Kybernetika

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Kybernetika, Vol. 48 (2012), No. 6, 1266--1288

Persistent URL: http://dml.cz/dmlcz/143130

KYBERNETIKA — VOLUME *48* (2012), NUMBER 6, PAGES 1266–1288

A GEOMETRIC ALGORITHM FOR THE OUTPUT FUNCTIONAL CONTROLLABILITY IN GENERAL MANIPULATION SYSTEMS AND MECHANISMS

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In this paper the control of robotic manipulation is investigated. Manipulation system analysis and control are approached in a general framework. The geometric aspect of manipulation system dynamics is strongly emphasized by using the well developed techniques of geometric multivariable control theory. The focus is on the (functional) control of the crucial outputs in robotic manipulation, namely the reachable internal forces and the rigid–body object motions. A geometric control procedure is outlined for decoupling these outputs and for their perfect trajectory tracking. The control of robotic manipulation is investigated. These are mechanical structures more complex than conventional serial–linkage arms. The robotic hand with possible inner contacts is a paradigm of general manipulation systems. Unilateral contacts between mechanical parts make the control of manipulation system quite involved. In fact, contacts can be considered as unactuated (passive) joints. The main goal of dexterous manipulation consists of controlling the motion of the manipulated object along with the grasping forces exerted on the object. In the robotics literature, the general problem of force/motion control is known as "hybrid control". This paper is focused on the decoupling and functional controllability of contact forces and object motions. The goal is to synthesize a control law such that each output vector, namely the grasping force and the object motion, can be independently controlled by a corresponding set of generalized input forces. The functional force/motion controllability is investigated. It consists of achieving force and motion tracking with no error on variables transients. The framework used in this paper is the geometric approach to the structural synthesis of multivariable systems.

Keywords: geometric approach, manipulators, functional controllability

Classification: 93D09, 19L64, 70Q05, 14L24

1. INTRODUCTION

This paper deals with general manipulation systems. These are mechanical structures more complex than conventional serial–linkage arms. The coordinated use of multiple fingers in a robot hand or, similarly, of multiple arms in cooperating tasks; the use of inner links of a robot arm (or finger) to hold an object and the exploitation of parallel mechanical structures are all examples of non–conventional usage of mechanisms for manipulation. Robotic hands can be considered as paradigms of general manipulation systems. The presence of unilateral contact phenomena between different parts of the mechanical structure is a special feature of manipulation systems. Mechanical contacts between the robotic parts and the environment can be viewed as actuated (passive) joints and, for this reason, they make manipulation system control quite involved. The analysis of dynamics and the control of manipulation systems become more complex when it is not possible to control contact forces in all directions. This usually happens when the number $\mathbf{q}$ of DoF's of the robotic device is smaller than t, the dimension of the contact force space. In [14], such a case is defined as "defective grasp". The importance of defective grasps has been underlined for the first time in "whole–hand" manipulation [17], where all links of the hand may be exploited to manipulate objects (see Figure 1). In industrial applications, kinematic defectivity is a common factor of almost all grippers used to grasp industrial parts. Consider, for instance, the simple mechanism in Figure 3 of Section 5. It will be shown that it exhibits a defective grasp. The main goal of dexterous manipulation tasks consists of controlling the motion of the manipulated object along with the grasping forces exerted on the object. In the robotics literature, the general problem of force/motion control is known as "hybrid control". For a broad overview on these topics, the reader is referred to [12, 18] and the references therein. In force/motion control, a very interesting aspect is the decoupling control. Roughly speaking, the multi–input, multi–output manipulation system is decoupled if each output vector, namely the grasping force and the object position vectors, can be *independently* controlled by a corresponding set of generalized input forces. Such a structure is desirable in a considerable number of advanced applications, including micromanipulation of tissues in surgery and in laparoscopy or assembly and manipulation of non–rigid (rubber or plastic) parts in industry. In all the examples above, it could be very dangerous to increase the squeezing force while giving rise to undesired, even if transient, object motions. Such a problem is common to all those hybrid controllers which do not rank noninteraction as a specific goal. In [7], the authors proved in a geometric setting that it is possible to decouple the object position and the squeezing force control for a wide class of manipulation systems by using a state–space feedback controller. This paper presents a systematic procedure in order to obtain the noninteracting controllability between force and motion and their functional controllability. Here, the noninteraction problem is investigated thoroughly in order to extend previous results to the functional force/motion controllability. Roughly speaking, it consists of achieving force and motion tracking with no error variables transients. To achieve a noninteraction a feedback control law is needed together with a feed-forward regulator. The functional controllability represents a structural property of the system which must be proven. In this paper noninteraction and functional controller is obtained. The relevance of the output functional controllability to manipulation control is justified by the necessity of very fast, loops of force control counteracting the grasp failure caused by possible disturbance actions. The framework used in this paper is the geometric approach to the structural synthesis of multivariable systems. For a broad overview the reader is referred to [3, 19] and the references therein. References [8, 11] and [16] mark progress in the analysis and synthesis of geometric controller for mechanical systems. The force/motion control problem has attracted significant attention over the last decade in the fields of robotic manipulation and mobile manipulators. Approaches exploiting input-output decoupling controllers are found, for instance, in the work [20]. The geometric approach allows very elegant solutions to control problems. Nevertheless, robustness analysis using a linear geometric control offers answers through rank conditions of matrices that are necessary conditions. These conditions are often not constructive ones. Even though the rank conditions offer simple "on-off" conditions, it is also possible to measure the robustness. The work in [10] investigates the geometric and structural characteristics involved in the control of general mechanisms and manipulation systems. These systems consist of multiple cooperating linkages that interact with a reference member of the mechanism (the "object") by means of contacts on any available part of their links. Grasp and manipulation of an object by the human hand is taken as a paradigmatic example for this class of manipulators. Special attention is devoted to the output specification and its controllability. The paper is organized as follows. Section 2 is devoted to the background. Section 3 shows some results on the noninteracting control. The main contribution of this paper is shown in Section 4 in which the functional controllability is presented as a structural property. The formal demonstrations of the results are shown in the appendix. At the end, a section dedicated to the case study presents, together with an example, a general procedure to calculate the geometric structures which are used for the control. The conclusions close the paper.

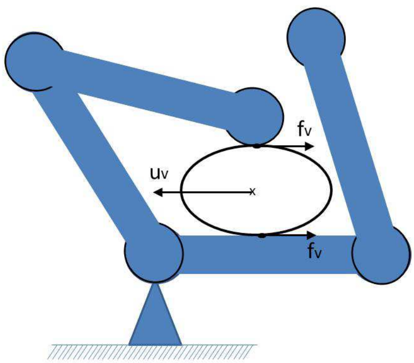

Fig. 1. Defective grasp: $\ker(\mathbf{J}^T) \neq \{0\}$. Contact force $\mathbf{f}_v$ and object position $\mathbf{u}_v$ which are not controllable by joint torques.

2. PRELIMINARIES

The manipulation system dynamics is linearized at an equilibrium configuration. The use of linearized model dynamics in the analysis of general manipulation systems is believed to be a significant advancement with respect to the literature, which is almost solely based on quasi–static models, especially for defective systems, and in fact provides richer results and better insights. For a detailed discussion of dynamics and the derivation of its linearized model the reader is referred to previous works by the authors, [15] and [14]. Notation and some results on the linearized dynamics of general manipulation systems, are summarized in this section. Let $\mathbf{q} \in \Re^q$ be the vector of joint positions, $\tau \in \Re^q$ the vector of joint forces and/or torques, $\mathbf{u} \in \Re^d$ the vector locally describing the position and the orientation of a frame attached to the object and finally $\mathbf{w} \in \Re^d$ the vector of external disturbances acting on the object. Let us further introduce the vector $\mathbf{t} \in \Re^t$ whose components include contact forces and torques. Assume that contact forces arise from a lumped–parameter model of visco–elastic phenomena at the contacts, summarized by the stiffness matrix $\mathbf{K}$ and the damping matrix $\mathbf{B}$. Jacobian matrix $\mathbf{J}$ and the grasp matrix $\mathbf{G}$ are usually defined as the linear maps relating the velocities of the contact points on the links and on the object, to the joint and object velocities, respectively. Besides advanced robotic tasks discussed in the introduction, whereas visco–elastic contact model is mandatory, it might be worthwhile to mention another reason, discussed in [13], for taking into account the visco–elastic contact model. It was shown that if the grasp is hyperstatic, i. e. $\mathsf{ker}(\mathbf{J}^T) \cap \mathsf{ker}(\mathbf{G}) \neq \mathbf{0}$, the rigid–body contact model leaves the nonlinear dynamics undetermined and, consequently, the visco–elastic model of contact interaction becomes mandatory. By the way, notice that kinematic deficiency ($\mathsf{ker}(\mathbf{J}^T) \neq \emptyset$) is a necessary condition for hyperstaticity. Consider a reference equilibrium configuration $(\mathbf{q}, \mathbf{u}, \dot{\mathbf{q}}, \dot{\mathbf{u}}, \tau, \mathbf{t}) = (\mathbf{q}_o, \mathbf{u}_o, \mathbf{0}, \mathbf{0}, \tau_o, \mathbf{t}_o)$, such that $\tau_o = \mathbf{J}^T\mathbf{t}_o$ and $\mathbf{w}_o = -\mathbf{G}\mathbf{t}_o$. In the neighbourhood of such an equilibrium the linearized dynamics of the manipulation system can be written as

