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Bahadur-Efficiency of Linear Rank Tests — a Survey

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A survey of the recent results on the Bahadur efficiency of two-sample linear rank tests is given. The classical results of Bahadur, Woodworth and Hájek are completed by the author's results on the behavior of the Bahadur efficiency for infinitely distant alternatives and these on the local Bahadur efficiency. A relation of the local Bahadur efficiency to the Pitman efficiency is mentioned.

Článek podává přehled nejnovějších výsledků o Bahadurově vydatnosti dvouvýběrových pořadových testů. Klasické Bahadurovy, Woodworthovy a Hájkovy výsledky jsou doplněny autorovými vlastními výsledky o chování Bahadurovy vydatnosti při nekonečně se vzdalujících alternativách a o lokálním chování Bahadurovy vydatnosti. Uvažuje se vztah lokální Bahadurovy vydatnosti k Pitmanově vydatnosti.

Дается обзор недавных результатов об эффективности двухвыборочных линейных ранговых критериев в смысле Бахадура. Классические результаты Бахадура, Гайека и Вудворса дополнены результатами автора о поведении эффективности Бахадура при бесконечно удаляющихся альтернативах и о локальном поведении эффективности Бахадура. Изучится отношение эффективности Бахадура к эффективности Питмана.

In 1960/67 the statistician Bahadur introduced an approximate and an exact measure for the asymptotic comparison of two tests, which is rather suitable for comparing nonparametric tests, especially linear rank tests. In the following I will give a survey of the main results of the theory of Bahadur-efficiency for the class of twosample linear rank tests. Similar results also hold for the one-sample symmetry and independence problems. Let me first give the general definition of the approximate and exact Bahadur-efficiencies.

1. The concept of Bahadur-efficiency

Let X be a random variable with distribution

$$\mathscr{L}(X) \in \{P_\vartheta, \vartheta \in \Theta\} .$$

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For the testing problem $\vartheta \in H$ against $\vartheta \in K$ $(\Theta = H + K)$ we regard one-sided asymptotic tests $\varphi = \{\varphi_n\}_{n \in \mathcal{N}}$,

$$\varphi_n := 1_{\{T_n > c_n\}} + 1_{\{T_n = c_n\}} \cdot \gamma_n ,$$

based on real-valued statistics $T_n = T_n(X)$. Suppose there exists a function $c : K \to \to (0, \infty)$ such that

$$-2 \,.\, n^{-1} \,.\, \ln L_n(X) \to c(\vartheta) \;\; P_\vartheta - \text{a.e.}\,, \quad \vartheta \in K$$

where $L_n(x)$ is the *level attained by* $T_n(x)$,

$$L_n(x) := \sup_{\vartheta \in H} P_\vartheta(T_n \geqq T_n(x)) \,.$$

Obviously $c(\vartheta)$ measures the speed of exponential convergence (to zero) of the attained level.

At $\vartheta \in K$ we clearly want to reject the hypothesis, this means the attained level should be small resp. the speed of convergence to zero should be high. Consequently among two tests the test with the larger $c(\vartheta)$ will be regarded as being (asymptotically) preferable.

Definition 1.1 (Bahadur (1967))
The function c/K is called *exact slope* of φ and for tests φ_i, $i = 1, 2$ with slopes c_i, $i = 1, 2$

$$e_{1,2}(\vartheta) := \frac{c_1(\vartheta)}{c_2(\vartheta)} \quad (\vartheta \in K)$$

exact Bahadur-efficiency (BE) of φ_1 relative to φ_2 at ϑ. □

To introduce the approximate concept, suppose that $F_n(t) := \inf_{\vartheta \in H} P_\vartheta(T_n \leqq t)$ is a distribution function (d.f.) with:

$$F_n(x) \to F(x) \quad \forall x \tag{1.1}$$

where F is a continuous d.f.. This suggests to approximate $L_n(x)$ by:

$$L_n^a(x) := 1 - F(T_n(x)) \,,$$

and we get:

Definition 1.2 (Bahadur 1960/67)
If there exists a function $c^a : K \to (0, \infty)$ such that

$$-2 \,.\, n^{-1} \,.\, \ln L_n^a(X) \to c^a(\vartheta) \;\; P_\vartheta - \text{a.e.}\,, \quad \vartheta \in K \,,$$

then c^a is called *approximate slope* of the test φ based on $\{T_n\}$. For two tests φ_i, $i = 1, 2$ with approximate slopes c_i^a the *approximate BE of* φ_1 *relative to* φ_2 is defined by:

$$e^a_{1,2}(\vartheta) := \frac{c^a_1(\vartheta)}{c^a_2(\vartheta)}, \quad \vartheta \in K. \quad \square$$

