David Saunders

Projective metrizability in Finsler geometry

Communications in Mathematics, Vol. 20 (2012), No. 1, 63--68

Persistent URL: http://dml.cz/dmlcz/143081

Communications in Mathematics 20 (2012) 63–68

Projective metrizability in Finsler geometry

David Saunders

Abstract. The projective Finsler metrizability problem deals with the question whether a projective-equivalence class of sprays is the geodesic class of a (locally or globally defined) Finsler function. This paper describes an approach to the problem using an analogue of the multiplier approach to the inverse problem in Lagrangian mechanics.

1 Introduction

Let M be a manifold of class C^∞ which is Hausdorff, second-countable and connected; let $\tau\colon T^\circ M \to M$ denote its slit tangent bundle; let (x^i) be local coordinates coresponding to some chart on M, and let (x^i, y^i) be the corresponding fibred coordinates on $T^\circ M$.

A Finsler function [1] is a smooth map $F\colon T^\circ M \to \mathbb{R}$ which is positive, *positively homogeneous* so that $F(kv) = kF(v)$ for $v \in T^\circ M$ whenever $k \in \mathbb{R}$, $k > 0$, and *strongly convex* so that at each point of $T^\circ M$ the matrix

$$g_{ij} = \tfrac{1}{2}\frac{\partial^2(F^2)}{\partial y^i\,\partial y^j}$$

is positive definite. Each Finsler function F gives rise to a variational problem on M of a special kind, where if $\gamma\colon (a,b) \to M$ is an extremal (in other words, a geodesic) then so is $\gamma \circ \phi$ where $\phi\colon (a,b) \to (a,b)$ with $\phi'(t) > 0$.

On the other hand, a spray [5] is a vector field Γ on $T^\circ M$ which is *second-order*, so that $S(\Gamma) = \Delta$ where S is the almost tangent structure on $T^\circ M$, and which is also *homogeneous*, so that $[\Delta, \Gamma] = \Gamma$ where Δ is the vector field on $T^\circ M$ given by the restriction of the dilation field on the tangent manifold TM. Locally

$$\Gamma = y^i\frac{\partial}{\partial x^i} - 2\Gamma^i\frac{\partial}{\partial y^i}$$

2010 MSC: 53C60

Key words: Finsler function, spray, projective equivalence, geodesic path, projective metrizability, Hilbert form

for some local functions Γ^i which are positively homogeneous of degree 2. Two sprays Γ_1, Γ_2 are said to be projectively related if $\Gamma_1 - \Gamma_2 = \alpha\Delta$ for some function α.

Every Finsler function F gives rise to a projective class of sprays in the following way. The *Hilbert form* of F is the 1-form $\theta_F = S(\mathrm{d}F)$ given locally by

$$\theta_F = \frac{\partial F}{\partial y^i}\mathrm{d}x^i$$

and having the property that if $\gamma\colon (a,b) \to M$ is a geodesic of F then $\gamma'\colon (a,b) \to T^\circ M$ is an integral curve of a spray $\Gamma \in \ker \mathrm{d}\theta_F$. Furthermore, if $\gamma \circ \phi$ is a reparametrized geodesic then $(\gamma \circ \phi)'$ is an integral curve of a projectively related spray $\Gamma - \alpha\Delta \in \ker \mathrm{d}\theta_F$, and indeed

$$\ker \mathrm{d}\theta_F = \langle \Gamma, \Delta \rangle \, .$$

The *projective metrizability problem* is about the converse question. Given a projective class $\{\Gamma\}$ of sprays on $T^\circ M$, when are these sprays derived from a Finsler function F on $T^\circ M$, either locally or globally? Here, 'locally' means on $T^\circ U$ where $v \in T^\circ M$ and U is an open neighbourhood of $\tau(v)$. There are several approaches to this problem; we consider only the *multiplier approach* as an analogue of a similarly-named approach to the inverse problem in Lagrangian mechanics (see [4] for a recent survey of this latter problem). We also restrict attention to $\dim M \geq 3$.

This paper is based on a talk given by the author at the satellite thematic session 'Geometric Methods in Calculus of Variations' of the 6th European Congress of Mathematics in Kraków, July 2012, and reports on joint work with Mike Crampin and Tom Mestdag [2][3].

