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Daniel Hlubinka

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The Support of $\mathscr{L}(\xi, \eta)$ When the Supports of $\mathscr{L}(\eta \mid \xi = x)$ And $\mathscr{L}(\xi)$ Are Given

DANIEL HLUBINKA

Praha*)

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An universally measurable Markov kernel $(x, \mathrm{P}^x) : \mathbb{X} \to M_1(\mathbb{Y})$ (where $\mathbb{X}, \mathbb{Y}$ are Polish spaces) is considered. Given $\lambda \in M_1(\mathbb{X})$ the paper treats the support of joint distribution $\mathscr{L}(\xi, \eta)$ where (ξ, η) is an $\mathbb{X} \times \mathbb{Y}$-valued random vector with $\mathscr{L}(\eta \mid \xi = x) = \mathrm{P}^x$, $\mathscr{L}(\xi) = \lambda$. In particular conditions implying $\operatorname{supp}(\mathscr{L}(\xi, \eta)) = \overline{\operatorname{supp}(\mathscr{L}(\xi)) \times \mathbb{Y} \cap \operatorname{Gr}(x \mapsto \operatorname{supp}(\mathscr{L}(\eta \mid \xi = x)))}$ are sought.

Introduction

The paper is intended as a supplement to recent studies [1] and [3] dealing with the existence of $\mathbb{X} \times \mathbb{Y}$-valued random vector with a given conditional structure. Throughout the paper a Markov kernel (x, P^x) and $\lambda \in M_1(\mathbb{X})$ are used to represent the conditional and marginal distribution of two-dimensional $\mathbb{X} \times \mathbb{Y}$-valued random vector (ξ, η). Assuming the Markov kernel to be universally measurable Proposition 3 provides the (general) relation between the supports of $\mathscr{L}(\xi, \eta)$ and $\mathscr{L}(\xi)$, $\mathscr{L}(\eta \mid \xi = x)$. Proposition 4 and its corollaries treat the case of a weakly continuous Markov kernel. In Propositions 5, 8 and Corollary 5A the case of a Markov kernel (x, P^x) such that $x \mapsto \operatorname{supp}(\mathrm{P}^x)$ is lower semicontinuous is studied and Propositions 6, 7 and Corollary 6A deal with upper semicontinuity of $x \mapsto \operatorname{supp}(\mathrm{P}^x)$.

Considering two random vectors $(\xi, \eta), (\xi', \eta')$ such that $\mathscr{L}(\xi) = \mathscr{L}(\xi')$ and $\operatorname{supp}(\mathscr{L}(\eta' \mid \xi' = x)) \subset \operatorname{supp}(\mathscr{L}(\eta \mid \xi = x))$ a.s. $[\mathscr{L}(\xi)]$ we are able to prove (Proposition 9) that $\operatorname{supp}(\mathscr{L}(\xi', \eta')) \subset \operatorname{supp}(\mathscr{L}(\xi, \eta))$.

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*) Department of Probability and Statistics, Faculty of Mathematics and Physics, Charles University, Sokolovská 83, 186 00 Praha 8 - Karlín, Czech Republic

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1. Definitions and basic results

Throughout the paper we shall use two fixed Polish spaces $\mathbb{X}$ and $\mathbb{Y}$. Denote by $\mathscr{G}(\mathbb{X})$, $\mathscr{F}(\mathbb{X})$, $\mathscr{B}(\mathbb{X})$, $\mathscr{A}(\mathbb{X})$, and $\mathscr{U}(\mathbb{X})$ classes of open, closed, Borel, analytical and universally measurable sets in $\mathbb{X}$ and denote the space of Borel probability measures on $\mathbb{X}$ by $M_1(\mathbb{X})$. Let us agree that speaking about topology on $M_1(\mathbb{X})$ we mean its standard weak topology that makes the space $M_1(\mathbb{X})$ Polish assuming $\mathbb{X}$ is Polish.

