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k -TORSIONLESS MODULES WITH FINITE
GORENSTEIN DIMENSION

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Abstract. Let R be a commutative Noetherian ring. It is shown that the finitely generated R -module M with finite Gorenstein dimension is reflexive if and only if $M_{\mathfrak{p}}$ is reflexive for $\mathfrak{p} \in \text{Spec}(R)$ with $\text{depth}(R_{\mathfrak{p}}) \leq 1$, and $\text{G-dim}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \leq \text{depth}(R_{\mathfrak{p}}) - 2$ for $\mathfrak{p} \in \text{Spec}(R)$ with $\text{depth}(R_{\mathfrak{p}}) \geq 2$. This gives a generalization of Serre and Samuel's results on reflexive modules over a regular local ring and a generalization of a recent result due to Belshoff. In addition, for $n \geq 2$ we give a characterization of n -Gorenstein rings via Gorenstein dimension of the dual of modules. Finally it is shown that every R -module has a k -torsionless cover provided R is a k -Gorenstein ring.

Keywords: torsionless module, reflexive module, Gorenstein dimension

MSC 2010: 13D05

1. INTRODUCTION

Let R be a commutative Noetherian ring with identity and let M be a finitely generated R -module. For an R -module M , the dual M^* of M is the R -module $\text{Hom}_R(M, R)$. There is a natural evaluation map $\delta_M: M \rightarrow M^{**}$ and M is called reflexive provided δ_M is an isomorphism. It is clear that every finitely generated free module is reflexive. In 1958, Serre proved that every reflexive module is free provided R is a regular local ring of dimension at most 2. Later, Samuel extended this result by showing that over a regular local ring of dimension at most 3, an R -module M is reflexive if and only if $\text{pd}_R M \leq 1$ and for every non-maximal prime ideal \mathfrak{p} , the localization $M_{\mathfrak{p}}$ is free over $R_{\mathfrak{p}}$. Recently, Belshoff considered an analogous question for modules over Gorenstein rings of low dimension. In particular, there is a notion of Gorenstein dimension for R -modules M , which satisfies the inequality

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$\text{G-dim}_R(M) \leq \text{pd}_R(M)$. This provides a natural setting for seeking extensions of the results of Serre and Samuel. In [3], Belshoff established the following:

- (1) If R is a Gorenstein local ring of dimension at most 2, then every reflexive module M has $\text{G-dim}_R(M) = 0$.
- (2) If R is a Gorenstein local ring of dimension 3, then an R -module M is reflexive if and only if $\text{G-dim}_R(M) \leq 1$ and the localization $M_{\mathfrak{p}}$ has Gorenstein dimension 0 over $R_{\mathfrak{p}}$ for every non-maximal prime ideal \mathfrak{p} .

In Section 2 we give a generalization of Belshoff's result (see Theorem 2.3). Recall that a ring R is called n -Gorenstein if the injective dimension of R is at most n . In Section 3, for $n \geq 2$ we give a characterization of n -Gorenstein rings via Gorenstein dimension of the dual of modules (see Theorem 3.4).

In Section 4 we bring a characterization of k -torsionless modules with finite Gorenstein dimension (see Theorem 4.5). Also this section concludes with a discussion of k -torsionless covers. Recall that for a class \mathcal{X} of modules and a module M , an R -homomorphism $\varphi: X \rightarrow M$ is an \mathcal{X} -cover of M provided it is left universal among all homomorphisms $Y \rightarrow M$ with $Y \in \mathcal{X}$ and, further, any $X \rightarrow M$ factors through φ via an automorphism $X \rightarrow X$. In addition, we investigate necessary and sufficient conditions which lead the tensor product of k -torsionless modules to be k -torsionless.

2. REFLEXIVE MODULES

This section contains some general remarks about reflexive modules with finite Gorenstein dimension. Indeed we present generalizations of the results of Belshoff [3], Serre [12] and Samuel [11].

First, we recall some necessary definitions which will be used in this section.