For computing the slope a method is given by:

Theorem 1.3 (Bahadur 1960/67)
(a) Suppose there exist
(1) a function $\tau : K \to (0, \infty)$ such that

$$(1.2) \qquad \frac{T_n}{\sqrt{n}} \to \tau(\vartheta) \; P_\vartheta - \text{a.e.}, \quad \vartheta \in K$$

(2) a function $I : \tau(K) \to (0, \infty)$ satisfying:

$$(1.3) \qquad n^{-1} . \ln \left[\sup_{\vartheta \in H} P_\vartheta(T_n \geqq \sqrt{(n)} . t_n)\right] \to -I(t),$$

$$\forall \{t_n\} : t_n \to t \in \tau(K).$$

Then the exact slope of φ is equal to $c = 2 . (I \circ \tau)$.
(b) Suppose we have (1.1), (1.2) and (instead of (1.3)):
(2′) there exists $d \in (0, \infty)$ such that

$$(1.4) \qquad -2 . \ln(1 - F(t)) = d . t^2 . (1 + 0(1)) \quad \text{for} \quad t \to \infty.$$

Then the approximate slope is equal to $c^a = d . \tau^2$ and $\{T_n\}$ is called *standard sequence*. $\square$

In order to verify (1.2) a *strong law of large numbers* and to prove (1.3) the *theory of large deviations* is needed, whereas (1.4) is often trivially satisfied. As the theory of large deviations was not yet explored profound enough, in the first papers on BE attention was focussed on the easier concept of the approximate efficiency. Examples showed that the approximate and exact efficiency usually differ much at alternatives far away from the hypothesis, so that the results of the approximate BE have to be regarded with caution. Nevertheless both efficiency measures often coinside at alternatives near the hypothesis, i.e. under so called local alternatives.

Definition 1.4 (Bahadur (1960), Kremer (1979))
Let Θ be a metric space and consider asymptotic tests $\varphi^{(i)}$, $i = 1, 2$, with exact and approximate BE $e_{1,2}$ and $e^a_{1,2}$. A sequence $\{\vartheta_j\}$ with $\vartheta_j \in K$, $\forall j$ such that there exists a ϑ_0 in the boundary of H with $\vartheta_j \to \vartheta_0$, is called *local alternative*. The value

$$E^{[a]}_{1,2}(\{\vartheta_j\}) := \liminf_{j \to \infty} e^{[a]}_{1,2}(\vartheta_j)$$

is denoted as *exact [approximate] local BE* of $\varphi^{(1)}$ relative to $\varphi^{(2)}$ under $\{\vartheta_j\}$. $\square$

Now, having introduced a measure for comparing two statistical procedures, the problem of determining an optimal procedure (according to the defined measure) arises.

Definition 1.5 (Bahadur 1960/67), Kremer (1979))
A test φ_1 is called *B-optimal at* $\vartheta \in K$, iff:

$$e_{1,2}(\vartheta) \geqq 1\,, \quad \forall \text{ tests } \varphi_2$$

For a metric parameter space Θ and local alternative $\{\vartheta_j\}$ the test φ_1 is called *local B-optimal under* $\{\vartheta_j\}$, iff:

$$E_{1,2}(\{\vartheta_j\}) \geqq 1\,, \quad \forall \text{ tests } \varphi_2\,. \quad \square$$

Later on we will see that the concept of approximate efficiency yields a useful tool for deriving results on exact local BE of linear rank tests, especially for proving local B-optimality.

In the sequel I restrict on:

2. The two sample-problem

Let $X_{11}, \ldots, X_{1n_1}, X_{21}, \ldots, X_{2n_2}$ be two samples of $n = n_1 + n_2$ independent random variables, X_{ij} having continuous d.f. $F_i/\mathbf{R}$. We consider the *two-sample testing problem of randomness versus positive stochastic deviation*, i.e. of deciding between the hypothesis and alternative:

(2.1) $$H = \{(F_1, F_2) : F_1 = F_2\} \quad \text{against} \quad K = \{(F_1, F_2) : F_1 \lneqq F_2\}$$

and compare asymptotic upper tests φ_b based on *linear rank statistics*

(2.2) $$T_n = n^{-1/2} \sum_{i=1}^{n_1} B_n(R_{ni})\,,$$

$(R_{n1}, \ldots, R_{nn})$ denoting the rank vector of the combined sample $(X_{11}, \ldots, X_{n1}, X_{21}, \ldots, X_{2n_2})$ and the scores $B_n(i) \in \mathbf{R}$, $i = 1, \ldots, n$ satisfying:

(2.3) $$b_n \xrightarrow{L_1} b\,.$$

where b_n is defined by:

$$b_n(u) := B_n(1 + [n \,.\, u])\,,$$

and the scores-generating function $b \in L_1$ is assumed (for simplicity) to be non-decreasing. Here (and in the sequel) "$\underset{\rightarrow}{L_K}$" denotes convergence in the function space $L_k = L_k(0, 1)$ ((0, 1) furnished with Lebesgue-measure λ). For deriving asymptotic results, we assume for $n_1 = n_1(n)$:

(2.4) $$\lim_{n \to \infty} n_1/n = s \;(\in (0, 1) \text{ fixed})\,.$$

Furthermore, for investigating local BE, we have to metrize our parameter space Θ, which is the product $\mathscr{F} \times \mathscr{F}$, $\mathscr{F}$ denoting the set of continuous distribution functions on **R**. In the sequel let Θ be furnished with a *metric d* generating the topology of convergence in distribution in both components.

3. The (exact) Bahadur-efficiency at fixed alternative

As mentioned above, the main difficulty in computing the exact BE lies in the verification of condition (1.3), i.e. in proving a suitable *large deviation theorem.* For special linear two-sample rank tests large deviation results were already derived by Hoadley in 1965 (Wilcoxon-test) and by Stone in 1967/68 (Wilcoxon- and normal-scores-test). Finally in 1970 Woodworth published a large deviation theorem for a rather general class of rank tests, including linear rank tests of the two-sample and independence problem. Under the conditions of section 2 Woodworth's theorem reduces to:

Theorem 3.1 (Woodworth (1970)
Consider an asymptotic test based on a linear rank statistic (2.2) with (nondecreasing) scores-generating function b for testing problem (2.1) (with (2.4)). Define:

$$\underline{t}(b, s) = s \cdot \int b \, d\lambda \,, \quad \bar{t}(b, s) := \int_{1-s}^{1} b \, d\lambda \,.$$

Then the large deviation statement (1.3) holds for $t < \bar{t}(b, s)$ with $I(t)$ defined according:

$$I_{b,s}(t) = 0 \,, \quad \text{for} \quad t \leqq \underline{t}(b, s) = \tag{3.1}$$

$$= r \cdot t + s \cdot \ln(z) - \int \ln((1 - s) + s \cdot \exp(r \cdot b)) \, d\lambda \,,$$

$$\text{for} \quad t \in (\underline{t}(b, s), \ \bar{t}(b, s))$$

where $r, z \geqq 0$ are the unique solutions of the integral equations:

$$\int s \cdot b \cdot \frac{z \cdot \exp(r \cdot b)}{(1 - s) + s \cdot z \cdot \exp(r \cdot b)} \, d\lambda = t \tag{3.2}$$

$$\int \frac{z \cdot \exp(r \cdot b)}{(1 - s) + s \cdot z \cdot \exp(r \cdot b)} \, d\lambda = 1 \,. \quad \square \tag{3.3}$$

The *proof* of the statement consists of two parts. First we assume that $b_n = b$ is a step function, which reduces (1.3) to a large deviation statement for a multinomial distributed statistic and as the consequence allows the application of a theorem of Hoeffding (1965). In the second part we utilize the assumption that the step function b_n approaches b and leads the general case back to part 1 of the proof. The resulting function $I(t)$ is defined according:

$$I(t) = \inf_{f \in F_t} \left[\int f \ln f \, d\lambda_2 \right]$$

where F_t is the set of all densities on $(0, 1)^2$ with uniform marginals and:

$$\int a \, . f \, d\lambda_2 \geqq t$$

(λ_2 denoting the Lebesgue measure on $(0, 1)^2$) with $a(u, v) := 1_{(0,s)}(u) \, . \, b(v)$. Solving the optimization problem by techniques of variational analysis, we get the formulas of Theorem 3.1. □

For computing the BE of our linear rank tests, a suitable *strong law of large numbers* only remains to develop.