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}_\tau \tau' + \mathbf{B}_w \mathbf{w}', \tag{1}$$

where state, input and disturbance vectors are defined as the departures from the reference equilibrium configuration:
$\mathbf{x} = \left[(\mathbf{q} - \mathbf{q}_o)^T \ (\mathbf{u} - \mathbf{u}_o)^T \ \dot{\mathbf{q}}^T \ \dot{\mathbf{u}}^T\right]^T$, $\tau' = \tau - \mathbf{J}^T\mathbf{t}_o$, $\mathbf{w}' = \mathbf{w} + \mathbf{G}\mathbf{t}_o$ and

$$\mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{L}_k & \mathbf{L}_b \end{bmatrix}, \ \mathbf{B}_\tau = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{M}_h^{-1} \\ \mathbf{0} \end{bmatrix}, \ \mathbf{B}_w = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{M}_o^{-1} \end{bmatrix}, \tag{2}$$

where $\mathbf{M}_h$ and $\mathbf{M}_o$ are the inertia matrices of the manipulator and the object, respectively. To simplify notation we will henceforth omit the prime in τ' and $\mathbf{w}'$. Neglecting rolling phenomena at the contacts, assuming a locally isotropic model of visco–elastic phenomena and assuming that local variations of the Jacobian and grasp matrices are small, simple expressions are obtained for $\mathbf{L}_k = -\mathbf{M}^{-1}\mathbf{P}_k$ and $\mathbf{L}_b = -\mathbf{M}^{-1}\mathbf{P}_b$, where $\mathbf{M} = \operatorname{diag}(\mathbf{M}_h, \mathbf{M}_o)$, $\mathbf{P}_k = \mathbf{S}^T\mathbf{K}\mathbf{S}$, $\mathbf{P}_b = \mathbf{S}^T\mathbf{B}\mathbf{S}$, and $\mathbf{S} = [\mathbf{J} \ \ -\mathbf{G}^T]$. According to the lumped visco–elastic model, the local description of the contact force vector is $\mathbf{t}' = \mathbf{t} - \mathbf{t}_o = \mathbf{C}_t\mathbf{x}$ with $\mathbf{C}_t = \left[\mathbf{K}\mathbf{J} \ -\mathbf{K}\mathbf{G}^T \ \mathbf{B}\mathbf{J} \ -\mathbf{B}\mathbf{G}^T\right]$. To our purposes, object, joint positions and forces are of interest as outputs. The corresponding output matrices are, respectively,

$$\mathbf{C}_u = [\ \mathbf{0} \ \ \mathbf{I} \ \ \mathbf{0} \ \ \mathbf{0}\ ], \ \mathbf{C}_q = [\ \mathbf{I} \ \ \mathbf{0} \ \ \mathbf{0} \ \ \mathbf{0}\ ], \ \mathbf{C}_t = \left[\ \mathbf{K}\mathbf{J} \ \ -\mathbf{K}\mathbf{G}^T \ \ \mathbf{B}\mathbf{J} \ \ -\mathbf{B}\mathbf{G}^T\ \right].$$

3. CONTROLLED OUTPUTS AND NONINTERACTING CONTROL

In this paper it has been assumed that contact points do not change. The manipulation is studied in those intervals of time when contact points hold without rolling and/or sliding. Thus, manipulation control goal involves mainly the control of grasp and the tracking of the desired object trajectory. With reference to the first control requirement let us introduce concept of *internal* forces. Usually, forces belonging to the null space of the grasp matrix $\mathbf{G}$ are referred to as “internal forces” (which are contact forces with zero resultant on the object). Such forces enable the robotic device to grasp the object and play a fundamental role in controlling the manipulation task. A suitable control of internal forces allows the manipulation system to counteract the possible grasp failure caused by disturbance actions on the object. Analytically, internal forces are defined as those forces belonging to the null space of the grasp matrix $\mathbf{G}$. In [14] manipulation systems with $\mathsf{ker}(\mathbf{G}) \neq \mathbf{0}$ were defined as *graspable* systems. With reference to object trajectories, rigid–body kinematics plays a particular role in manipulation control. Rigid–body kinematics has been studied in a quasi–static setting in [4] and in terms of unobservable subspaces in [5]. In both cases rigid kinematics was described by the base matrix $\mathbf{\Gamma}$ whose columns form a basis for $\mathsf{ker}\left[\mathbf{J}\ \ -\mathbf{G}^T\right] = \mathsf{im}(\mathbf{\Gamma})$ where

$$\mathbf{\Gamma} = \left[\mathbf{\Gamma}_{qc}^T\ \mathbf{\Gamma}_{uc}^T\right]^T, \quad \text{and} \quad \mathbf{J}\mathbf{\Gamma}_{qc} = \mathbf{G}^T\mathbf{\Gamma}_{uc}. \tag{3}$$

Observe that, for the sake of brevity, it is assumed here that the system is not *redundant*: $\mathsf{ker}(\mathbf{J}) = \{\mathbf{0}\}$ and that it is not *indeterminate*: $\mathsf{ker}(\mathbf{G}^T) = \{\mathbf{0}\}$, see [4] for further details. The column space of $\mathbf{\Gamma}$ consists of coordinated rigid–body motions of the mechanism, for the manipulator ($\mathbf{\Gamma}_{qc}$) and the object ($\mathbf{\Gamma}_{uc}$) components. They do not involve visco–elastic deformations at contacts and can be regarded as low–energy motions. In this sense, they represent the easiest way to move the object. In the following, a special subspace of internal forces and the rigid–body object motions are characterized as output matrices of the linearized dynamics, see Section 2. These outputs, namely $\mathbf{t}'$ and $\mathbf{u}'$ (henceforth $\mathbf{t}$ and $\mathbf{u}$), represent variations of contact force and object position vectors from the relative equilibrium values. Before introducing the controlled outputs, let us recall the concept of *contact–kinematics defectivity*, or briefly *defectivity*. According to [14] and [6], a given grasp is called *contact–kinematics defective* if $\mathsf{ker}(\mathbf{J}^T) \neq \{\mathbf{0}\}$. As pointed out, the grasp *defectivity* deeply affects contact forces and the object motion controllability which, in general, is lost. Figure 1 describes some uncontrollable directions of contact forces $\mathbf{f}_v$ and object motions $\mathbf{u}_v$ for a simple 3–DoF’s defective device.

Recall that whenever the number of joints is lower than the number of elements of the contact force, as in the simple grippers of Figures 1 and 3, it ensues that $\mathsf{ker}(\mathbf{J}^T) \neq \{\mathbf{0}\}$ and the grasp is defective. Although, in the presence of defectivity, contact forces $\mathbf{t}$ and object motions $\mathbf{u}$ loose the output controllability. It was shown in [14] that the output controllability property holds for their projection on the subspace of *reachable internal* forces $\mathbf{t}_i$ and of *rigid–body object* motions $\mathbf{u}_c$. Moreover, if the output vector is chosen by grouping such projections $\mathbf{y} = \left(\mathbf{t}_i^T\ \mathbf{u}_c^T\right)^T$, not only $\mathbf{y}$ is consistent, i. e. output controllable, but it also exhausts the control capability by making the input–output representation of dynamics square. The *reachable internal* contact forces $\mathbf{t}_i$ are defined as the projection of the force vector $\mathbf{t}$ into the null space of $\mathbf{G}$: Then the output matrix is defined as follows

$$\mathbf{e}_{ti} = \mathbf{E}_{ti}\mathbf{x}; \quad \text{with} \quad \mathbf{E}_{ti} = (\mathbf{I} - \mathbf{K}\mathbf{G}^T(\mathbf{G}\mathbf{K}\mathbf{G}^T)^{-1}\mathbf{G})\mathbf{C}_t = \begin{bmatrix} \mathbf{Q}_k & \mathbf{0} & \mathbf{Q}_\beta & \mathbf{0} \end{bmatrix}, \quad (4)$$

where

$$\mathbf{Q}_k = (\mathbf{I} - \mathbf{K}\mathbf{G}^T(\mathbf{G}\mathbf{K}\mathbf{G}^T)^{-1}\mathbf{G})\mathbf{K}\mathbf{J} \quad (5)$$

and

$$\mathbf{Q}_\beta = (\mathbf{I} - \mathbf{B}\mathbf{G}^T(\mathbf{G}\mathbf{B}\mathbf{G}^T)^{-1}\mathbf{G})\mathbf{B}\mathbf{J}. \quad (6)$$