Definition 2.1. Let R be a ring and let M be an R -module. The dual of M is the module $\text{Hom}_R(M, R)$, which we usually denote by M^* , the bidual then is M^{**} , and analogous conventions apply to homomorphisms. The bilinear map $M \times M^* \rightarrow R$, $(x, \varphi) \mapsto \varphi(x)$, induces a natural homomorphism $\delta_M: M \rightarrow M^{**}$. We say that M is torsionless if δ_M is injective, and that M is reflexive if δ_M is bijective.

In [1], Auslander and Bridger introduced the Gorenstein dimension, $\text{G-dim}_R(M)$, for every finitely generated R -module as follows:

Definition 2.2. A finitely generated R -module M is said to have G -dimension zero ($\text{G-dim}_R(M) = 0$) if and only if M satisfies the following three properties:

- (i) M is reflexive,
- (ii) $\text{Ext}_R^i(M, R) = 0$ for each $i \geq 1$,
- (iii) $\text{Ext}_R^i(M^*, R) = 0$ for each $i \geq 1$.

Also we say that $\text{G-dim}_R(M) \leq n$ if there is an exact sequence

$$0 \longrightarrow G_n \longrightarrow \dots \longrightarrow G_2 \longrightarrow G_1 \longrightarrow G_0 \longrightarrow M \longrightarrow 0$$

such that $\text{G-dim}_R(G_i) = 0$ for each $i \geq 0$.

For every R -module M the inequality $\text{G-dim}_R(M) \leq \text{pd}_R(M)$ is proved by Auslander and Bridger in [1]. They showed that the equality holds if $\text{pd}_R(M)$ is finite. Furthermore, the Gorenstein dimension of an R -module is closely related to its depth, i.e., $\text{G-dim}_R(M) = \text{depth}(R) - \text{depth}_R(M)$. This equality is called the Auslander-Bridger formula.

The following theorem is special case of [4, Proposition 1.4.1] and gives a generalization of [3, Proposition 1.1, Proposition 1.7].

Theorem 2.3. *Let R be a ring and let M be an R -module with $\text{G-dim}_R(M) < \infty$. Then the following statements hold.*

- (1) *M is torsionless if and only if*
 - (i) *$M_{\mathfrak{p}}$ is torsionless for $\mathfrak{p} \in \text{Ass}(R)$, and*
 - (ii) *$\text{G-dim}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \leq \text{depth}(R_{\mathfrak{p}}) - 1$ for $\mathfrak{p} \in \text{Spec}(R)$ with $\text{depth}(R_{\mathfrak{p}}) \geq 1$.*
- (2) *M is reflexive if and only if*
 - (i) *$M_{\mathfrak{p}}$ is reflexive for $\mathfrak{p} \in \text{Spec}(R)$ with $\text{depth}(R_{\mathfrak{p}}) \leq 1$, and*
 - (ii) *$\text{G-dim}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \leq \text{depth}(R_{\mathfrak{p}}) - 2$ for $\mathfrak{p} \in \text{Spec}(R)$ with $\text{depth}(R_{\mathfrak{p}}) \geq 2$.*

Proof. (1) Assume that M is torsionless. By [4, Proposition 1.4.1], we must show that $\text{G-dim}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \leq \text{depth}(R_{\mathfrak{p}}) - 1$ for $\mathfrak{p} \in \text{Spec}(R)$ with $\text{depth}(R_{\mathfrak{p}}) \geq 1$. Note that according to [4, Proposition 1.4.1], $\text{depth}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \geq 1$ for $\mathfrak{p} \in \text{Spec}(R)$ with $\text{depth}(R_{\mathfrak{p}}) \geq 1$. Now by the Auslander-Bridger formula we have

$$\text{G-dim}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) = \text{depth}(R_{\mathfrak{p}}) - \text{depth}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \leq \text{depth}(R_{\mathfrak{p}}) - 1.$$

Conversely, by [4, Proposition 1.4.1], it suffices to show that $\text{depth}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \geq 1$ for $\mathfrak{p} \in \text{Spec}(R)$ with $\text{depth}(R_{\mathfrak{p}}) \geq 1$. By hypothesis and the Auslander-Bridger formula we have

$$\text{depth}(R_{\mathfrak{p}}) - \text{depth}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) = \text{G-dim}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \leq \text{depth}(R_{\mathfrak{p}}) - 1,$$

therefore, $\text{depth}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \geq 1$ for $\mathfrak{p} \in \text{Spec}(R)$ with $\text{depth}(R_{\mathfrak{p}}) \geq 1$.