Theorem 3.2 (Woodworth (1970), Hájek (1974))
Under the assumptions of section 2 one has:

(3.5) $$\lim_{n \to \infty} T_n/\sqrt{n} = \tau_{b,s}(F_1, F_2) \quad \text{a.e.}$$

with

$$\tau_{b,s}(F_1, F_2) := s \, . \int b(G) \, dF_1$$

$$G := s \, . \, F_1 + (1 - s) \, . \, F_2 \, . \quad \square$$

Let us give the idea of the *proof*. We approximate T_n by a statistic of type

$$S_n = n^{-1/2} \, . \sum_{i=1}^{n_1} \tilde{b} \left(\frac{R_{ni}}{n + 1} \right),$$

where $\tilde{b}$ is a uniformly continuous function on $(0, 1)$. For proving that:

$$S_n/\sqrt{n} \to s \, . \int \tilde{b}(G) \, dF_1 \quad \text{a.e.}$$

we use the representation:

$$S_n/\sqrt{n} = \frac{n_1}{n} \int \tilde{b} \left(\frac{n}{n + 1} \, . \, G_n \right) dF_{1n} \, ,$$

with $G_n = (n_1/n) F_{1n} + (n_2/n) F_{2n}$, F_{in} denoting the empirical d.f. of X_{ij}, $j = 1, \ldots$ $\ldots, n_i$, and apply the Glivenko-Cantelli-theorem. □

Remark 3.3

The Theorem 3.2 was originally proved by Hájek in 1974 under the additional assumptions:

(1) F_i is dominated by the Lebesgue-measure

(2) b_n, $n \in \mathbf{N}$ have uniformly bounded variation on closed subintervals of $(0, 1)$, which are superfluous.

Now we are in position to calculate the *BE of two linear rank tests* φ_{b_i}, $i = 1, 2$ with scores-generating functions b_i, $i = 1, 2$. The *slope of test* φ_b is according to theorems 1.3, 3.1, 3.2 given by:

$$c_b((F_1, F_2), s) = 2 \cdot I_{b,s}(\tau_{b,s}(F_1, F_2)), \tag{3.6}$$

at $(F_1, F_2) \in K$ with

$$\tau_{b,s}(F_1, F_2) < \bar{t}(b, s).$$

For example one can choose $b(u) = u$ (*Wilcoxon-test*), $b(u) = \Phi^{-1}(u)$ (*normal-scores-test*), $b(u) = \text{sign}(u - 1/2)$ (*median-test*). Let us have a look at the resulting efficiency curves for the special cases of

(a) normal-shift-alternatives

(b) logistic-shift-alternatives.

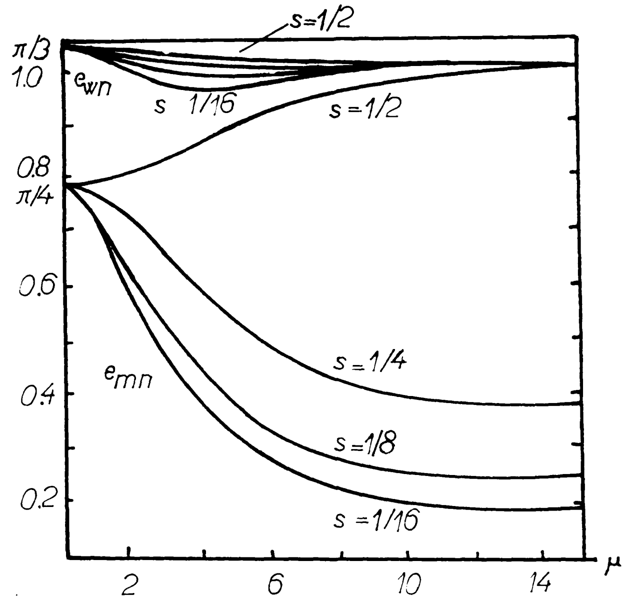

Figure 1. Normal shift alternatives (e_{mn}: BE of the median relative to the normal scores test; e_{wn}: BE of the Wilcoxon relative to the normal scores test.

For small translation parameter μ the BE obviously is indepedent of s, but strongly dependent on the special alternative (F_1, F_2). For moderate μ the BE is strongly dependent on s, the dependence on (F_1, F_2) has decreased. Finally for large μ the BE is independent of the special alternative, the BE of the Wilcoxon relative to the normal-scores test is equal to 1 independently of s, whereas the efficiency of the median-relative to the normal scores test is strongly dependent on s and equal to 1 only for $s = 1/2$.