It is well known fact that for $\mu \in M_1(\mathbb{X})$ there are two equivalent definitions of *the support* of μ

$$\operatorname{supp}(\mu) := \bigcap_{\mathscr{F}(\mathbb{X})} \{F : \mu(F) = 1\} = \{x \in \mathbb{X} : \mu(G) > 0, \forall x \in G \in \mathscr{G}(\mathbb{X})\}.$$

For the rest of the paper we shall denote by $\mathrm{U}(x)$, and D°, $\bar{D}$, ∂D and $\complement D$ an open neighbourhood of x, and interior, closure, boundary and complement of $D \subset \mathbb{X}$. For $D \subset \mathbb{X} \times \mathbb{Y}$ we shall denote its sections by $D_x := \{y \in \mathbb{Y} : (x, y) \in D\}$.

Any map $\Psi : \mathbb{X} \to 2^{\mathbb{Y}}$ is referred to as a *multifunction* from $\mathbb{X}$ to $\mathbb{Y}$. We shall write $\Psi : \mathbb{X} \rightrightarrows \mathbb{Y}$ and denote $\operatorname{Gr}(\Psi) := \{(x,y) : y \in \Psi(x)\}$ its graph. A multifunction $\Psi(x) : \mathbb{X} \rightrightarrows \mathbb{Y}$ will be called *correspondence* if $\Psi(x) \in \mathscr{F}(\mathbb{Y})$ for all $x \in \mathbb{X}$.

Define Ψ to be *upper semicontinuous* (USC) and *lower semicontinuous* (LSC) if

$$\{x \in \mathbb{X} : \Psi(x) \subset G\} \in \mathscr{G}(\mathbb{X})\ \forall G \in \mathscr{G}(\mathbb{Y}), \text{ and}$$
$$\{x \in \mathbb{X} : \Psi(x) \cap G \neq \emptyset\} \in \mathscr{G}(\mathbb{X})\ \forall G \in \mathscr{G}(\mathbb{Y}) \text{ respectively.}$$

It is easy to verify equivalent conditions for upper and lower semicontinuity;
Ψ is USC iff $\forall \mathrm{U}(\Psi(x))\ \exists \mathrm{U}(x) : \Psi(y) \subset \mathrm{U}(\Psi(x))\ \forall y \in \mathrm{U}(x)$, and
Ψ is LSC iff $\forall y \in \Psi(x)\ \forall \{x_n\}, x_n \to x\ \exists \{y_n\} : y_n \in \Psi(x_n), \lim y_n = y$.

An example of LSC correspondence is provided by

Lemma 1. *The correspondence* $\Sigma : M_1(\mathbb{X}) \rightrightarrows \mathbb{X}$, $\Sigma(\mu) = \operatorname{supp}(\mu)$ *is lower semicontinuous.*

Proof.

$$\complement\{\mu : \operatorname{supp}(\mu) \cap G \neq \emptyset\} = \{\mu : \operatorname{supp}(\mu) \subset \complement G\} \in \mathscr{F}(M_1(\mathbb{X})), \forall G \in \mathscr{G}(\mathbb{X}).$$

□

Definition 1. An universally meassurable map $x \mapsto \mathrm{P}^x$ from $\mathbb{X}$ to $M_1(\mathbb{X})$ will be called *universally measurable Markov kernel* (abbreviated UMK) and denoted here by (x, P^x).

A map $x \mapsto \operatorname{supp}(\mathrm{P}^x)$ from $\mathbb{X}$ to $\mathbb{Y}$ will be called *support correspondence* of (x, P^x).

Remark. Due to proposition 8.4.6. in [2] the measurability w.r.t. $(\mathscr{U}, \mathscr{U})$ is equivalent to the measurability w.r.t. $(\mathscr{U}, \mathscr{B})$.

Definition 2. Define $P^\lambda \in M_1(\mathbb{X} \times \mathbb{Y})$ *measure induced by* (x, P^x) *and* $\lambda \in M_1(\mathbb{X})$ by

$$P^\lambda(U) = \int_{\mathbb{X}} P^x(U_x)\, \lambda(dx)\,, \quad U \in \mathscr{U}(\mathbb{X} \times \mathbb{Y})\,.$$

Remark. For an arbitrary universally measurable $f : \mathbb{X} \times \mathbb{Y} \to [0, +\infty]$ holds

$$\iint_{\mathbb{X}\times\mathbb{Y}} f(x, y)\, P^\lambda(dx, dy) = \int_{\mathbb{X}}\int_{\mathbb{Y}} f(x, y)\, P^x(dy)\, \lambda(dx)\,.$$

Definition 3. Define a multifunction $\Psi : \mathbb{X} \rightrightarrows \mathbb{Y}$ to be *U-measurable* if $\{x : \Psi(x) \cap G \neq \emptyset\} \in \mathscr{U}(\mathbb{X})$ for $G \in \mathscr{G}(\mathbb{Y})$.