(2) is proved along the same lines as (1). □

The following two results which can be found in [3], [12] and [11], follow from Theorem 2.3.

Corollary 2.4. *Let (R, \mathfrak{m}) be a regular or Gorenstein ring with $\dim(R) \leq 2$. If M is a reflexive R -module, then $\text{pd}_R(M) = 0$ or $G\text{-dim}_R(M) = 0$.*

Proof. By [4, Theorem 2.2.7] or [6, Theorem 1.27] the projective dimension of M is finite or the G -dimension of M is finite, since R is regular or Gorenstein, respectively. By assumption $\text{depth}(R_{\mathfrak{p}}) \leq 2$ for all $\mathfrak{p} \in \text{Spec}(R)$. If $\text{depth}(R_{\mathfrak{p}}) = 2$, then by Theorem 2.3(2), $\text{pd}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) = 0$ ($G\text{-dim}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) = 0$). Otherwise, $M_{\mathfrak{p}}$ is reflexive by Theorem 2.3(2) and by [4, Exercise 1.4.19] we have

$$\text{depth}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) = \text{depth}_{R_{\mathfrak{p}}}(\text{Hom}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}^*, R_{\mathfrak{p}})) \geq \min\{2, \text{depth}(R_{\mathfrak{p}})\}.$$

So $\text{pd}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) = 0$ ($G\text{-dim}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) = 0$). □

Corollary 2.5. *Let (R, \mathfrak{m}) be a regular or Gorenstein ring of dimension 3. An R -module M is reflexive if and only if $\text{pd}_R(M) \leq 1$ or $G\text{-dim}_R(M) \leq 1$ and $\text{pd}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) = 0$ or $G\text{-dim}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) = 0$, respectively, for every prime ideal \mathfrak{p} distinct from \mathfrak{m} .*

Proof. It is proved along the same lines as Corollary 2.4. □

3. CHARACTERIZATION OF n -GORENSTEIN RINGS

In this section, for $n \geq 2$ we give a characterization of n -Gorenstein rings via Gorenstein dimension of the dual of modules. Before that, we recall some definitions and properties. We follow standard notation and terminology from [7].

Definition 3.1. A Gorenstein ring with $\text{id}_R(R)$ at most n is called n -Gorenstein.

Definition 3.2. A submodule T of an R -module N is said to be a pure submodule if $0 \rightarrow A \otimes_R T \rightarrow A \otimes_R N$ is exact for all R -modules A , or equivalently, if $\text{Hom}_R(A, N) \rightarrow \text{Hom}_R(A, N/T) \rightarrow 0$ is exact for all finitely presented R -modules A .

An exact sequence $0 \rightarrow T \rightarrow N \rightarrow N/T \rightarrow 0$ (or $0 \rightarrow T \rightarrow N$) is said to be pure exact if T is a pure submodule of N . An R -module M is said to be pure injective if for every pure exact sequence $0 \rightarrow T \rightarrow N$ of R -modules, the induced sequence $\text{Hom}_R(N, M) \rightarrow \text{Hom}_R(T, M) \rightarrow 0$ is exact.

The following lemma and theorem improve results due to Belshoff [3], where he studied the case $n = 2$. Now we generalize them for $n \geq 2$.

Lemma 3.3. *Let N be an R -module and let $n \geq 2$ be an integer. Then $\text{id}_R(N) \leq n$ provided that $\text{Ext}_R^{n-1}(M^*, N) = 0$ for every finitely generated R -module M .*

Proof. Let

$$E^\bullet: 0 \rightarrow N \rightarrow E^0 \xrightarrow{\alpha^1} E^1 \xrightarrow{\alpha^2} E^2 \xrightarrow{\alpha^3} \dots$$

be an injective resolution of N . Since the Hom evaluation morphism

$$\theta_{MRE^\bullet}: M \otimes_R \text{Hom}_R(R, E^\bullet) \longrightarrow \text{Hom}_R(\text{Hom}_R(M, R), E^\bullet)$$

is an isomorphism of complexes (see [5]) and $\text{Ext}_R^{n-1}(M^*, N) = 0$, so the $(n-1)$ -st cohomology module of $M \otimes E^\bullet$ is zero. Therefore $0 \rightarrow M \otimes_R \text{Im}(\alpha^n) \rightarrow M \otimes_R E^n$ is exact. This means that $\text{Im}(\alpha^n)$ is a pure submodule of the injective module E^n , and now by [7, Lemma 9.1.5], $\text{Im}(\alpha^n)$ is injective and we get the assertion. \square

It is straightforward that for a Gorenstein ring R , the following statements hold.