One can remark that $\mathsf{im}(\mathbf{Q_k}) = \mathsf{im}(\mathbf{Q}_\beta)$ under the hypothesis $\mathsf{im}(\mathbf{K}) = \mathsf{im}(\mathbf{B})$ and the *rigid–body* object motions $\mathbf{u}_c$ are defined as the projection of the object displacement $\mathbf{u}$ onto the column space of $\mathbf{\Gamma}_{uc}$:

$$\mathbf{e}_{uc} = \mathbf{E}_{uc}\mathbf{x}, \quad \text{where } \mathbf{E}_{uc} = \mathbf{\Gamma}_{uc}(\mathbf{\Gamma}_{uc}^T\mathbf{\Gamma}_{uc})^{-1}\mathbf{\Gamma}_{uc}^T \left[\mathbf{0}\ \mathbf{I}\ \mathbf{0}\ \mathbf{0}\right]. \quad (7)$$

Notice that to simplify notation, matrices $(\mathbf{Q}^T\mathbf{Q})^{-1}$ and $(\mathbf{\Gamma}_{uc}^T\mathbf{\Gamma})^{-1}$ will be omitted in the following.

3.1. Noninteracting control

In [9] the following decoupling theorem was stated,

Theorem 1. (Noninteraction) Consider the linearized manipulation system of Section 2. If $\mathsf{ker}(\mathbf{G}^T) = \{\mathbf{0}\}$, there exists a stabilizing state–feedback control law, $\tau = \mathbf{F}\mathbf{x} + \tau^*$ and an input partition $\tau^* = \mathbf{U}_{ti}\mathbf{u}_{ti} + \mathbf{U}_{uc}\mathbf{u}_{uc}$ which decouples reachable internal forces $\mathbf{t}_i$ and rigid–body object motions $\mathbf{u}_c$.

Remark 1. Theorem 1 states that a control law and a joint torques partition exist such that, for zero initial conditions each input affects only the relative output.

The geometric concept from which the previous result develops is originally developed in [3] the $\mathcal{S}$*–constrained controllability.* It consists of those state space vectors reachable through trajectories entirely lying in the constraining subspace $\mathcal{S}$. It was shown that, for the aforementioned outputs $\mathbf{t}_i$ and $\mathbf{u}_c$, there exists a decoupling and stabilizing state feedback matrix $\mathbf{F}$, along with two input partition matrices $\mathbf{U}_{ti}$ and $\mathbf{U}_{uc}$ such that, for the following two triples

$$\begin{aligned} &(\mathbf{E}_{ti},\, \mathbf{A} + \mathbf{B}_\tau\mathbf{F},\, \mathbf{B}_\tau\mathbf{U}_{ti}), \\ &(\mathbf{E}_{uc},\, \mathbf{A} + \mathbf{B}_\tau\mathbf{F},\, \mathbf{B}_\tau\mathbf{U}_{uc}), \end{aligned} \quad (8)$$

it holds:

$$\begin{aligned} &\mathcal{R}_{ti} = min\mathcal{I}(\mathbf{A} + \mathbf{B}_\tau\mathbf{F},\, \mathbf{B}_\tau\mathbf{U}_{ti}) \subseteq \mathsf{ker}(\mathbf{E}_{uc}), \\ &\mathbf{E}_{ti}\mathcal{R}_{ti} = \mathsf{im}(\mathbf{E}_{ti}), \end{aligned} \quad (9)$$

$$\begin{aligned} &\mathcal{R}_{uc} = min\mathcal{I}(\mathbf{A} + \mathbf{B}_\tau\mathbf{F},\, \mathbf{B}_\tau\mathbf{U}_{uc}) \subseteq \mathsf{ker}(\mathbf{E}_{ti}), \\ &\mathbf{E}_{uc}\mathcal{R}_{uc} = \mathsf{im}(\mathbf{E}_{uc}). \end{aligned} \quad (10)$$

Here,

$$min\mathcal{I}(\mathbf{A}, \mathsf{im}(\mathbf{B})) = \sum_{i=0}^{n-1} \mathbf{A}^i \mathsf{im}(\mathbf{B}) \tag{11}$$

is the minimum $\mathbf{A}$–invariant subspace containing $\mathsf{im}(\mathbf{B})$ with respect to a general system defined by the triple $(\mathbf{A}, \mathbf{B}, \mathbf{C})$. Moreover, the partition matrices $\mathbf{U}_{uc}$ and $\mathbf{U}_{ti}$ satisfy the following relations

$$\begin{array}{rcl} \mathsf{im}(\mathbf{B}_\tau \mathbf{U}_{uc}) & = & \mathsf{im}(\mathbf{B}_\tau) \cap \mathcal{R}_{uc}, \\ \mathsf{im}(\mathbf{B}_\tau \mathbf{U}_{ti}) & = & \mathsf{im}(\mathbf{B}_\tau) \cap \mathcal{R}_{ti}, \end{array} \tag{12}$$

and the stabilizing matrix $\mathbf{F}$ is such that

$$\begin{array}{rcl} (\mathbf{A} + \mathbf{B}_\tau \mathbf{F})\mathcal{R}_{uc} & \subseteq & \mathcal{R}_{uc}, \\ (\mathbf{A} + \mathbf{B}_\tau \mathbf{F})\mathcal{R}_{ti} & \subseteq & \mathcal{R}_{ti}. \end{array} \tag{13}$$

The decoupling controller is that sketched in Figure 2.

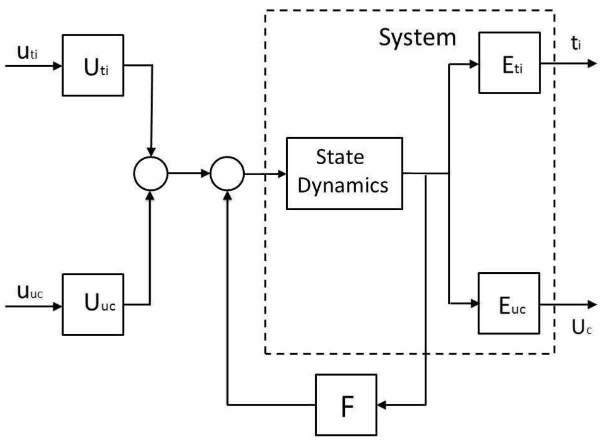

Fig. 2. Force/motion decoupling controller.

4. FORCE/MOTION FUNCTIONAL CONTROLLABILITY

This section is aimed at the analysis of the output functional controllability of manipulation systems. As already pointed out, we are interested in controlling the internal contact forces and the rigid–body object motions to achieve force and motion tracking without transients of error variables. In the whole, in robotic manipulation, the exact trajectory tracking is paramount and this is particularly emphasized by the advanced manipulation tasks recalled in Section 1. It is our belief that noninteraction should be a basic requirement of internal–force and object–motion control, thus the objective of the control becomes twofold and an effort is made to achieve both decoupling and functional controllability of reachable internal forces and rigid body motions in general manipulation systems. To attack the problem, the natural approach is to analyze the constrained output controllability idea, cf. [2] and [3], formalized below.

Definition 1. (Perfect output controllability) Given the triple $(\mathbf{A}, \mathbf{B}, \mathbf{C})$, the output subspace $\mathcal{L}^i$ is said to be perfect output functionally (OF) controllable with respect to ith derivative and with respect to the subspace of states $\mathcal{S}$ if $\mathcal{L}^i = \mathbf{C}\mathcal{S}$ and, for every initial state $\mathbf{x}_0 \in \mathcal{S}$, it is possible, by means of proper bounded and measurable control function, to follow in $\mathcal{L}$ any trajectory arbitrarily given in the class of functions which admit ith derivative with respect to time, while the state evolves into $\mathcal{S}$.