Remark. $\Psi : \mathbb{X} \rightrightarrows \mathbb{Y}$ is U-measurable iff $\{x : \Psi(x) \subset F\} \in \mathscr{U}(\mathbb{X})$ for $F \in \mathscr{F}(\mathbb{Y})$. See also [1] and [4] to find more about this concept.

We borrow a part of Lemma 1 in [3]

Lemma 2. *Let* $\Psi : \mathbb{X} \rightrightarrows \mathbb{Y}$ *be an U-measurable correspondence. Then* $\mathrm{Gr}(\Psi) \in \mathscr{U}(\mathbb{X}) \otimes \mathscr{B}(\mathbb{Y})$.

Applying Lemmata 1 and 2 we can conclude.

Proposition 1. *Let* (x, P^x) *be an universally measurable Markov kernel. Then its support correspondence* S *is U-measurable and* $\mathrm{Gr}(S) \in \mathscr{U}(\mathbb{X}) \otimes \mathscr{B}(\mathbb{Y})$.

Proof. $\{x : \mathrm{supp}(P^x) \subset F\} = (x, P^x)^{-1} \underbrace{\{\mu : \mathrm{supp}(\mu) \subset F\}}_{\in \mathscr{F}(M_1(\mathbb{Y}))} \in \mathscr{U}(\mathbb{X}), \forall F \in \mathscr{F}(\mathbb{Y})$.

□

Remark. It follows from Lemma 1 in [1] that Borel measurability of (x, P^x) also implies that $\mathrm{Gr}(S) \in \mathscr{A}(\mathbb{X} \times \mathbb{Y})$.

Now we are prepared to proceed to the main topic of our study – relation between $\mathrm{supp}(P^\lambda)$ and $\mathrm{supp}(\lambda)$, $\mathrm{supp}(P^x)$, $x \in \mathrm{supp}(\lambda)$.

Proposition 2. $(x, y) \in \mathrm{supp}(P^\lambda) \Rightarrow x \in \mathrm{supp}(\lambda)$.

Proof. Suppose $x \notin \mathrm{supp}(\lambda)$. Then there exists $U(x)$ such that $\lambda(U(x)) = 0$ and consequently $P^\lambda(U(x) \times \mathbb{Y}) = 0$. Hence for any $y \in \mathbb{Y}$ there exists $U(x, y)$ such that $P^\lambda(U(x, y)) = 0$. Therefore $(x, y) \notin \mathrm{supp}(P^\lambda)$. □

Counter example showing that $(x, y) \in \mathrm{supp}(P^\lambda) \not\Rightarrow y \in \mathrm{supp}(P^x)$ and $x \in \mathrm{supp}(\lambda)$, $y \in \mathrm{supp}(P^x) \not\Rightarrow (x, y) \in \mathrm{supp}(P^\lambda)$.

Choose $\mathbb{X} = \mathbb{Y} = [0, 1]$ and

$$P^x = \begin{cases} \varepsilon_x & \text{for} \quad x \in [0, 1) \\ \varepsilon_0 & \text{for} \quad x = 1, \end{cases}$$

where ε_x is the probability measure concentrated at x. Let $\lambda = \lambda_1$ where λ_1 denotes the Lebesgue measure on $\mathbb{X}$. Obviously $(1, 1) \in \mathrm{supp}(P^\lambda)$ but $1 \notin \mathrm{supp}(P^1)$, and on the other hand $(1, 0) \notin \mathrm{supp}(P^\lambda) = \mathrm{Diag}\,[0, 1]^2$, while $1 \in \mathrm{supp}(\lambda)$, $\{0\} = \mathrm{supp}(P^1)$.

Proposition 3. *Let (x, P^x) be an universally measurable Markov kernel. Then*

$$\operatorname{supp}(\mathrm{P}^\lambda) \subset \overline{\operatorname{Gr}(S) \cap \operatorname{supp}(\lambda) \times \mathbb{Y}}$$

holds.