2 The comparison with Lagrangian mechanics

Lagrangian mechanics, in the time-independent case, considers a function L on the tangent manifold TM, and the corresponding local Euler-Lagrange equations

$$\frac{\partial L}{\partial x^j} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial y^j}\,;$$

by writing $z^i = \dot{y}^i = \ddot{x}^i$ the total derivative $\mathrm{d}/\mathrm{d}t$ on the right-hand side may be replaced to give the explicit formulation

$$\frac{\partial L}{\partial x^j} = z^i \frac{\partial^2 L}{\partial y^i \, \partial y^j}\, .$$

If the Hessian matrix $h_{ij} = \partial L/\partial y^i \, \partial y^j$ is regular then this equation may be solved locally for the second derivatives z^i, and there is a unique vector field Γ on TM satisfying $S(\Gamma) = \Delta$ and with the property that if γ is a solution of the Euler-Lagrange equations (an extremal of the variational problem defined by L) then γ' is an integral curve of Γ.

The inverse problem of Lagrangian mechanics is to start with a vector field Γ satisfying $S(\Gamma) = \Delta$, and to determine whether Γ arises from a Lagrangian in this way. Any such vector field may again be written locally as

$$\Gamma = y^i \frac{\partial}{\partial x^i} - 2\Gamma^i \frac{\partial}{\partial y^i}$$

(without, of course, any homogeneity condition on the functions Γ^i), and any integral curve of Γ will be the derivative of a curve in M satisfying the second-order equation $z^i + 2\Gamma^i = 0$. Comparing this with the Euler-Lagrange equations $z^i h_{ij} = \partial L/\partial x^j$ for a possible Lagrangian L shows the importance of the regularity of the multiplier matrix h_{ij} in the study of this problem.

3 Positivity and strong convexity

The projective metrizability problem for Finsler geometry is, on the face of it, quite similar to the inverse problem of Lagrangian mechanics. A spray is a vector field on $T^\circ M \subset TM$ of the required form, and a Finsler function may be regarded as a Lagrangian. The difference is that a Finsler function is required to be positively homogeneous, and so its Hessian matrix can never be regular; indeed

$$y^j \frac{\partial^2 F}{\partial y^i\, \partial y^j} = 0\,.$$

We shall, though, need some kind of regularity, and we can see how to approach this by writing

$$h_{ij} = \frac{\partial^2 F}{\partial y^i\, \partial y^j}\,, \qquad g_{ij} = \tfrac{1}{2}\frac{\partial^2 (F^2)}{\partial y^i\, \partial y^j} = h_{ij} F + \frac{\partial F}{\partial y^i}\frac{\partial F}{\partial y^j}\,.$$

Define h_{ij} to be *positive quasidefinite* if $h_{ij}(y)v^i v^j \geq 0$, with equality only when $v = \lambda y$; say that a function F on $T^\circ M$ is a *pseudo-Finsler function* if it is positively homogeneous and if its Hessian h_{ij} is positive quasidefinite. The following result is essentially Theorem 1 of [2].

Theorem 1. *If F is a pseudo-Finsler function on $T^\circ M$ then locally there is a Finsler function $\tilde{F}$ such that $F - \tilde{F}$ is a total derivative, so that F and $\tilde{F}$ satisfy the same Euler-Lagrange equations and therefore have the same geodesics. If in addition F is positive then g_{ij} is positive definite, so that F is itself a Finsler function. If F is absolutely homogeneous, so that $F(kv) = |k|F(v)$ for any $k \neq 0$ rather than only for $k > 0$, then F is necessarily positive, so that again it is a Finsler function.*

4 Projective classes of sprays

The projective metrizability problem considers a projective class $\{\Gamma\}$ of sprays, and asks whether there is a corresponding Finsler function F. (Given F, one may select a distinguished spray from the class by requiring $\Gamma(F) = 0$; this gives rise to a different inverse problem, starting with a single spray, which we do not consider here.)

We approach this problem by adapting a technique which has been used to study the inverse problem in Lagrangian mechanics. Every spray on $T^\circ M$ gives rise to a nonlinear connection on τ with horizontal projector

$$H_\Gamma = \tfrac{1}{2}(I - \mathcal{L}_\Gamma S) = \mathrm{d}x^i \otimes \left(\frac{\partial}{\partial x^i} - \frac{\partial \Gamma^j}{\partial y^i}\frac{\partial}{\partial y^j}\right);$$

the connection allows us to define the horizontal lift $X^{\mathrm{h}} = H_\Gamma(X)$ of a vector field X along τ (that is, of a section of the pull-back bundle $\tau^* TM \to T^\circ M$). We may also use the almost tangent structure to define the vertical lift $X^{\mathrm{v}} = S(X)$; in coordinates, if $X = X^i \partial/\partial x^i$ where X^i are locally defined functions on $T^\circ M$ then

$$X^{\mathrm{h}} = X^i \left(\frac{\partial}{\partial x^i} - \frac{\partial \Gamma^j}{\partial y^i} \frac{\partial}{\partial y^j} \right), \qquad X^{\mathrm{v}} = X^i \frac{\partial}{\partial y^i} .$$