- (i) If $\dim(R) = 0$, then all R -modules are reflexive,
- (ii) If $\dim(R) = 1$, then all torsionless R -modules are reflexive.

Now we are ready to give a characterization of n -Gorenstein rings via Gorenstein dimension of the dual of modules.

Theorem 3.4. *For any integer $n \geq 2$, R is an n -Gorenstein ring if and only if for every finitely generated R -module M , $\text{G-dim}_R(M^*) \leq n - 2$.*

Proof. Let R be an n -Gorenstein ring and let M be a finitely generated R -module. By [13, Corollary 1.5], M^* is reflexive. It is straightforward that $(M_{\mathfrak{p}}^*)$ is a reflexive $R_{\mathfrak{p}}$ -module, for every prime ideal \mathfrak{p} of R . By Theorem 2.3 we have $\text{G-dim}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}^*) \leq n - 2$, since $\dim(R_{\mathfrak{p}}) \leq n$. According to [6, Proposition 1.15] $\text{G-dim}_R(M^*) \leq n - 2$. Conversely, let M be a finitely generated R -module. By hypothesis $\text{G-dim}_R(M^*)$ is finite, so we have

$$n - 2 \geq \text{G-dim}_R(M^*) = \sup\{i: \text{Ext}_R^i(M^*, R) \neq 0\},$$

therefore $\text{Ext}_R^{n-1}(M^*, R) = 0$. Now by Lemma 3.3 we conclude that $\text{id}_R(R) \leq n$, so R is an n -Gorenstein ring. \square

In [7, Theorem 9.1.11], Enochs and Jenda showed that the property of being n -Gorenstein imposes nice conditions on the homological properties of modules over such rings. In the following we improve Enochs and Jenda's result.

Corollary 3.5. *Let $n \geq 2$ be an integer. For a ring R the following statements are equivalent.*

- (1) R is n -Gorenstein.
- (2) $G\text{-dim}_R(M^*) \leq n - 2$ for all R -modules M .
- (3) $\text{id}_R(F) \leq n$ for all flat R -modules F .
- (4) $\text{id}_R(P) \leq n$ for all projective R -modules P .
- (5) $\text{fd}_R(E) \leq n$ for all injective R -modules E .
- (6) $\text{pd}_R(E) \leq n$ for all injective R -modules E .

Proof. By Theorem 3.4, (1) and (2) are equivalent and by [7, Theorem 9.1.11], (1), (3), (4), (5) and (6) are equivalent. □

4. k -TORSIONLESS MODULES

In [10], Mašek defined the k -torsionless modules for $k \geq 0$. Indeed he gave a generalization of the torsionless and reflexive modules, i.e., torsionless modules are 1-torsionless and reflexive modules are 2-torsionless. The first result of this section is to give a generalization of [4, Proposition 1.4.1] for k -torsionless modules. As an application we show that the class of maximal Cohen-Macaulay modules and the class of k -torsionless modules are equivalent over Gorenstein local ring with dimension k . Finally, we show that every module over a k -Gorenstein ring has a k -torsionless cover.

Definition 4.1. Let M be a module, and let

$$(\pi): P_1 \xrightarrow{u} P_0 \xrightarrow{f} M \longrightarrow 0$$

be a projective presentation of M . The Auslander dual, $D(M)$, of M is defined as

$$D(M) = \text{coker}(u^*: P_0^* \longrightarrow P_1^*),$$

in other words, dualizing (π) we get an exact sequence

$$(\pi^*): 0 \longrightarrow M^* \xrightarrow{f^*} P_0^* \xrightarrow{u^*} P_1^* \longrightarrow D(M) \longrightarrow 0.$$

Clearly, $D(M)$ depends on which projective presentation (π) is used in the definition. In [10], Mašek proved the uniqueness of $D(M)$ up to projective equivalence. Moreover, Mašek proved that for an R -module M and natural R -homomorphism $\delta_M: M \rightarrow M^{**}$ we have

$$\ker(\delta_M) \cong \text{Ext}_R^1(D(M), R), \quad \text{coker}(\delta_M) \cong \text{Ext}_R^2(D(M), R).$$

In addition, $\text{Ext}_R^i(D(M), R) \cong \text{Ext}_R^{i-2}(M^*, R)$, $\forall i \geq 3$. The following definition is from [10].