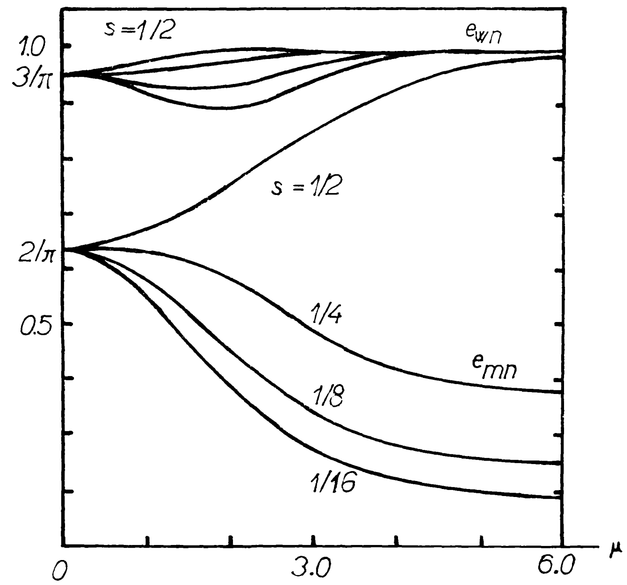

Figure 2. Logistic shift alternatives.

In the sequel I will present more general results on the BE of two-sample-rank tests and give general explanations for some of the above appearances. First of all, let me cite an interesting *result on B-optimal linear rank tests*, which is (under additional conditions) due to Hájek (1974).

Theorem 3.4 (Hájek (1974))
For $\vartheta = (F_1, F_2) \in K$, $G := s \, . \, F_1 + (1 - s) \, . \, F_2$ define densities:

$$f_i := (F_i \circ G^{-1}) / \mathrm{d}\lambda$$

and

$$b_\vartheta := \ln \left(\frac{f_1}{f_2} \right) .$$

Then a two-sample linear rank test φ_{b_ϑ} (i.e. with scores-generating function b_ϑ) is B-optimal at ϑ with:

$$\tau_{b_\vartheta, s}(\vartheta) = s \, . \int b \, . \, f_1 \, \mathrm{d}\lambda \in \left(\underline{t}(b_\vartheta, s), \; \bar{t}(b_\vartheta, s) \right) . \quad \square$$

The simple *proof* is based on an inequality of Bahadur and Raghavachari, stating for the slope $c(\vartheta, s)$ of an asymptotic two-sample test:

$$c(\vartheta, s) \leqq 2 \,.\, K(\vartheta, s)\,, \tag{3.7}$$

where K is the *Kullback-Leibler-information number*:

$$K(\vartheta, s) := s \,.\, \int f_1 \ln(f_1)\, d\lambda + (1 - s) \,.\, \int f_2 \ln(f_2)\, d\lambda\,. \tag{3.8}$$

As a consequence we have to show only $c_{b_\vartheta}(\vartheta, s) = 2 \,.\, K(\vartheta, s)$. System (3.2), (3.3) (with $b = b_\vartheta$) is solved by $r = z = 1$, substituting these solutions, $b = b_\vartheta$, $t = \tau_{b_\vartheta, s}(\vartheta)$ into (3.1), one directly gets the statement:

$$I_{b_\vartheta, s}(\tau_{b_\vartheta, s}(\vartheta)) = K(\vartheta, s)\,. \quad \square$$

The theorem implies as B-optimal test at the normal-shift-alternative $(F_1(y), F_2(y)) = (\Phi(y - \mu), \Phi(y))$ for example the linear rank test with scores-generating function:

$$b(u) = G^{-1}(u)\,, \quad G = s \,.\, F_1 + (1 - s) \,.\, F_2\,,$$

which obviously is (strongly) dependent on the translation-parameter μ. In general it is impossible to prove the optimality of the test φ_{b_ϑ} in a larger subclass of alternatives so that the theorem is of little value for deciding which linear rank test to choose in the practice. Nevertheless, the theorem can be used (see Behnen and Neuhaus (1981/82)) as the motivation of a practicable adaptive rank-test.

Now let us turn to the behavior of BE (of linear rank tests) at alternatives far away from the hypothesis. It is obvious that a reasonable asymptotic test should have at least (nearly) optimal BE efficiency at such alternatives. Does this hold true for the standard linear rank tests?