Proof. The set $\operatorname{Gr}(S) \cap \operatorname{supp}(\lambda) \times \mathbb{Y}$ is (Proposition 1) universally measurable. Thus we can write

$$\mathrm{P}^\lambda(\operatorname{Gr}(S) \cap \operatorname{supp}(\lambda) \times \mathbb{Y}) = \int_{\operatorname{supp}(\lambda)} \mathrm{P}^x(\operatorname{supp}(\mathrm{P}^x))\, \lambda(\mathrm{d}x) = \lambda(\operatorname{supp}(\lambda)) = 1\,.$$

□

Up to now we have used the assumption of universal measurability of (x, P^x) only. Now we shall proceed to results requiring additional conditions on the UMK.

2. Weakly continuous Markov kernels

Definition 4. An universally measurable Markov kernel (x, P^x) is called *(weakly) continuous* if $x_n \to x \Rightarrow \mathrm{P}^{x_n} \xrightarrow{w} \mathrm{P}^x$.

Proposition 4. *Assume that an UMK (x, P^x) is weakly continuous. Then the inclusion*

$$\operatorname{Gr}(S) \cap \operatorname{supp}(\lambda) \times \mathbb{Y} \subset \operatorname{supp}(\mathrm{P}^\lambda)\,,$$

where S is the support correspondence of the UMK, holds.

Proof. Consider $x \in \operatorname{supp}(\lambda)$, $y \in \operatorname{supp}(\mathrm{P}^x)$, and $\mathrm{U}(x, y)$. Then there exist $\tilde{\mathrm{U}}(x)$ and $\mathrm{U}(y)$ such that $\tilde{\mathrm{U}}(x) \times \mathrm{U}(y) \subset \mathrm{U}(x, y)$ and $\mathrm{P}^x(\partial \mathrm{U}(y)) = 0$. Continuity of (x, P^x) implies the existence of $\mathrm{U}(x) \subset \tilde{\mathrm{U}}(x)$ such that $z \in \mathrm{U}(x) \Rightarrow \mathrm{P}^z(\mathrm{U}(y)) > 0$ and hence

$$\mathrm{P}^\lambda(\mathrm{U}(x, y)) \geq \mathrm{P}^\lambda(\mathrm{U}(x) \times \mathrm{U}(y)) = \int_{\mathrm{U}(x)} \mathrm{P}^z(\mathrm{U}(y))\, \lambda(\mathrm{d}z) > 0\,.$$

□

Using Propositions 3 and 4 we may conclude.

Corollary 4A. *Assume that an UMK (x, P^x) is weakly continuous. Then*

$$\overline{\operatorname{Gr}(S) \cap \operatorname{supp}(\lambda) \times \mathbb{Y}} = \operatorname{supp}(\mathrm{P}^\lambda)$$

holds.

Proof. Use Propositions 3 and 4, and closeness of $\operatorname{supp}(\mathrm{P}^\lambda)$ □

Lemma 3. *The support correspondence of a continuous UMK (x, P^x) is lower semicontinuous.*

Proof. Consider continuous (x, P^x) and its support correspondence S. Then

$$\{x\colon \operatorname{supp}(\mathrm{P}^x) \cap G \neq \emptyset\} = (x, \mathrm{P}^x)^{-1} \underbrace{\{\mu\colon \operatorname{supp}(\mu) \cap G \neq \emptyset\}}_{\in \mathscr{G}(M_1(\mathbb{Y}))} \in \mathscr{G}(\mathbb{X}).$$

follows from the assumption of continuity of (x, P^x). □

Remark. Continuity of UMK does not imply upper semicontinuity of its support correspondence. Neither upper nor lower semicontinuity of support correspondence does imply weak continuity of its Markov kernel. An example is provided by choice $\mathbb{X} = \mathbb{Y} = [0, 1]$, $\lambda = \lambda_1$ and $\mathrm{P}^x = x\Phi + (1 - x)\,\varepsilon_0$, where Φ is any distribution with $\{0\} \subsetneqq \operatorname{supp}(\Phi)$.