Recall, cf. [2], that the output functional controllability is strictly related to the geometric–type extension of the relative degree for multivariable systems and that each subspace $\mathcal{S}$ satisfying Definition 1 is an $(\mathbf{A}, \mathbf{B})$–controlled invariant. The last observation highlights the relationships between the noninteracting controller and the output functional controllability will help to prove next theorem which states the OF–controllability of general manipulation systems to be decoupled according to the previous section.

Theorem 2. (Output Functional Controllability and Noninteraction) Consider the linearized System (1) with $\mathsf{ker}(\mathbf{G}^T) = \mathbf{0}$. The output subspaces $\mathsf{im}(\mathbf{E}_{ti})$, $\mathsf{im}(\mathbf{E}_{uc})$ are OF controllable with respect to the 1st and 3rd derivative and with respect to the constrained reachable subspaces $\mathcal{R}_{ti}$ and $\mathcal{R}_{uc}$, respectively. Moreover the state–feedback decoupling controller of Section 3.1 (eqs. 8, 12 and 13) makes the system, with outputs $\mathbf{t}_i$ and $\mathbf{u}_c$, noninteracting and OF–controllable.

The proof is reported in Appendix III.

Remark 2. Regarding the functional controllability of rigid–body as object motions, the 3–rd order of derivative means that the output $\mathbf{u}_c$ can perfectly track any desired trajectory $\mathbf{u}_{cd}$ which has a piecewise continuous 3–rd derivative. This is true for all initial states $\mathbf{x}_o$ in $\mathcal{R}_{ti} + \mathcal{R}_{uc}$ and with piecewise continuous control functions $\mathbf{u}_c(t)$. Furthermore, it could be easily shown that order 3 of the rigid–body object motions $\mathbf{u}_c$ is not due to the particular choice of the subspace $\mathcal{R}_{uc}$ but it is an inherent property of the system. It is related to the relative degree of the relationship between the rigid–body object motion and the joint–torques.

In [14] it has been proven that the input–output representation, $\mathbf{y}(s) = \mathbf{G}(s)\tau(s)$, of linearized dynamics of manipulation systems is invertible. In this paper the problem of the output functional controllability is approached after having solved the force/motion decoupling problem by means of a state–feedback controller. The improvement consists in obtaining a more robust controller. If the inversion algorithm fails, the decoupling structure is able to fix at least the possible force/motion coupling problems. Moreover, regarding computational aspects, in general the inversion of a block–diagonal transfer function involves less operations than the inversion of the undecoupled $\mathbf{G}(s)$.

5. CASE STUDY

In this section numerical results are reported for the simple defective gripper pictorially described in Figure 3. It is a planar 3–DoF's Cartesian manipulator and has been chosen

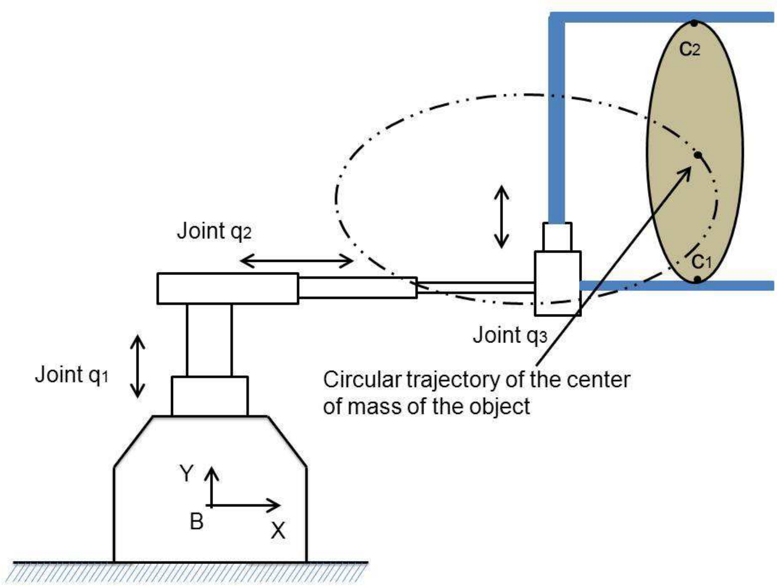

Fig. 3. Planar 3–DoF's Cartesian manipulator. It exhibits a defective ($\ker(\mathbf{J}^T) = \mathbf{0}$) grasp.

in order to show the effectiveness of previous results for industrial grippers. In the base frame B, the contact *centroids*, cf. [4], are $\mathbf{c}_1 = (2,\, 2)$, $\mathbf{c}_2 = (2, 3)$ and object center of mass is $c_b = (2,\, 2.5)$ while the transpose of the Jacobian and the grasp matrix assume the following values

$$\mathbf{J}^T = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}; \quad \mathbf{G} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0.5 & 0 & -0.5 & 0 \end{bmatrix}.$$

The inertia matrices of the object and manipulator along with stiffness and damping matrices at the contacts are assumed to be normalized to the identity matrix. The controlled outputs are (a) the projection $\mathbf{t}_i$ of the contact forces along the 1–dimensional subspace of reachable contact force $\mathsf{im}([0\,1\,0\,-1]^T)$ and (b) the projection of the rigid–body motion in the 2–dimensional subspace of object motions $\mathsf{im}\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ which, since $\mathbf{u} = [\delta\mathbf{x}\ \delta\mathbf{y}\ \delta\theta]^T$, corresponds to translations of the object.

5.1. General procedure

The objective of the control is twofold. First, force and motion control must be decoupled, then the perfect tracking of desired trajectories $\mathbf{t}_{id}$ and $\mathbf{u}_{cd}$ can be achieved. The decoupling controller is pictorially described in Figure 2 and has been synthesized, according to Section 3.1, eqs. (8), (12) and (13). State–feedback matrix $\mathbf{F}$ and input partition matrix $\mathbf{U} = \begin{bmatrix} \mathbf{U}_{ti} & \mathbf{U}_{uc} \end{bmatrix}$ are obtained respectively according to the following procedure:

- Step 1: According to (9), calculate the reachable subspace of the internal contact force with the following expression:

$$\mathcal{R}_{ti} = (\mathbf{E}_{ti}^T \mathbf{E}_{ti})^{-1} \mathbf{E}_{ti}^T \mathsf{im}(\mathbf{E}_{ti}). \tag{14}$$

- Step 2: Once $\mathcal{R}_{ti}$ is calculated, the following calculation allows to calculate partition $\mathbf{U}_{ti}$ according to (12):

$$\mathsf{im}(\mathbf{U}_{ti}) = (\mathbf{B}_\tau^T \mathbf{B}_\tau)^{-1} \mathbf{B}_\tau^T \mathsf{im}(\mathbf{B}_\tau) \cap \mathcal{R}_{ti}. \tag{15}$$

- Step 3: According to (9) calculate state–feedback matrix $\mathbf{F}_{ti}$ solving the following linear problem:

$$\mathcal{R}_{ti} = min\mathcal{I}(\mathbf{A} + \mathbf{B}_\tau \mathbf{F}_{ti}, \, \mathbf{B}_\tau \mathbf{U}_{ti}) \subseteq \mathsf{ker}(\mathbf{E}_{uc}). \tag{16}$$

- Step 4: According to (10) calculate the reachable subspace of the internal coordinated movements with the following expression:

$$\mathcal{R}_{uc} = (\mathbf{E}_{uc}^T \mathbf{E}_{uc})^{-1} \mathbf{E}_{uc}^T \mathsf{im}(\mathbf{E}_{uc}). \tag{17}$$

- Step 5: Once $\mathcal{R}_{uc}$ is calculated, the following calculation allows to calculate partition $\mathbf{U}_{uc}$ according to (12):

$$\mathsf{im}(\mathbf{U}_{uc}) = (\mathbf{B}_\tau^T \mathbf{B}_\tau)^{-1} \mathbf{B}_\tau^T \mathsf{im}(\mathbf{B}_\tau) \cap \mathcal{R}_{uc}. \tag{18}$$

- Step 6: According to (10) calculate state–feedback matrix $\mathbf{F}_{uc}$ solving the following linear problem:

$$\mathcal{R}_{uc} = min\mathcal{I}(\mathbf{A}_{ti} + \mathbf{B}_\tau \mathbf{F}_{uc}, \, \mathbf{B}_\tau \mathbf{U}_{uc}) \subseteq \mathsf{ker}(\mathbf{E}_{ti}). \tag{19}$$