4. Bahadur efficiency at infinity

Throughout this section let $\{(F_{n1}, F_{n2})\}$ be a sequence with $(F_{n1}, F_{n2}) \in K$ and:

$$\lim_{n \to \infty} [\sup_{x \in \mathbf{R}} (F_{n2}(x) - F_{n1}(x))] = 1\,.$$

i.e. we investigate alternatives running away from the hypothesis. We get for our linear rank test φ_b:

Theorem 4.1 (Kremer (1980))
(a) If $(b(u) < b(1 - s),\ \forall u < 1 - s)$ or $(b(v) > b(1 - s),\ \forall v > 1 - s)$:

$$\lim_{n \to \infty} c_b((F_{n1}, F_{n2}), s) = 2 \,.\, h(s)\,.$$

(b) Otherwise:

$$\lim_{n\to\infty} c_b((F_{n1}, F_{n2}), s) = 2 \cdot \left[h(s) - \lambda(b = b(1-s)) \cdot h\left(\frac{s - \lambda(b > b(1-s))}{\lambda(b = b(1-s))}\right)\right]$$

where

$$h(v) := -(v \cdot \ln(v) + (1-v)\ln(1-v)), \quad v \in [0, 1]. \quad \square$$

We easily show:

$$\lim_{n\to\infty} \tau_{b,s}(F_{n1}, F_{n2}) = \bar{t}(b, s),$$

as a consequence we have to calculate only

$$\lim_{t\to\bar{t}(b,s)} I_{b,s}(t).$$

Since the evaluation of the limit is a little bit technical and does not give new insights, I do not like to go into details. □

Furthermore one can prove for the Kullback-Leibler-information number (see (3.8)):

$$\lim_{n\to\infty} K((F_{n1}, F_{n2}), s) = h(s).$$

Now the inequality (3.7) of Bahadur and Raghavachari implies, that *our linear rank test φ_b is asymptotically optimal under $\{(F_{n1}, F_{n2})\}$, iff b satisfies the conditions of Theorem 4.1 (a). This means that such a linear rank test is (nearly) B-optimal at alternatives far away from the hypothesis.* Examples are the Wilcoxon- and normal-scores-tests, explaining that their BE is nearly equal to one for large shift parameter. We derive for the asymptotic slope of the median-test:

$$c_\infty(s) := \lim_{n\to\infty} c_b((F_{n1}, F_{n2}), s) = 2 \cdot h(s) - h(2 \cdot \min(s, 1-s))$$

which is equal to the optimal value $2 \cdot h(s)$, iff $s = 1/2$. Furthermore:

$$\frac{c_\infty(s)}{2 \cdot h(s)} \to 0 \quad \text{for} \quad s \to 0 \quad \text{or} \quad s \to 1.$$

Consequently, the median test behaves comparably bad at alternatives far away from the hypothesis, if the asymptotic sample size ratio s differs (much) from 1/2. This explains the nonregular behavior of the efficiency curves of the median relative to the normal scores test in our two examples of the shift-alternatives (see figures 1,2) and clearly disqualifies the median test.

In the classical theory of rank tests linear rank tests are usually compared under local alternatives. Some statisticians (Hájek, Behnen) derived a general theory of asymptotic comparison for linear rank tests under local alternatives based on the concept of contiguity. I am going to show in the following section that similar results can be derived with the concept of BE.

5. Local Bahadur-efficiency

The concept of approximate BE is the basic tool for deriving results on the (exact) BE near the null hypothesis. I make instead of (2.3) the stronger assumption in the sequel:

$$b_n \xrightarrow{L_2} b\,, \quad b \in L_2\,.$$

Now we have:

Theorem 5.1 (Kremer (1979), (1982)
The approximate slope of the linear rank test φ_b is given by:

$$c_b^a((F_1, F_2), s) = \frac{(\tau_{b,s}(F_1, F_2) - \underline{t}(b, s))^2}{s \cdot (1 - s) \cdot \sigma^2(b)}$$

with:

$$\sigma^2(b) = \int \left(b - \int b \, \mathrm{d}\lambda\right)^2 \mathrm{d}\lambda\,. \quad \square$$

This result follows by suitable normalizing the linear rank statistic T_n and showing that the resulting sequence is a standard sequence. Condition (1.2) of Theorem 1.3 holds by Theorem 3.2. Conditions (1.1), (1.4) are satisfied for the normal d.f. $F = \Phi$ (consequently $d = 1$) according to theorems of Hájek, Behnen on asymptotic normality of linear rank tests under the null hypothesis. $\square$

The following figure 3 presents the approximate efficiencies of the median, Wilcoxon and normal scores two-sample tests for the normal shift alternatives.