Definition 4.2. A module M is k -torsionless if $\text{Ext}_R^i(D(M), R) = 0$, $\forall i = 1, \dots, k$. So 1-torsionless is torsionless, and 2-torsionless is reflexive. For $k \geq 3$, M is k -torsionless if M is reflexive and $\text{Ext}_R^i(M^*, R)$, $\forall i = 1, \dots, k - 2$.

Note that M is k -torsionless if and only if $M_{\mathfrak{p}}$ is k -torsionless over $R_{\mathfrak{p}}$ for all $\mathfrak{p} \in \text{Spec}(R)$.

Definition 4.3. An R -module M possesses property (\tilde{S}_k) if

$$\text{depth}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \geq \min\{k, \text{depth}(R_{\mathfrak{p}})\}, \forall \mathfrak{p} \in \text{Spec}(R).$$

Proposition 4.4. Let M be an R -module such that $G\text{-dim}_R(M) < \infty$ and let $k \geq 0$ be an integer. Then the following statements are equivalent.

- (i) M is k -torsionless.
- (ii) M possesses property (\tilde{S}_k) .

Proof. By [10, Theorem 42], (i) and (ii) are equivalent if G -dimension of M is locally finite. On the other hand, by [2, Corollary 6.3.4] “ G -dimension of M is finite” is equivalent to “ G -dimension of M is locally finite”. \square

The following theorem is one of the main results of this section. It gives a generalization of [4, Proposition 1.4.1] for k -torsionless modules.

Theorem 4.5. Let R be a ring and let M be an R -module with $G\text{-dim}_R(M) < \infty$. Then M is k -torsionless if and only if

- (i) $M_{\mathfrak{p}}$ is k -torsionless for $\mathfrak{p} \in \text{Spec}(R)$ with $\text{depth}(R_{\mathfrak{p}}) \leq k - 1$, and
- (ii) $\text{depth}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \geq k$ for $\mathfrak{p} \in \text{Spec}(R)$ with $\text{depth}(R_{\mathfrak{p}}) \geq k$.

Furthermore, by the Auslander-Bridger formula, M is k -torsionless if and only if $M_{\mathfrak{p}}$ is k -torsionless for $\mathfrak{p} \in \text{Spec}(R)$ with $\text{depth}(R_{\mathfrak{p}}) \leq k - 1$, and $G\text{-dim}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \leq \text{depth}(R_{\mathfrak{p}}) - k$ for $\mathfrak{p} \in \text{Spec}(R)$ with $\text{depth}(R_{\mathfrak{p}}) \geq k$.

Proof. Assume that M is a k -torsionless R -module, then (i) is straightforward and by Proposition 4.4, M possesses property (\tilde{S}_k) . So (ii) holds.

Conversely, by Proposition 4.4, it suffices to show that M satisfies property (\tilde{S}_k) , since $G\text{-dim}_R(M) < \infty$. If $\text{depth}(R_{\mathfrak{p}}) \geq k$, then by (ii), $\text{depth}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \geq k$ and so $\text{depth}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \geq \min\{k, \text{depth}(R_{\mathfrak{p}})\}$. Otherwise by (i), $M_{\mathfrak{p}}$ is a k -torsionless $R_{\mathfrak{p}}$ -module, hence $M_{\mathfrak{p}}$ possesses property (\tilde{S}_k) . So M is k -torsionless. \square

Corollary 4.6. *Let R be a Gorenstein local ring of dimension k and let M be a nonzero R -module. Then the following statements are equivalent.*

- (i) M is k -torsionless.
- (ii) $G\text{-dim}_R(M) = 0$.
- (iii) M is maximal Cohen-Macaulay.