We now define the *dynamical covariant derivative* ∇ and the *Jacobi endomorphism* Φ acting on a vector field X along τ by

$$[\Gamma, X^{\mathrm{h}}] = (\nabla X)^{\mathrm{h}} + (\Phi X)^{\mathrm{v}} , \qquad [\Gamma, X^{\mathrm{v}}] = -X^{\mathrm{h}} + (\nabla X)^{\mathrm{v}} .$$

With these tools at hand, we can now state a result which is essentially Theorem 2 of [2].

Theorem 2. *Suppose given a projective class of sprays. If, in a contractible chart, a positive quasidefinite matrix of functions h_{ij} satisfies the Helmholtz conditions*

$$h_{ji} = h_{ij} , \qquad \frac{\partial h_{ij}}{\partial y^k} = \frac{\partial h_{ik}}{\partial y^j} , \qquad h_{ij} y^j = 0$$

and

$$(\nabla h)_{ij} = 0 , \qquad h_{ij} \Phi^k_j = h_{kj} \Phi^k_i ,$$

where ∇h and Φ^k_j are the dynamical covariant derivative and Jacobi endomorphism of any spray in the class, then there is a local pseudo-Finsler function F with Euler-Lagrange equations satisfied by the geodesics of the sprays.

It follows from Theorem 1 that, when these conditions are satisfied, there is a local Finsler function with Euler-Lagrange equations satisfied by the geodesics of the sprays.

5 Global aspects

The result of Theorem 2 has been given in coordinates and is essentially local, although it is valid for complete fibres (it is 'y-global' in the terminology of Finsler geometry). To consider the existence of a pseudo-Finsler function globally on $T^\circ M$, we use the techniques of Čech cohomology.

If $\{U_\lambda\}$ is an open cover of M, then we say that $\{U_\lambda\}$ is a *good cover* if all nonempty finite intersections of the sets U_λ are contractible. It may be shown that if there is a spray defined on M then M admits a good cover by the domains of coordinate charts ([2], Appendix B); the proof uses Whitehead's result on the existence of geodesically convex sets [6][7].

Let $\{U_\lambda\}$ be such a cover. Given a projective class of sprays and a (0,2) tensor field h along τ whose components in each chart satisfy the conditions of Theorem 2, there is a pseudo-Finsler function F_λ defined on each U_λ. If $U_\lambda \cap U_\mu$ is nonempty then

$$F_\lambda - F_\mu = y^i \frac{\partial \phi_{\lambda\mu}}{\partial x^i}$$

for some function $\phi_{\lambda\mu}$ defined on $T^\circ(U_\lambda \cap U_\mu)$ which is unique to within a constant. Also, if $U_\lambda \cap U_\mu \cap U_\nu$ is nonempty then

$$\phi_{\mu\nu} - \phi_{\lambda\nu} + \phi_{\lambda\mu} = k_{\lambda\mu\nu}$$

is constant on the connected set $T^\circ(U_\lambda \cap U_\mu \cap U_\nu)$, and if $U_\kappa \cap U_\lambda \cap U_\mu \cap U_\nu$ is nonempty then

$$k_{\lambda\mu\nu} - k_{\kappa\mu\nu} + k_{k\lambda\nu} - k_{\kappa\lambda\mu} = 0$$

on $T^\circ(U_\kappa \cap U_\lambda \cap U_\mu \cap U_\nu)$. We see from this that the obstruction to the construction of a global pseudo-Finsler function lies in the second Čech cohomology group of the cover, and as we have taken a good cover this is isomorphic to the de Rham cohomology group $H^2(M)$. The following result is essentially the second part of Theorem 3 of [2].

Theorem 3. *Suppose given a projective class of sprays. If there is a (0,2) tensor field h along τ such that*

- *in each chart of a good atlas the components h_{ij} satisfy the Helmholtz conditions and are positive quasidefinite, and*
- $H^2(M) = 0$,

then there is a global pseudo-Finsler function F with Euler-Lagrange equations satisfied by the geodesics of the sprays, and each point of $T^\circ M$ has a neighbourhood on which there is a corresponding local Finsler function.