- Step 7 : The final state–feedback decoupling matrix is the following:

$$\mathbf{F} = \mathbf{F}_{ti} + \mathbf{F}_{uc}. \tag{20}$$

  End

Note the matrix $\mathbf{F}_{ti}$ makes the subspace of the internal contac forces invariant. This means that the internal contact forces do not influnce the coordinated movements. The practical meaning of that is that it is possible to sqeeze the object without moving it. Through state–feedback matrix $\mathbf{F}_{ti}$ the following matrix which represents the system is obtained:

$$\mathbf{A}_{ti} = \mathbf{A} + \mathbf{B}_\tau \mathbf{F}_{ti}. \tag{21}$$

Note that, matrix $\mathbf{A}_{ti}$ is that defined in (21), and matrix $\mathbf{F}_{uc}$ makes the subspace of the object motions invariant. This means that the object motions do not influence the internal contact forces. The practical meaning of that is that, it is possible to move the object without squeezing it. Through the feedback matrix $\mathbf{F}_{uc}$ the following matrix which finally represents the state of the noninteracting system is obtained:

$$\mathbf{A}_{dec} = \mathbf{A}_{ti} + \mathbf{B}_{\tau}\mathbf{F}_{uc}. \tag{22}$$

Relation (20) can be derived considering that $\mathbf{A}_{ti} = \mathbf{A} + \mathbf{B}_{\tau}\mathbf{F}_{ti}$ and that $\mathbf{A}_{dec} = \mathbf{A}_{ti} + \mathbf{B}_{\tau}\mathbf{F}_{uc}$. In fact, combining these two relations the following mathematical expression is obtained:

$$\mathbf{A}_{dec} = \mathbf{A} + \mathbf{B}_{\tau}(\mathbf{F}_{ti} + \mathbf{F}_{uc}), \tag{23}$$

and expression (20) comes directly from (23). After the numerical calculations the following matrices are obtained:

$$\mathbf{F} = \begin{bmatrix} -7 & 6.5 & -6 & -1 & -41 & 0 & -7.5 & -0.02 & -5.5 & -3 & -22 & 0 \\ 10 & -120 & 10 & -72 & 5 & 0 & 0.29 & -16 & 0.29 & 7.2 & -6.2 & 0 \\ -6.1 & 6.5 & -7.1 & -0.97 & -41 & 0 & -5.5 & -0.021 & -7.5 & -3.1 & -22 & 0 \end{bmatrix},$$

$$\mathbf{U}_{ti} = \begin{bmatrix} -0.707 \\ 0 \\ 0.707 \end{bmatrix}, \qquad \mathbf{U}_{uc} = \begin{bmatrix} 0 & -0.707 \\ 1 & 0 \\ 0 & -0.707 \end{bmatrix}.$$

The control task is set to follow a circle with angular velocity of 0.1rad/sec, starting from point $\mathbf{u}_o$ of coordinates $(2.5, 1)$, see Figure 3, while keeping the contact force constant to the value $\mathbf{t}_o = [0; 1; 0; -1]^T$. The computed prismatic joint forces $\tau^* = \mathbf{U}_{ti}\mathbf{u}_{ti} + \mathbf{U}_{uc}\mathbf{u}_{uc}$ realizing the perfect tracking of desired object motions and internal force are reported in Figure 4 along with the perfectly tracked internal force parametrization $\mathbf{t}_i$. The object trajectory perfect tracking, which is a circular trajectory of the center of mass of the object, is reported in Figure 3. It is worthwhile to remark that for the simple possibly industrial gripper, under the reasonable hypothesis that the angular dynamics of the object can be disregarded, linearized dynamics represents the complete description of manipulation system dynamics.

6. CONCLUSIONS

The decoupling and trajectory tracking procedure discussed in this paper applies to robotic manipulation systems whose dynamics can be modelled according to Section 2. The class of manipulation systems under investigation is wide enough to include a considerable number of grasp configurations, such as those using internal and/or extremal links to grasp objects, those with contact kinematic redundancy and so forth.

Due to the possible presence of defectivity, the control outputs were suitably chosen as the reachable internal forces and the rigid–body object motions. The main results of this paper are summarized in two theorems. The first one is related to noninteracting control of general manipulation systems. The second one focuses on the perfect output functional controllability of internal force and object motions. The problem approached in this paper is relevant to the robotic community and is commonly addressed as *hybrid control*. With respect to the solution for the system inversion proposed in [14], the improvement presented here consists of the synthesis of a more robust controller. In fact its inner decoupling structure is able to fix the possible coupling problems, whatever the joint input signals are.

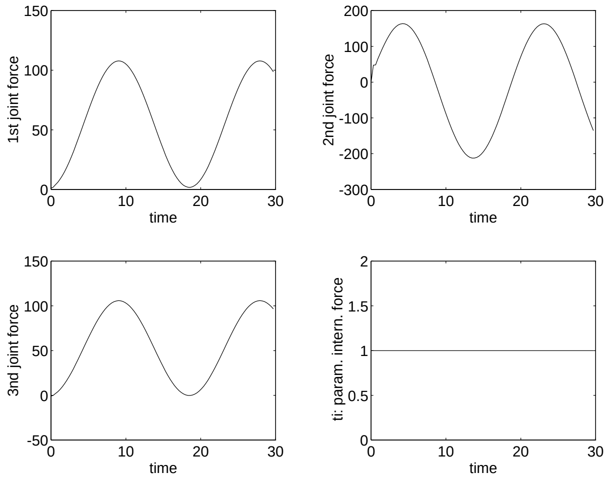

Fig. 4. Internal force $\mathbf{t}_i$ perfectly tracks the constant internal force while the object center of mass perfectly tracks the unit circle as depicted in Figure 3.

APPENDIX

Appendix I: Some geometrical properties

This part of the appendix is devoted to calculate some structures which will be useful in order to show the proposed theorem on the functional controllability. Let $\mathbf{\Gamma}_h$ be the basis for the identical internal forces subspace as explained in [15], thus

$$\mathsf{im}(\mathbf{\Gamma}_h) \ : \ \mathsf{im}(\mathbf{M}_h^{-1}\mathbf{J}^T) \cap max\mathcal{I}(\mathbf{M}_h^{-1}\mathbf{J}^T\mathbf{KJ}, \mathsf{ker}(\mathbf{GKJ})). \tag{24}$$

Here,

$$max\mathcal{I}(\mathbf{A}, \mathsf{ker}(\mathbf{C})) = \bigcap_{i=0}^{n-1} \mathbf{A}^i \mathsf{ker}(\mathbf{C}) \tag{25}$$

is maximum $\mathbf{A}$–invariant subspace contained in $\mathsf{ker}(\mathbf{C})$ with respect to a general system defined by the triple $(\mathbf{A}, \mathbf{B}, \mathbf{C})$. More in depth about the meaning of equation (24), it is possible to observe that this intersection states a basis for the identical internal forces subspace. In fact these forces are all those forces which do not generate movements of the object. In fact, the maximal invariant subspace of the forces through $\mathbf{M}_h^{-1}\mathbf{J}^T\mathbf{KJ}$, see the dynamic matrix of the system represented in (2), contained in $\mathsf{ker}(\mathbf{GKJ})$) is considered. Subspace $\mathsf{ker}(\mathbf{GKJ})$) represents the null of the object motion (no motions), see (2).

Remark 3. The subspace null of $\mathbf{Q}$ can be calculated very easily, in fact $\mathsf{ker}(\mathbf{Q}) = \mathsf{ker}(\mathbf{J}) + \mathcal{V}$ where $\mathcal{V} = \{\mathbf{v} | \mathbf{KJv} \in \mathsf{ker}(\mathbf{I} - \mathbf{KG}^T(\mathbf{GKG}^T)^{-1}\mathbf{G}) = \mathsf{im}(\mathbf{KG}^T), \mathbf{v} \notin \mathsf{ker}(\mathbf{J})\}$.