Semicontinuity of the support correspondence S is discussed in next section.

3. Markov kernels with semicontinuous support correspondence

Proposition 5. *Assume that an UMK (x, P^x) is such that its support correspondence S is lower semicontinuous. Then*

$$x \in \operatorname{supp}(\lambda),\ y \in \operatorname{supp}(\mathrm{P}^x) \Rightarrow (x, y) \in \operatorname{supp}(\mathrm{P}^\lambda)$$

Proof. Fix an arbitrary open $\mathrm{U}(y)$. The lower semicontinuity of S implies that the set $\{z\colon S(z) \cap \mathrm{U}(y) \neq \emptyset\}$ is open. Hence there exists $\mathrm{U}(x)$ such that $z \in \mathrm{U}(x) \Rightarrow S(z) \cap \mathrm{U}(y) \neq \emptyset \Rightarrow \mathrm{P}^z(\mathrm{U}(y)) > 0$ and therefore

$$\mathrm{P}^\lambda(\mathrm{U}(x) \times \mathrm{U}(y)) = \int_{\mathrm{U}(x)} \underbrace{\mathrm{P}^z(\mathrm{U}(y))}_{>0} \lambda(\mathrm{d}z) > 0\,.$$

□

Remark. We have proved in Lemma 3 that for a continuous Markov kernel its support correspondence is lower semicontinuous. From this point of view Proposition 5 improves Proposition 4 above.

The assumption of upper semicontinuity leads to implication reverse to that of Proposition 5.

Proposition 6. *Assume that an UMK (x, P^x) is such that its support correspondence S is upper semicontionuous. Then*

$$(x, y) \in \operatorname{supp}(\mathrm{P}^\lambda) \Rightarrow x \in \operatorname{supp}(\lambda),\ y \in \operatorname{supp}(\mathrm{P}^x)\,.$$

Proof. The proposition is a corollary to Proposition 3 using the closeness of graph of an upper semicontinuous correspondence. □

Proposition 7. *Assume that an UMK (x, P^x) is such that for $x \in \mathbb{X}$, $y \in \mathbb{Y}$ the*

$$(x, y) \in \operatorname{supp}(\mathrm{P}^\lambda) \Leftrightarrow x \in \operatorname{supp}(\lambda),\ y \in \operatorname{supp}(\mathrm{P}^x)\,.$$

holds. Then the support correspondence S of (x, P^x) is upper semicontinuous on supp (λ).

Proof. Fix arbitrary $x \in \operatorname{supp}(\lambda)$ and $U(\operatorname{supp}(P^x))$. Put $F = \complement U(\operatorname{supp}(P^x))$. Then $\{x\} \times F \subset \complement\operatorname{supp}(P^\lambda)$, and therefore exist $U(x)$, $U(\{x\} \times F)$ such that $U(x) \times F \subset U(\{x\} \times F) \subset \complement\operatorname{supp}(P^\lambda)$. It follows that $z \in U(x) \cap \operatorname{supp}(\lambda) \Rightarrow \operatorname{supp}(P^z) \cap F = \emptyset \Rightarrow \operatorname{supp}(P^z) \subset U(\operatorname{supp}(P^x))$. □

4. Properties holding a.s. $[\lambda]$

In the section we shall study a Markov kernel such that either its support correspondence or the kernel itself is semicontinuous a.s. $[\lambda]$ or weakly continuous a.s. $[\lambda]$ respectively.

Definition 5. A multifunction $\Psi : \mathbb{X} \rightrightarrows \mathbb{Y}$ is called *upper (lower) semicontinuous a.s.* $[\lambda]$ if Ψ is upper (lower) semicontinuous on $\mathbb{X} \setminus N$ and $\lambda(N) = 0$.

An UMK (x, P^x) is called *weakly continuous a.s.* $[\lambda]$ if (x, P^x) is weakly continuous on $\mathbb{X} \setminus N$ and $\lambda(N) = 0$.

The set N is called *set of non-continuity.*

Corollary 5A. *Assume that an UMK (x, P^x) is such that its support correspondence S is lower semicontinuous a.s. $[\lambda]$. Then*

$$x \in \operatorname{supp}(\lambda),\ y \in \operatorname{supp}(P^x) \Rightarrow (x, y) \in \operatorname{supp}(P^\lambda)$$

for $x \in \mathbb{X} \setminus N$, where N is the set of non-contituity.

