverting $Q = I + A$. We can consider λ, λ^* resp. in (1.4), (1.5) as a physical parameter (the so called criticality parameter) instead as a mathematical eigenvalue of the matrix pencils $P = \lambda Q$, $P^* - \lambda^* Q^*$ resp. and introduce formally a new eigenvalue $\mu = \mu(\lambda)$, $\nu = \nu(\varrho)$ resp., setting

(2.14) $$\lambda \neq 0, \quad \lambda^* = \varrho \neq 0$$

and

(2.14a) $$R(\lambda)\, x = \mu x\,, \quad \text{where} \quad R(\lambda) = \left(\frac{1}{\lambda} P - A\right) \text{ (neutron balance condition)}$$

(2.14b) $$\mu \equiv \mu\,(\lambda) = 1 \quad \text{(criticality condition)}$$

instead (1.4), as an inverse eigenvalue problem for the corresponding right eigenvector $x > 0$ of $R(\lambda)$,

(2.15) $$S(\varrho)\, y = \nu y, \quad \text{where} \quad S(\varrho) = \left(\frac{1}{\varrho} P^* - A^*\right) \text{ (neutron balance condition)}$$

(2.15a) $$\nu \equiv \nu(\varrho) = 1 \quad \text{(criticality condition)}$$

instead (1.5), as an inverse eigenvalue problem for the corresponding right eigenvector $y > 0$ of $S(\varrho)$.

Let us assume that, for $\lambda \in \langle \lambda_{\min}, \lambda_{\max} \rangle$, $\varrho \in \langle \varrho_{\min}, \varrho_{\max} \rangle$, the operators $R(\lambda)$ and $S(\varrho)$ are of Perron-Frobenius type, and both the dominant eigenvalue functions $\mu(\lambda), \nu\,(\varrho)$ are monotone, with $1 - \varepsilon_1 \leq \mu(\lambda), \nu(\varrho) \leq 1 + \varepsilon_2$, $\varepsilon_1, \varepsilon_2 > 0$. We denote $\lambda_{\text{crit.}}$, $\varrho_{\text{crit.}}$ resp. the critical values of the criticality parameters λ, ϱ, for which the criticality conditions (2.14b), (2.15a) resp. are just fulfilled for the dominant eigenvalues, i.e.,

(2.16) $$\mu_d = \mu(\lambda_{\text{crit.}}) = 1\,,$$

(2.17) $$\nu_d = \nu(\varrho_{\text{crit.}}) = 1$$

where μ_d, ν_d resp. denote the simple positive dominant eigenvalue of $R(\lambda_{\text{crit}})$, $S(\varrho_{\text{crit.}})$ resp. By $x_d = x(\lambda_{\text{crit.}}), y_d = y(\varrho_{\text{crit.}})$, $\|x_d\| = \|y_d\| = 1$ let us denote the corresponding normalized positive eigenvectors.

Clearly, we have, denoting ζ^* the complex conjugate of $\zeta \in \mathbf{C}$, and by R^* the adjoint operator to $R \in \mathscr{A}(E_N)$

(2.18) $$[R(\lambda)]^* = \frac{1}{\lambda^*} P^* - A^* = S(\lambda^*)$$

In general, we shall have obviously

(2.19) $$\lambda^*_{\text{crit.}} \equiv [\lambda^*_{\text{crit.}}]^* \neq \varrho$$

so that, despite the validity of the relation (2.18), the positive normalized eigenvectors x, y shall not be, in general, elements of biorthonormal bases of right and left eigenvectors of $R(\lambda_{\text{crit.}})$ in E_N.

But if, in a special case, the relation

$$\lambda^*_{\text{crit.}} \equiv [\lambda_{\text{crit.}}]^* = \varrho_{\text{crit.}} \tag{2.20}$$

shall be valid, and if the matrix $R(\lambda_{\text{crit.}}) = \frac{1}{\lambda_{\text{crit.}}} P - A$ shall have a complete system of normalized right eigenvectors

$$\{x_k \neq 0\}_{k=1}^N, \quad R(\lambda)_{\text{crit.}}\, x_k = \mu_k x_k, \quad \text{span}\,\{x_k\} = E_N,$$

then the vectors y_i, $i = 1, 2, \ldots, N$, $\langle y_i, x_k \rangle = \delta_{ik}$ of the corresponding biorthonormal base in E_N shall be normalized left eigenvectors of $R(\lambda_{\text{crit.}})$, i.e., we shall have $[R(\lambda_{\text{crit.}})]^* \, y_i = \mu_i^* y_i$, $y_i \neq 0$, $i = 1, 2, \ldots, N$, $\mu_d^* = 1$.