From (7) it is easy to show that $\mathcal{V} = \mathsf{im}(\mathbf{\Gamma}_{qc})$ and thus that

$$\mathsf{ker}(\mathbf{Q}) = \mathsf{im}(\mathbf{\Gamma}_r) + \mathsf{im}(\mathbf{\Gamma}_{qc}). \tag{26}$$

The following two Lemmas show the following condition:

$$\mathbf{E}_{ti}\mathcal{R}_{B_{ti}} = \mathsf{im}(\mathbf{E}_{ti}). \tag{27}$$

To show condition (27) it is equivalent to show that

$$\mathsf{im}(\mathbf{Q}\begin{bmatrix} \mathbf{\Gamma}_h & \mathbf{S}_q\mathbf{Z} \end{bmatrix}) = \mathsf{im}(\mathbf{Q}), \tag{28}$$

with

$$\mathsf{im}(\mathbf{S}_q) = min\mathcal{I}(\mathbf{M}_h^{-1}\mathbf{J}^T\mathbf{K}\mathbf{J}, \mathbf{M}_h^{-1}\mathbf{J}^T\mathbf{K}\mathbf{G}^T), \tag{29}$$

and where $\mathbf{Z}$ is such that

$$\mathsf{im}(\mathbf{M}_o^{-1}\mathbf{G}\mathbf{K}\mathbf{J}\mathbf{S}_q\mathbf{Z}) = \mathsf{im}(\mathbf{M}_o^{-1}\mathbf{G}\mathbf{K}\mathbf{J}\mathbf{S}_q) \cap \mathsf{ker}(\mathbf{\Gamma}_{uc}^T). \tag{30}$$

Remark 4. About the meaning of the equivalence of equation (27) with (28) it is to observe as follows. Eq. (30) states the intersection between the controllable object motions from the manipulator movements, through subspace $\mathsf{im}(\mathbf{M}_o^{-1}\mathbf{G}\mathbf{K}\mathbf{J})$ of the object motions, and the object motions which belong to the orthogonal subspace with respect to the object motions themselves. The orthogonal subspace with respect to the object motions subspace is represented by $\mathsf{ker}(\mathbf{\Gamma}_{uc}^T)$. In fact, it is known from the linear algebra that $\mathsf{ker}(\mathbf{\Gamma}_{uc}^T) = (\mathsf{im}(\mathbf{\Gamma}_{uc}))^{\perp}$. This means that $\mathbf{S}_q\mathbf{Z}$ states all the object motions which are not coordinate with the manipulator movements. Expression $\mathsf{im}(\mathbf{M}_o^{-1}\mathbf{G}\mathbf{K}\mathbf{J}\mathbf{S}_q)$ of eq. (30) represents the subspaces of the controllable object motions from the manipulator movements. In fact, subspace $\mathsf{im}(\mathbf{S}_q)$ is the minimal invariant subspace of the dynamics of the manipulator, which is represented by subspace $\mathsf{im}(\mathbf{M}_h^{-1}\mathbf{J}^T\mathbf{K}\mathbf{J})$, see (2), which includes the subspace of the object motions through subspace $\mathsf{im}(\mathbf{M}_h^{-1}\mathbf{J}^T\mathbf{K}\mathbf{G}^T)$, see (2).

To prove (27), the following two relationships are to be shown.

$$\mathsf{ker}(\mathbf{Q}) \cap \mathsf{im}\begin{bmatrix} \mathbf{\Gamma}_h & \mathbf{S}_q\mathbf{Z} \end{bmatrix} = \mathbf{0}, \tag{31}$$

$$\mathsf{rank}(\begin{bmatrix} \mathbf{\Gamma}_h & \mathbf{S}_q\mathbf{Z} \end{bmatrix}) = \mathsf{rank}(\mathbf{Q}). \tag{32}$$

Lemma 1. Let $\mathsf{im}(\mathbf{\Gamma}_h)$, $\mathsf{ker}(\mathbf{Q})$, and $\mathsf{im}(\mathbf{S}_q\mathbf{Z})$ be subspaces as given in (24), (26), and (30), respectively, then the following relation holds:

$$\mathsf{ker}(\mathbf{Q}) \cap \mathsf{im}\begin{bmatrix} \mathbf{\Gamma}_h & \mathbf{S}_q\mathbf{Z} \end{bmatrix} = \mathbf{0}. \tag{33}$$

P r o o f. From the previous remark, (33) can be verified by checking if vectors $\mathbf{x}$, $\mathbf{y}$, $\mathbf{v}$ and $\mathbf{w}$ such that

$$\boldsymbol{\Gamma}_r\mathbf{x} + \boldsymbol{\Gamma}_{qc}\mathbf{y} = \boldsymbol{\Gamma}_h\mathbf{v} + \mathbf{S}_q\mathbf{Z}\mathbf{w} \tag{34}$$

exist. The demonstration consists of showing that these vectors do not exist. From the definition of $\mathsf{im}(\boldsymbol{\Gamma}_{qc})$ and $\mathsf{im}(\boldsymbol{\Gamma}_h)$ it is possible to see that $\mathsf{im}(\mathbf{S}_q)$ is included in $\mathsf{im}({\mathbf{M}_h}^{-1}\mathbf{J}^T)$. $\mathsf{im}(\boldsymbol{\Gamma}_r)$ is not included in $\mathsf{im}({\mathbf{M}_h}^{-1}\mathbf{J}^T)$ because this is included in $\mathsf{ker}(\mathbf{J})$[1]. Thus eq. (34) could be written in the following way:

$$\boldsymbol{\Gamma}_{qc}\mathbf{y} = \boldsymbol{\Gamma}_h\mathbf{v} + \mathbf{S}_q\mathbf{Z}\mathbf{w}. \tag{35}$$

If eq. (35) is true, then $\mathbf{M}_o^{-1}\mathbf{G}\mathbf{K}\mathbf{J}$ and

$$\mathbf{M}_o^{-1}\mathbf{G}\mathbf{K}\mathbf{J}\boldsymbol{\Gamma}_{qc}\mathbf{y} = \mathbf{M}_o^{-1}\mathbf{G}\mathbf{K}\mathbf{J}\boldsymbol{\Gamma}_h\mathbf{v} + \mathbf{M}_o^{-1}\mathbf{G}\mathbf{K}\mathbf{J}\mathbf{S}_q\mathbf{Z}\mathbf{w}. \tag{36}$$

Considering that $\mathsf{im}(\boldsymbol{\Gamma}_h) \subseteq \mathsf{ker}(\mathbf{G}\mathbf{K}\mathbf{J})$, then

$$\mathbf{M}_o^{-1}\mathbf{G}\mathbf{K}\mathbf{G}^T\boldsymbol{\Gamma}_{uc}\mathbf{y} = \mathbf{M}_o^{-1}\mathbf{G}\mathbf{K}\mathbf{J}\mathbf{S}_q\mathbf{Z}\mathbf{w}, \tag{37}$$

but this is never verified. In fact, because of choice of $\mathbf{Z}$, $\mathsf{im}(\mathbf{M}_o^{-1}\mathbf{G}\mathbf{K}\mathbf{J}\mathbf{S}_q\mathbf{Z}) \subseteq \mathsf{ker}(\boldsymbol{\Gamma}_{uc}^T)$ and it is very easy to show that if $\mathsf{im}(\mathbf{M}_o^{-1}\mathbf{G}\mathbf{K}\mathbf{G}^T\boldsymbol{\Gamma}_{uc}) \subseteq \mathsf{ker}(\boldsymbol{\Gamma}_{uc}^T)$, then matrix $\mathbf{M}_o^{-1}\mathbf{G}\mathbf{K}\mathbf{G}^T$ would be an orthogonal projector, but it is not true because this is not a projector form[2]. This means that vectors which satisfy eq. (34) do not exist. This shows that (33) is proven. □

Lemma 2. Let $\mathsf{im}(\boldsymbol{\Gamma}_h)$ and $\mathsf{im}(\mathbf{S}_q\mathbf{Z})$ be subspaces as given in (24) and (30), respectively, then the following relation holds:

$$\mathsf{rank}\begin{bmatrix} \boldsymbol{\Gamma}_h & \mathbf{S}_q\mathbf{Z} \end{bmatrix} = \mathsf{rank}(\boldsymbol{\Gamma}_h) + \mathsf{rank}(\mathbf{S}_q\mathbf{Z}) = q - r - c. \tag{38}$$

P r o o f. The first equality comes from the null intersection between $\mathsf{im}(\boldsymbol{\Gamma}_h)$ and $\mathsf{im}(\mathbf{S}_q\mathbf{Z})$. In fact, from (24) $\mathsf{im}(\boldsymbol{\Gamma}_h)$ is a subspace of $max\mathcal{I}(\mathbf{M}_h^{-1}\mathbf{J}^T\mathbf{K}\mathbf{J}, \mathsf{ker}(\mathbf{G}\mathbf{K}\mathbf{J}))$ which, from (29), is orthogonal with respect to $\mathsf{im}(\mathbf{M}_h^{-1}\mathbf{S}_q)$[3]. The proof of the second equality of the lemma begins with the following relation.

$$max\mathcal{I}(\mathbf{M}_h^{-1}\mathbf{J}^T\mathbf{K}\mathbf{J}, \mathsf{ker}(\mathbf{G}\mathbf{K}\mathbf{J})) = \mathsf{im}(\mathbf{M}_h^{-1}\mathbf{S}_q)^{\perp}, \tag{39}$$

and it follows that

$$\mathsf{im}(\mathbf{M}_h^{-1}\mathbf{J}^T) \subseteq max\mathcal{I}(\mathbf{M}_h^{-1}\mathbf{J}^T\mathbf{K}\mathbf{J}, \mathsf{ker}(\mathbf{G}\mathbf{K}\mathbf{J})) \oplus \mathsf{im}(\mathbf{M}_h^{-1}\mathbf{S}_q). \tag{40}$$