Proof. Follow the proof of Proposition 5 using $U'(x) = U(x) \cap \complement N$ instead of $U(x)$. □

Corollary 6A. *Assume that an UMK (x, P^x) is such that its support correspondence S is upper semicontinuous a.s. $[\lambda]$. Then*

$$(x, y) \in \operatorname{supp}(P^\lambda) \Rightarrow x \in \operatorname{supp}(\lambda),\ y \in \operatorname{supp}(P^x)\,.$$

for $x \in \mathbb{X} \setminus N$, where N is the set of non-continuity.

Proof. Fix $x \in (\mathbb{X} \setminus N) \cap \{x : \exists y \in \mathbb{Y},\ (x, y) \in \operatorname{supp}(P^\lambda)\}$. It follows from Proposition 2 that $x \in \operatorname{supp}(\lambda)$. Consider any $y \in \complement\operatorname{supp}(P^x)$. Then exist $U_1 = U(\operatorname{supp}(P^x))$ and $U_2 = U(x) \setminus N$ such that $y \notin \overline{U_1}$ and $z \in U_2 \Rightarrow \operatorname{supp}(P^z) \subset U_1$. The set $U(x) \times (\complement U_1)^\circ$ is an open neighbourhood of (x, y) and

$$P^\lambda(U(x) \times \complement U_1^\circ) \le \lambda(N) + P^\lambda(U_2 \times \complement U_1^\circ) = \int_{U_2} P^z(\complement U_1)^\circ\, \lambda(dz) = 0\,,$$

what implies $(x, y) \notin \operatorname{supp}(P^\lambda)$. □

Corollary 4B. *If an UMK* (x, P^x) *is weakly continuous a.s.* $[\lambda]$ *and* $x \in \mathbb{X} \setminus N$ *(N is the set on non-continuity) then*

$$x \in \operatorname{supp}(\lambda),\ y \in \operatorname{supp}(\mathrm{P}^x) \Rightarrow (x, y) \in \operatorname{supp}(\mathrm{P}^\lambda)$$

Proof. The support correspondence S of (x, P^x) is lower semicontinuous on $\mathbb{X} \setminus N$, i.e. S is lower semicontinuous a.s. $[\lambda]$. □

Corollary 4C. *Assume that an UMK* (x, P^x) *is continuous a.s.* $[\lambda]$ *and such that*

$$\forall x \in N \cap \operatorname{supp}(\mathrm{P}^\lambda)\ \exists \mathrm{U}(x) : z \in \mathrm{U}(x) \setminus N \Rightarrow \operatorname{supp}(\mathrm{P}^x) \subset \operatorname{supp}(\mathrm{P}^z)\ \textit{a.s.}\ [\mathrm{P}^x],$$

where N is the set of non-continuity. Then

$$\overline{Gr(S) \cap \operatorname{supp}(\lambda) \times \mathbb{Y}} = \operatorname{supp}(\mathrm{P}^\lambda).$$

Proof. Fix $x \in N \cap \operatorname{supp}(\lambda)$, $y \in \operatorname{supp}(\mathrm{P}^x)$. Then

$$\begin{aligned}\exists \mathrm{U}(x) : z \in \mathrm{U}(x) \setminus N &\Rightarrow \mathrm{P}^z(\mathrm{U}(y)) > 0 \\ &\Rightarrow \mathrm{P}^\lambda(\mathrm{U}(x) \times \mathrm{U}(y)) \geq \int_{\mathrm{U}(x)\setminus N} \mathrm{P}^z(\mathrm{U}(y))\, \lambda(\mathrm{d}z) > 0\end{aligned}$$

for all $\mathrm{U}(y)$. Using Corollary 4B for $x \in \operatorname{supp}(\lambda) \setminus N$ we finish the proof. □