Proof. (i) implies (ii) by Theorem 4.5.

(ii) \Rightarrow (iii): Assume that $G\text{-dim}_R(M) = 0$, so by the Auslander-Bridger formula the assertion holds.

(iii) \Rightarrow (i): Suppose that M is a maximal Cohen-Macaulay R -module, so $M_{\mathfrak{p}}$ is a maximal Cohen-Macaulay $R_{\mathfrak{p}}$ -module for all $\mathfrak{p} \in \text{Supp}(M)$. Therefore M possesses property (\tilde{S}_k) and hence by Proposition 4.4, M is k -torsionless. \square

In the following we study the covering properties of the class of k -torsionless modules. This result improves [3, Theorem 2.2].

Definition 4.7. Let \mathcal{X} be the class of finitely generated k -torsionless R -modules. An \mathcal{X} -precover (it will be called a k -torsionless precover) of a finitely generated R -module M is defined to be an R -homomorphism $\varphi: C \rightarrow M$, for some $C \in \mathcal{X}$ such that for any R -homomorphism $f: D \rightarrow M$ where $D \in \mathcal{X}$, there is a homomorphism $g: D \rightarrow C$ such that $\varphi g = f$. An \mathcal{X} -precover $\varphi: C \rightarrow M$ is called an \mathcal{X} -cover (it will be called a k -torsionless cover) if whenever $g: C \rightarrow C$ is such that $\varphi g = \varphi$, then g is an automorphism of C .

It is known that a projective precover of a module M always exists and when the ring R has the property that the direct limit of projective modules is projective, then M has a projective cover [7, Corollary 5.2.7]. Flat covers exist for all modules over any ring [7, Theorem 7.4.4]. In [3, Theorem 2.2], Belshoff proved that over a Gorenstein local ring of dimension at most 2, every finitely generated module has a reflexive cover. The next theorem gives a generalization of this result.

Theorem 4.8. *Let R be a k -Gorenstein ring, and let M be an R -module. Then M has a k -torsionless cover $C \rightarrow M$.*

Proof. By [7, Theorem 11.6.9], M has a Gorenstein projective cover $C \rightarrow M$ and C is finitely generated. It follows from Corollary 4.6 that $C \rightarrow M$ is the k -torsionless cover of M . \square

In [8, Corollary 2.6] and [9], Huneke and Wiegand proved the following result: *Let R be a complete intersection ring and let M and N be nonzero R -modules such that $\text{Tor}_i^R(M, N) = 0$ for all $i \geq 1$. If $M \otimes_R N$ is maximal Cohen-Macaulay, then so are M and N .*

In the following, we provide necessary and sufficient conditions which lead the tensor product of k -torsionless modules to be k -torsionless.

Theorem 4.9. *Let R be a complete intersection ring with $\dim(R) = k$ and let M and N be nonzero R -modules such that $\mathrm{Tor}_i^R(M, N) = 0$ for all $i \geq 1$. Then $M \otimes_R N$ is k -torsionless if and only if M and N are k -torsionless.*

Proof. “ \Rightarrow ” Assume that $M \otimes_R N$ is k -torsionless. By Corollary 4.6, we get that $M \otimes_R N$ is maximal Cohen-Macaulay. Now by [8, Corollary 2.6], M and N are maximal Cohen-Macaulay, so by Corollary 4.6, M and N are k -torsionless.

“ \Leftarrow ” Let $\mathfrak{p} \in \mathrm{Spec}(R)$. Then M possesses property (\widetilde{S}_k) , since M is k -torsionless. So we have

$$\begin{aligned} \mathrm{depth}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} N_{\mathfrak{p}}) &= \mathrm{depth}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) + \mathrm{depth}_{R_{\mathfrak{p}}}(N_{\mathfrak{p}}/(\mathfrak{p}R_{\mathfrak{p}})N_{\mathfrak{p}}) \\ &\geq \mathrm{depth}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \\ &\geq \min\{k, \mathrm{depth}(R_{\mathfrak{p}})\}. \end{aligned}$$

Therefore $M \otimes_R N$ possesses property (\widetilde{S}_k) and then by Proposition 4.4, the assertion is proved. \square

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