Now, from (29) $\mathsf{im}(\mathbf{M}_h^{-1}\mathbf{S}_q) \subseteq \mathsf{im}(\mathbf{M}_h^{-1}\mathbf{J}^T)$. From the above mentioned inclusion and from definition (67) it follows that

$$\begin{aligned} \mathsf{im}(\mathbf{M}_h^{-1}\mathbf{J}^T) &= \mathbf{M}_h^{-1}\mathbf{J}^T \cap \big(max\mathcal{I}(\mathbf{M}_h^{-1}\mathbf{J}^T\mathbf{K}\mathbf{J}, \mathsf{ker}(\mathbf{G}\mathbf{K}\mathbf{J})) \oplus \mathsf{im}(\mathbf{M}_h^{-1}\mathbf{S}_q)\big) \\ &= \big(\mathbf{M}_h^{-1}\mathbf{J}^T \cap max\mathcal{I}(\mathbf{M}_h^{-1}\mathbf{J}^T\mathbf{K}\mathbf{J}, \mathsf{ker}(\mathbf{G}\mathbf{K}\mathbf{J}))\big) \oplus \mathsf{im}(\mathbf{M}_h^{-1}\mathbf{S}_q) \\ &= \mathsf{im}(\boldsymbol{\Gamma}_h) \oplus \mathsf{im}(\mathbf{M}_h^{-1}\mathbf{S}_q). \end{aligned} \tag{41}$$

---

[1] In general for a linear application $\mathbf{L}$ the following relationship holds: $\mathsf{im}(\mathbf{L}^T) + \mathsf{ker}(\mathbf{L}) = \mathsf{im}(\mathbf{I})$.

[2] Given a subspace $\mathcal{L}$ of which the basis matrix is $\mathbf{L}$, then the orthogonal projector is $(\mathbf{I} - \mathbf{L}(\mathbf{L}^\mathbf{T}\mathbf{L})^{-\mathbf{1}}\mathbf{L}^\mathbf{T})$.

[3] Given a subspace $\mathcal{L}$ of which the basis matrix is $\mathbf{L}$, then $\mathsf{ker}(\mathbf{L}^T) = (\mathsf{im}(\mathbf{L}))^{\perp}$.

It follows that

$$\mathsf{rank}(\mathbf{\Gamma}_h) + \mathsf{rank}(\mathbf{S}_q) = \mathsf{rank}(\mathbf{M}_h^{-1}\mathbf{J}^T) = \mathsf{rank}(\mathbf{J}) = q - r \tag{42}$$

and

$$\mathsf{rank}(\mathbf{\Gamma}_h) = q - r - \mathsf{rank}(\mathbf{S}_q). \tag{43}$$

It remains to calculate $\mathsf{rank}(\mathbf{S}_q\mathbf{Z})$. Recalling that $\mathbf{S}_q$ and $\mathbf{Z}$ are basis matrices and from (30) $\mathsf{rank}(\mathbf{Z}) \leq \mathsf{rank}(\mathbf{S}_q)$, then

$$\mathsf{rank}(\mathbf{S}_q\mathbf{Z}) = \mathsf{rank}(\mathbf{Z}). \tag{44}$$

From the definition of $\mathbf{Z}$ in (30) it follows that

$$\mathsf{rank}(\mathbf{Z}) = \mathsf{rank}(\mathbf{S}_q) - \mathsf{rank}(\mathbf{Z}^\perp), \tag{45}$$

where $\mathsf{rank}(\mathbf{S}_q)$ is the number of components $\mathbf{z} \in \mathbf{Z}$. The last part of this demonstration consists of estimating $\mathsf{rank}(\mathbf{Z}^\perp)$, which from (30) is

$$\mathsf{rank}(\mathbf{Z}^\perp) = \mathsf{rank}(\mathbf{S}_q^T\mathbf{J}^T\mathbf{K}\mathbf{G}^T\mathbf{M}_o^{-1}\mathbf{\Gamma}_{uc}). \tag{46}$$

From (29), it is easy to show that $\mathsf{ker}(\mathbf{S}_q^T) \subseteq \mathsf{ker}(\mathbf{GKJ})$, and thus

$$\mathsf{ker}(\mathbf{S}_q^T) \cap \mathsf{im}(\mathbf{J}^T\mathbf{K}\mathbf{G}^T) = \mathbf{0} \tag{47}$$

and

$$\mathsf{rank}(\mathbf{Z}^\perp) = \mathsf{rank}(\mathbf{J}^T\mathbf{K}\mathbf{G}^T\mathbf{M}_o^{-1}\mathbf{\Gamma}_{uc}). \tag{48}$$

Now

$$\mathsf{rank}(\mathbf{Z}^\perp) = \mathsf{rank}(\mathbf{J}^T\mathbf{K}\mathbf{G}^T\mathbf{M}_o^{-1}\mathbf{\Gamma}_{uc}) = \mathsf{rank}(\mathbf{\Gamma}_{uc}) = c. \tag{49}$$

If (48) is transposed, then

$$\mathsf{rank}(\mathbf{Z}^\perp) = \mathsf{rank}(\mathbf{\Gamma}_{uc}^T\mathbf{M}_o^{-1}\mathbf{GKJ}), \tag{50}$$

and from (3)

$$\mathsf{rank}(\mathbf{Z}^\perp) = \mathsf{rank}(\mathbf{\Gamma}_{uc}^T\mathbf{M}_o^{-1}\mathbf{GKG}^T\mathbf{\Gamma}_{uc}) = \mathsf{rank}(\mathbf{\Gamma}_{uc}), \tag{51}$$

where the last equality follows because matrix $\mathbf{\Gamma}_{uc}^T\mathbf{M}_o^{-1}\mathbf{GKG}^T\mathbf{\Gamma}_{uc}$ has full rank. Finally, from (44), (45) and (49), it can be concluded:

$$\mathsf{rank}(\mathbf{S}_q\mathbf{Z}) = \mathsf{rank}(\mathbf{S}_q) - c. \tag{52}$$

Now, if this last result is compared with (43), then

$$\mathsf{rank}\begin{bmatrix} \mathbf{\Gamma}_h & \mathbf{S}_q\mathbf{Z} \end{bmatrix} = q - r - c. \tag{53}$$

□

Remark 5. Eq. (32) is proven only if in case of kinematic defectivity, which can be mathematically expressed as $\mathsf{ker}(\mathbf{J}^T) \neq \mathbf{0}$, i. e., being $\mathbf{J} \in \Re^{(t\times q)}$, thus only in case of $t > q$. It is easy to prove that in case of $t \leq q$ is only a trivial extension. Let r e c be the ranks of matrices $\mathbf{\Gamma}_r$ and $\mathbf{\Gamma}_{uc}$, respectively. Then $\mathsf{rank}(\mathbf{J}) = q - r$. From Lemma 2 we have that $\mathsf{rank}\begin{bmatrix} \mathbf{\Gamma}_h & \mathbf{S}_q\mathbf{Z} \end{bmatrix} = \mathsf{rank}(\mathbf{\Gamma}_h) + \mathsf{rank}(\mathbf{S}_q\mathbf{Z}) = q - r - c$. In conclusion in (32) it is shown that

$$\mathsf{rank}(\mathbf{Q}) = q - r - c, \tag{54}$$

and this comes trivially from (26). In fact, $\mathsf{rank}(\mathbf{Q}) = \mathsf{rank}(\mathbf{Q}^T) = q - \mathsf{rank}(\mathsf{ker}(\mathbf{Q})) = q - (r + c)$.