The example of the spray

$$\Gamma = y^1 \frac{\partial}{\partial x^1} + y^2 \frac{\partial}{\partial x^2} + y^3 \frac{\partial}{\partial x^3} + \sqrt{(y^1)^2 + (y^2)^2 + (y^3)^2} \left(y^1 \frac{\partial}{\partial y^2} - y^2 \frac{\partial}{\partial y^1} \right)$$

defined on $T^\circ\mathbb{R}^3$, which is in the projective class of sprays arising from the global pseudo-Finsler function

$$F = \sqrt{(y^1)^2 + (y^2)^2 + (y^3)^2} + \tfrac{1}{2}(x^2 y^1 - x^1 y^2)\,,$$

shows that there need not be a global Finsler function giving rise to the projective class.

6 Multiplier tensors and 2-forms

In a global formulation, the multiplier matrix h_{ij} is the coordinate representation of a symmetric (0,2) tensor field h along the projection $T^\circ M \to M$ (that is, locally $h = h_{ij}\, \mathrm{d}x^i \otimes \mathrm{d}x^j$). This tensor field is closely related to a 2-form on $T^\circ M$ which, given the existence of a Finsler function F, will be the differential $d\theta_F$ of its Hilbert form. We can therefore translate the conditions on h given above into conditions on the 2-form; these results are essentially Theorems 5 and 6 of [3].

Theorem 4. *Suppose given a spray Γ and a 2-form ω on $T^\circ M$, and let $\{\mathrm{d}x^i\,, \phi^i = H_\Gamma(\mathrm{d}y^i)\}$ be a local basis of 1-forms on $T^\circ M$. If*

- $\langle \Gamma, \Delta \rangle \subset \ker\omega$ *and* $\mathcal{L}_\Gamma \omega = 0$,
- $\omega(V_1, V_2) = 0$ *if V_1, V_2 are vertical, and*
- $\mathrm{d}\omega(H, V_1, V_2) = 0$ *if V_1, V_2 are vertical and H horizontal*

then in any chart we may write

$$\omega = h_{ij}\mathrm{d}x^i \wedge \phi^j$$

where h_{ij} satisfies the Helmholtz conditions. It is also the case that a 2-form ω satisfying the stated conditions must be closed.

It follows that if the matrix h_{ij} obtained above is positive quasidefinite on a contractible chart then there will be a local pseudo-Finsler function for Γ.

Theorem 5. *Suppose given a projective class of sprays. If there is a 2-form ω satisfying the conditions of Theorem 4 for any spray in the class, and if the functions h_{ij} are positive quasidefinite, and if furthermore $H^2(M) = 0$, then there is a global pseudo-Finsler function F with Euler-Lagrange equations satisfied by the geodesics of the sprays.*

Acknowledgement

The author wishes to acknowledge the support of grant no. 201/09/0981 for Global Analysis and Geometry of Fibred Spaces from the Czech Science Foundation, and also the joint IRSES project GEOMECH (EU FP7, nr 246981).

References

[1] D. Bao, S.-S. Chern, Z. Shen: *An Introduction to Riemann-Finsler Geometry*. Springer (2000).

[2] M. Crampin, T. Mestdag, D. J. Saunders: The multiplier approach to the projective Finsler metrizability problem. *Diff. Geom. Appl.* 30 (2012) 604–621.

[3] M. Crampin, T. Mestdag, D. J. Saunders: Hilbert forms for a Finsler metrizable projective class of sprays. Diff. Geom. Appl., to appear.

[4] O. Krupková, G. E. Prince: Second order ordinary differential equations in jet bundles and the inverse problem of the calculus of variations. In: D. Krupka and D. J. Saunders: *Handbook of Global Analysis*. Elsevier (2008) 837–904.

[5] Z. Shen: *Differential Geometry of Spray and Finsler Spaces*. Kluwer (2001).

[6] J. H. C. Whitehead: Convex regions in the geometry of paths. *Quart. J. Math.* 3 (1932) 33–42.

[7] J. H. C. Whitehead: Convex regions in the geometry of paths – addendum. *Quart. J. Math.* 4 (1933) 226–227.

Author's address:
Department of Mathematics, Faculty of Science, The University of Ostrava, 30. dubna 22, 701 03 Ostrava, Czech Republic

E-mail: `david@symplectic.demon.co.uk`

Received: 10 September, 2012