The comparison with the figure 1 of the exact BE shows that the approximate efficiency yields incorrect results for large shift-parameter μ. The equality of the limit of the exact and approximate BE for $\mu \to 0$ is a special case of the basic

Theorem 5.2 (Kremer (1979), (1982))
Assume:

$$b \in L_3$$

and let $\{(F_{n1}, F_{n2})\}$ be a local alternative. Then:

$$\lim_{n\to\infty} \frac{c_b((F_{n1}, F_{n2}), s)}{c_b^a((F_{n1}, F_{n2}), s)} = 1\,,$$

i.e. the exact and approximate slopes are locally equivalent.

The *proof* consists in deriving an asymptotic formula for $I_{b,s}(\tau_{b,s}(F_{n1}, F_{n2}))$, which can easily be compared with $c_b^a((F_{n1}, F_{n2}), s)$. As a first step we can show:

$$\tau_{b,s}(F_{n1}, F_{n2}) \to \underline{t}(b, s) =: \underline{t}$$

and by using suitable Taylor expansions of (3.1)–(3.3):

$$I_{b,s}(t) = \frac{1}{2}\,\frac{(t - \underline{t})^2}{s\,.\,(1 - s)\,\sigma^2(b)} + o((t - \underline{t})^2) \quad \text{for} \quad t \to \underline{t}\,.$$

so we have:

$$c_b((F_{n1}, F_{n2}), s) = \frac{(\tau_{b,s}(F_{n1}, F_{n2}) - \underline{t})^2}{s\,.\,(1 - s)\,.\,\sigma^2(b)} + o((\tau_{b,s}(F_{n1}, F_{n2}) - \underline{t})^2)\,.$$

for $n \to \infty$.

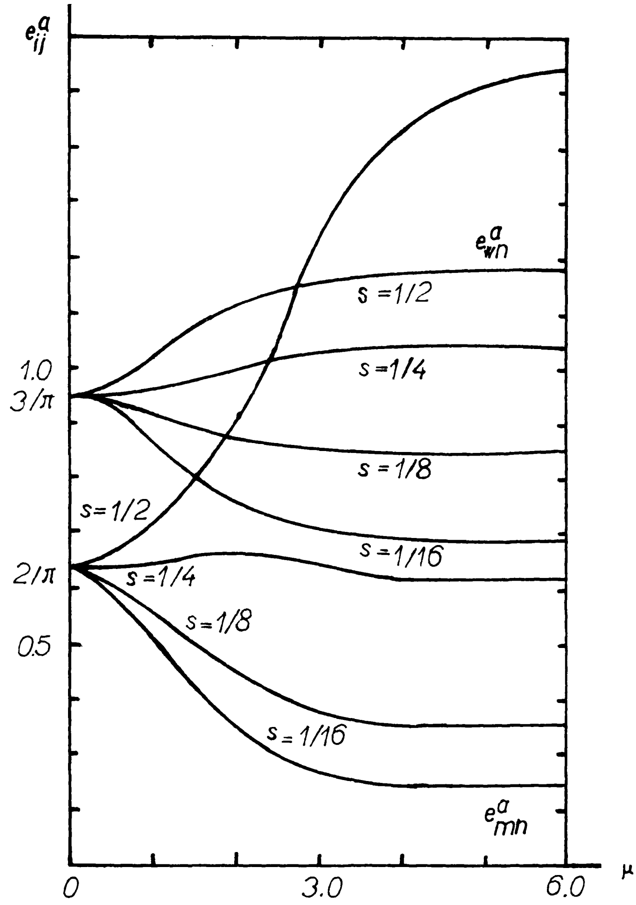

Figure 3. Approximate BE of the Wilcoxon- relative to the normal-scores test (e^a_{wn}) resp. of the median- relative to the normal-scores-test (e^a_{mn}) at the normal-shift-alternatives ($(\phi(y - \mu), \phi(y))$).

As we know from Theorem 5.1 that the first term on the rhs is equal to the approximate slope, the statement follows at once. □

Consequently we have for two linear rank tests φ_1, φ_2 fulfilling the assumptions of the theorem:

$$E_{1,2}(\{(F_{n1}, F_{n2})\}) = E^a_{1,2}(\{F_{n1}, F_{n2})\}) ,$$

i.e. exact and approximate local BE are identical. This result now enables us to derive a *general theory of local comparison* for linear rank tests, as near the null hypothesis we may regard the far easier approximate instead of the exact BE.