3. An illustrating numerical example

For numerical illustration of the above mentioned results in the Euclidean space E_3, let us take [5]

$$P = \begin{pmatrix} 2{,}2 & 1 & 1{,}8 \\ 1{,}1 & 2 & 0{,}9 \\ 1{,}1 & 1 & 1{,}8 \end{pmatrix} \tag{3.1}$$

as the neutron production matrix, and

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \tag{3.1}$$

as the neutron absorbtion matrix, so that the matrix $Q = I + A$ is nonsingular ($\det(Q) = 3 \neq 0$) and has positive entries.

The matrices P and A were chosen so, that the matrix $M = Q^{-1}P$ is symmetric with nonnegative entries and has three simple real eigenvalues 0,9, 1 and 1,1, with $\lambda_d = 1{,}1$ as simple positive dominant eigenvalue, and $x_d = z_d = (1, 0, 0)$ as the corresponding nonnegative right and left eigenvectors, so that M is of Perron-Frobenius type.

The matrix $N = PQ^{-1}$ is a nonsymmetric one. Because N is similar to M, its eigenvalues shall be the same as for M, i.e., also real and simple, again with $\lambda_d = 1{,}1$ as simple positive dominant eigenvalue, to which there correspond in this case the right eigenvector $w_d = \left(1, \frac{1}{2}, \frac{1}{2}\right)$ and the left eigencvetor $y_d = \left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right)$, so that N is not of Perron-Frobenius type.

We have clearly

(3.3)

$$P^* = \begin{pmatrix} 2{,}2 & 1{,}1 & 1{,}1 \\ 1 & 2 & 1 \\ 1{,}8 & 0{,}9 & 1{,}8 \end{pmatrix}, Q = \begin{pmatrix} 2 & 1 & 2 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}, Q^* = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 2 & 1 & 2 \end{pmatrix}, Q^{-1} = \begin{pmatrix} -1 & 0 & -1 \\ -\frac{1}{3} & \frac{2}{3} & 0 \\ -\frac{1}{3} & -\frac{1}{3} & 1 \end{pmatrix}$$

so that

$$M = Q^{-1}P = \begin{pmatrix} 1{,}1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0{,}9 \end{pmatrix} \tag{3.4}$$

$$N = PQ^{-1} = \frac{1}{30}\begin{pmatrix} 38 & 2 & -12 \\ 4 & 31 & -6 \\ 5 & 2 & 21 \end{pmatrix} \tag{3.5}$$

and

$$Qx_d = (2 \;\; 1 \;\; 1), \quad Px_d = (2{,}2 \;\; 1{,}1 \;\; 1{,}1) = 1{,}1\, Qx_d \tag{3.6}$$

$$Q^* y_d = \left(\frac{1}{\sqrt{2}}, 0, 0\right), \quad P^* y_d = \left(\frac{1{,}1}{\sqrt{2}}, 0, 0\right) = 1{,}1\, Q^* y_d \tag{3.7}$$

so that x_d, y_d resp. are the right (left) eigenvector resp. of the matrix pencil $P - \lambda Q$, corresponding to its dominant eigenvalue $\lambda_d = 1{,}1$. Verifying the relations (1.9) and (1.10), we compute

$$Q^* y_d = \frac{1}{\sqrt{2}}(1, 0, 0) = \frac{1}{\sqrt{2}} z_d \tag{3.8}$$

and

$$Qx_d = (2 \quad 1 \quad 1) = 2\left(1, \frac{1}{2}, \frac{1}{2}\right) = 2w_d \tag{3.9}$$

We see that, using the relations (1.9) and (1.10), we need not compute z_d and w_d from the homogeneous systems $(M^* - \lambda_d)\, z_d = 0$, $(N - \lambda_d)\, w_d = 0$, the norming constants for z_d and w_d being arbitrary.

References

[1] VARGA R. S.: Matrix iterative analysis, Prentice Hall Inc., Englewood Cliffs, N. Jersey (1962).

[2] ROČEK J., MIASNIKOV A.: Report ÚJV Řež 2196-R, P (in Czech). See also: Heterogeneous method and reactor criticality calculations (in Czech). Acta Polytechnica 7, IV, 151—158 (1973).

[3] PEASE M. G. III: Methods of Matrix Algebra, Academic Press, N. York, London (1965).

[4] GELFAND I. M.: Linear algebra (in Czech). Ed. ČSAV, Prague (1953).

[5] ROČEK J., MIASNIKOV A., ZEZULA R.: Numerical comparison of two methods for solving generalized nonselfadjoint matrix eigenvalue problems. (To appear).