Definition 6. A support correspondence S is called *lower semicontinuous on a set of positive measure* $[\lambda]$ at $x \in \operatorname{supp}(\lambda)$ if $\forall y \in \operatorname{supp}(\mathrm{P}^x)$ $\forall \mathrm{U}(y)$ exists $Z_{\mathrm{U}(y)}$ such that

$$\lambda(\mathrm{U}(x) \cap Z_{\mathrm{U}(y)}) > 0\ \ \forall \mathrm{U}(x),\ \mathrm{P}^z(\mathrm{U}(y)) > 0\ \text{ for } z \in Z_{\mathrm{U}(y)}.$$

Proposition 8. *A support correspondence* $S : x \mapsto \operatorname{supp}(\mathrm{P}^x)$ *is lower semicontinuous on a set of positive measure* $[\lambda]$ *on* $\operatorname{supp}(\lambda) \Leftrightarrow$

$$x \in \operatorname{supp}(\lambda),\ y \in \operatorname{supp}(\mathrm{P}^x) \Rightarrow (x, y) \in \operatorname{supp}(\mathrm{P}^\lambda)$$

Proof. "⇒" Fix arbitrary open $\mathrm{U}(x)$, $\mathrm{U}(y)$. Then

$$\mathrm{P}^\lambda(\mathrm{U}(x) \times \mathrm{U}(y)) \geq \int_{\mathrm{U}(x)\cap Z_{\mathrm{U}(y)}} \underbrace{\mathrm{P}^z(\mathrm{U}(y))}_{>0} \lambda(\mathrm{d}z) > 0$$

"⇐" Choose $\mathrm{U}(y)$ and define $Z_{\mathrm{U}(y)} := \{x : \mathrm{P}^x(\mathrm{U}(y)) > 0\}$. The set Z is nonempty since $x \in Z_{\mathrm{U}(y)}$, and for fixed arbitrary $\mathrm{U}(x)$

$$0 < \mathrm{P}^\lambda(\mathrm{U}(x) \times \mathrm{U}(y)) = \int_{\mathrm{U}(x)\cap Z_{\mathrm{U}(y)}} \underbrace{\mathrm{P}^z(\mathrm{U}(y))}_{>0} \lambda(\mathrm{d}z) + \int_{\mathrm{U}(x)\setminus Z_{\mathrm{U}(y)}} \underbrace{\mathrm{P}^z(\mathrm{U}(y))}_{=0} \lambda(\mathrm{d}z)$$

holds. Therefore $\lambda(\mathrm{U}(x) \cap Z_{\mathrm{U}(y)}) > 0$. □

Finally we turn our attention to the relation of supports for two random vectors.

Proposition 9. *Assume two universally measurable Markov kernels* (x, P^x) *and* (x, Q^x) *such that* $\operatorname{supp}(\mathrm{P}^x) \subset \operatorname{supp}(\mathrm{Q}^x)$ *a.s.* $[\lambda]$. *Then* $\operatorname{supp}(\mathrm{P}^\lambda) \subset \operatorname{supp}(\mathrm{Q}^\lambda)$.

Proof. Denote $N := \{x : \operatorname{supp}(\mathrm{P}^x) \not\subset \operatorname{supp}(\mathrm{Q}^x)\}$ and note that $\lambda(N) = 0$. Fix arbitrary $(x, y) \in \operatorname{supp}(\mathrm{P}^\lambda)$. Then

$$0 < \mathrm{P}^\lambda(\mathrm{U}(x) \times \mathrm{U}(y)) = \int_{\mathrm{U}(x) \setminus N} \mathrm{P}^z(\mathrm{U}(y))\, \lambda(\mathrm{d}z) + \underbrace{\int_{\mathrm{U}(x) \cap N} \mathrm{P}^z(\mathrm{U}(y))\, \lambda(\mathrm{d}z)}_{0}.$$

$\mathrm{P}^z(G) > 0,\ z \in \complement N \Rightarrow \mathrm{Q}^z(G) > 0$ for $G \in \mathscr{G}(\mathbb{Y})$ and

$$\mathrm{Q}^\lambda(\mathrm{U}(x) \times \mathrm{U}(y)) = \int_{\mathrm{U}(x) \setminus N} \mathrm{Q}^z(\mathrm{U}(y))\, \lambda(\mathrm{d}z) > 0$$

since $\mathrm{P}^z(\mathrm{U}(y)) > 0$ for $z \in S \subset \mathrm{U}(x) \setminus N,\ \lambda(S) > 0$. □

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