Let $b \mid (0, 1)$ be a function with

$$b \in L_2 , \quad \int b \, \mathrm{d}\lambda = 0$$

and consider functions $b_\Delta \mid (0, 1)$, $\Delta > 0$ satisfying:

$$b_\Delta \xrightarrow{L_2} b , \quad \Delta \to 0$$

$$\Delta \, . \sup_t |b_\Delta(t)| \to 0 , \quad \Delta \to 0$$

$$\int_0^t b_\Delta \, \mathrm{d}\lambda \begin{cases} =0 , & t = 1 \\ \leqq 0 , & \forall t \in (0, 1) \\ \neq 0 , & \exists t \in (0, 1) \end{cases}$$

Choosing $\eta \in (0, \infty)$ such that:

$$\eta \, . \, \Delta \, . \min \left[\sqrt{\left(\frac{1-s}{s}\right)} , \sqrt{\left(\frac{s}{1-s}\right)} \right] . \sup_t |b_\Delta(t)| \leqq 1 \quad \forall \Delta \in (0, \Delta_0) ,$$

we denote for $(F, F) \in H$ by $(F_{\Delta 1}, F_{\Delta 2})$ the alternative, $F_{\Delta 1}$ resp. $F_{\Delta 2}$ having F-density:

$$f_{\Delta 1} = 1 + \eta \, . \, \Delta \, . \sqrt{\left(\frac{1-s}{s}\right)} \, . \, b_\Delta(F) \quad \text{resp.} \quad f_{\Delta 2} = 1 - \eta \, . \, \Delta \, . \sqrt{\left(\frac{s}{1-s}\right)} \, . \, b_\Delta(F) \, .$$

With this notation we get:

Theorem 5.3 (Kremer (1979), (1982))

(a) Let φ_i, $i = 1, 2$ be linear rank tests with scores-generating functions $b_i \in L_3$. Then:

$$E_{1,2}(\{F_{\Delta_j,1}, F_{\Delta_j,2})\}) = \frac{\sigma^2(b_2)}{\sigma^2(b_1)} \left[\frac{\int b_1 b \, \mathrm{d}\lambda}{\int b_2 b \, \mathrm{d}\lambda} \right]^2 , \quad \forall \{\Delta_j\} : \Delta_j \to 0 , \quad \Delta_j > 0 \, .$$

(b) Assume furthermore b to be nondecreasing and

$$b \in L_3 , \quad \lim_{\Delta \to 0} \Delta \, . \int |b_\Delta^3| \, \mathrm{d}\lambda = 0 \, .$$

Then a linear rank test with scores-generating function b is local optimal under each $\{(F_{\Delta_j,1}, F_{\Delta_j,2})\}$.

We may by Theorem 5.2 restrict on investigating the approximate BE resp. slope for *proof.* Part (b) once again is an application of the inequality of Bahadur & Raghavachari, i.e. we only have to show that the approximate slope is asymptotically equivalent to the corresponding Kullback-Leibler-information number. □

Furthermore we can derive a general theorem on the existence of bounds on the local BE for our two-sample linear rank tests.

Theorem 5.4 (Kremer (1979), (1982))
Under the conditions of Theorem 5.3 (a) the following statements are equivalent:

(1) $b := \frac{b_1}{\sigma(b_1)} - c \cdot \frac{b_2}{\sigma(b_2)}$ is λ – a.e. nondecreasing for $c \in (0, \infty)$,

(2) $E_{1,2}(\{(F_{n1}, F_{n2})\}) \geqq c^2$, for all local alternatives $\{(F_{n1}, F_{n2})\}$.

The Theorems 5.3, 5.4 are quite similar to the former results derived by Hájek and Behnen under the assumption of contiguity with the classical *Pitman efficiency*. This correspondence is not surprising, since Wieand (1976) gave general conditions under which Pitman efficiency is equal to the approximate local BE. I do not like to go further into the discussion on the connection between Pitman efficiency and local BE. I only want to remark that for our linear rank tests both local efficiency measures yield the same results under the assumptions of the so called *Chernoff-Savage-approach* (cf. Chernoff & Savage (1958), Kremer (1979)).

The question might arise, why to develop a theory of local comparison by use of BE, having already a nice theory based on Pitman-efficiency. The justification may be found in the rather weak assumptions of the above approach. Pitman-efficiency considerations are based on the assumption of contiguous alternatives or alternatively on the smoothness assumptions (on the scores-generating function) of the Chernoff-Savage-approach. In the above theory we investigate general local alternatives and at the same time require only fairly weak assumptions on the scores-generating function.

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