Appendix II: Calculation of the controlled invariant subspaces

Part a)

The following subspace must be calculated:

$$max\mathcal{V}(\mathbf{A}, \mathsf{im}(\mathbf{B}_\tau), \mathsf{im}(\mathbf{B}_{uc})). \tag{55}$$

According to equations (4), it is possible to write as follows. Considering

$$\mathsf{ker}(\mathbf{E}_{ti}) = \mathsf{ker}\begin{bmatrix} \mathbf{Q}_k & \mathbf{0} & \mathbf{Q}_\beta & \mathbf{0} \end{bmatrix}, \tag{56}$$

it could be useful to remark that $\mathsf{ker}(\mathbf{Q}_k) = \mathsf{ker}(\mathbf{Q}_\beta)$ under the hypothesis of proportionality enunciated above, then

$$\mathsf{ker}(\mathbf{E}_{ti}) \supseteq \mathsf{im}(\mathbf{B}_{uc}), \tag{57}$$

with

$$\mathbf{B}_{uc} = \begin{bmatrix} \mathbf{\Gamma}_{qc} & \mathbf{0} & \mathbf{\Gamma}_{qc} & \mathbf{0} & \mathbf{\Gamma}_{qc} & \mathbf{0} & \mathbf{\Gamma}_{qc} & \mathbf{0} & .. \\ \mathbf{\Gamma}_{uc} & \mathbf{0} & -\mathbf{\Gamma}_{uc} & \mathbf{0} & -\mathbf{H}\mathbf{\Gamma}_{uc} & \mathbf{0} & -\mathbf{H}^2\mathbf{\Gamma}_{uc} & \mathbf{0} & .. \\ \mathbf{0} & \mathbf{\Gamma}_{qc} & \mathbf{0} & \mathbf{\Gamma}_{qc} & \mathbf{0} & \mathbf{\Gamma}_{qc} & \mathbf{0} & \mathbf{\Gamma}_{qc} & .. \\ \mathbf{0} & \mathbf{\Gamma}_{uc} & \mathbf{0} & -\mathbf{\Gamma}_{uc} & \mathbf{0} & -\mathbf{H}\mathbf{\Gamma}_{uc} & \mathbf{0} & -\mathbf{H}^4\mathbf{\Gamma}_{uc} & .. \end{bmatrix}, \tag{58}$$

where $\mathbf{H} = \mathbf{M}_o^{-1}\mathbf{G}\mathbf{B}\mathbf{G}^\mathbf{T}$. To verify that

$$max\mathcal{V}(\mathbf{A}, \mathsf{im}(\mathbf{B}_\tau), \mathsf{im}(\mathbf{B}_{uc})) = \mathsf{im}(\mathbf{B}_{uc}), \tag{59}$$

it results very easy. In fact, $\mathsf{im}(\mathbf{B}_{uc})$ is a controlled invariant subspace. The inclusion condition (57) comes from the inspection of the two subspaces considering relationship (26).

The following subspace must be calculated:

$$max\mathcal{V}(\mathbf{A}, \mathsf{im}(\mathbf{B}_\tau), \mathsf{im}(\mathbf{B}_{ti})). \tag{60}$$

According to equations (7), it is possible to write as follows. Considering that

$$\mathbf{E}_{uc} = (\mathbf{\Gamma}_{uc}^T\mathbf{\Gamma})^{-1}\mathbf{\Gamma}_{uc}^T\left[\mathbf{0}\ \mathbf{I}\ \mathbf{0}\ \mathbf{0}\right], \tag{61}$$

then

$$\mathsf{ker}(\mathbf{E}_{uc}) = \mathsf{im}\begin{bmatrix} \mathbf{I}_{q\times q} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathsf{ker}(\mathbf{\Gamma}_{uc}^T) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_{q\times q} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}_{u\times u} \end{bmatrix}. \tag{62}$$

Considering the following subspace:

$$\mathsf{im}(\mathbf{B}_{ti}) = \mathsf{im}\begin{bmatrix} \mathbf{\Gamma}_h & \mathbf{0} & \mathbf{S}_q & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathsf{ker}(\mathbf{\Gamma}_{uc}^T)\cap\mathbf{S}_u & \mathbf{0} \\ \mathbf{0} & \mathbf{\Gamma}_h & \mathbf{0} & \mathbf{S}_q & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathsf{ker}(\mathbf{\Gamma}_{uc}^T)\cap\mathbf{S}_u \end{bmatrix}, \tag{63}$$

then it follows that

$$\mathsf{ker}(\mathbf{E}_{uc}) \supseteq \mathsf{im}(\mathbf{B}_{ti}), \tag{64}$$

just by observing that

$$\mathsf{im}(\mathbf{S}_q) = min\mathcal{I}(\mathbf{M}_h^{-1}\mathbf{J}^T\mathbf{K}\mathbf{J}, \mathbf{M}_h^{-1}\mathbf{J}^T\mathbf{K}\mathbf{G}^T), \tag{65}$$

$$\mathsf{im}(\mathbf{S}_u) = min\mathcal{I}(\mathbf{M}_o^{-1}\mathbf{G}\mathbf{K}\mathbf{G}^T, \mathbf{M}_o^{-1}\mathbf{G}\mathbf{K}\mathbf{J}), \tag{66}$$

and finally that $\mathbf{\Gamma}_h$ is a basis matrix of

$$\mathrm{im}(\mathbf{M}_o^{-1}\mathbf{J}^T) \cap max\mathcal{I}(\mathbf{M}_h^{-1}\mathbf{J}^T\mathbf{K}\mathbf{J}, \mathsf{ker}(\mathbf{G}\mathbf{K}\mathbf{J})). \tag{67}$$

From these considerations it follows that (63) is a subspace included in $\mathsf{ker}(\mathbf{E}_{uc})$.

Remark 6. It is to observe that subspace $\mathsf{im}(\mathbf{S}_u)$ represents the controllable object motions from the manipulator movements. In fact, $\mathsf{im}(\mathbf{S}_u)$ is the minimal invariant subspace of the object motions which is represented by subspace $\mathsf{im}(\mathbf{M}_o^{-1}\mathbf{G}\mathbf{K}\mathbf{G}^{\mathbf{T}})$, see (2), which includes the subspace of the manipulator movements through subspace $\mathsf{im}(\mathbf{G}\mathbf{K}\mathbf{J})$, see (2).

To calculate the subspace defined in (60) it will be sufficient to find a subspace $\mathsf{im}(\mathbf{V})$ controlled invariant in $(\mathbf{A}, \mathbf{B}_\tau)$ and included in $\mathsf{im}(\mathbf{B}_{ti})$ with the following structure[4]:

$$\mathbf{V} = \begin{bmatrix} \mathbf{\Gamma}_h & \mathbf{0} & \mathbf{S}_q\mathbf{Z} & \mathbf{0} & \mathbf{M}_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{M}_b & \mathbf{M}_2 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{\Gamma}_h & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{S}_q\mathbf{Z} & \mathbf{0} & \mathbf{M}_1 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{M}_b & \mathbf{M}_2 \end{bmatrix}. \tag{68}$$

With $\mathbf{Z}$ such that

$$\mathsf{im}(\mathbf{M}_o^{-1}\mathbf{G}\mathbf{K}\mathbf{J}\mathbf{S}_q\mathbf{Z}) = \mathsf{im}(\mathbf{M}_o^{-1}\mathbf{G}\mathbf{K}\mathbf{J}\mathbf{S}_q) \cap \mathsf{ker}(\mathbf{\Gamma}_{uc}^T). \tag{69}$$

Subspace $\mathrm{im}(\mathbf{V})$ must be controlled invariant and it is necessary that

$$\mathbf{A}\mathsf{im}(\mathbf{V}) \subseteq \mathsf{im}(\mathbf{V}) + \mathsf{im}(\mathbf{B}_\tau), \tag{70}$$

$$\mathsf{im}(\mathbf{V}) \subseteq \mathsf{im}(\mathbf{B}_{ti}). \tag{71}$$

Eq. (71) of the previous relations is satisfied if:

$$\mathsf{im}(\mathbf{M}_1) \subseteq \mathsf{im}(\mathbf{S}_q), \tag{72}$$

$$\mathsf{im}(\mathbf{M}_2) \subseteq \mathsf{ker}(\mathbf{\Gamma}_{uc}^T), \tag{73}$$

$$\mathsf{im}(\mathbf{M}_b) \subseteq \mathsf{ker}(\mathbf{\Gamma}_{uc}^T), \tag{74}$$

while eq. (70) it is satisfied if:

$$\mathsf{im}(\mathbf{M}_o^{-1}\mathbf{G}\mathbf{K}\mathbf{J}\mathbf{S}_q\mathbf{Z}) \subseteq \mathsf{im}\begin{bmatrix} \mathbf{M}_b & \mathbf{M}_2 \end{bmatrix}, \tag{75}$$

[4] It will be enough to consider a subspace included in $\mathsf{im}(\mathbf{B}_{ti})$. This choice will help to design the controller. In fact, this choice is constructive and the solvent subspace must be controlled invariant.

$$\mathsf{im}(-\mathbf{M}_o^{-1}\mathbf{GKG}^T\mathbf{M}_b) \subseteq \mathsf{im}\left[\begin{array}{cc}\mathbf{M}_b & \mathbf{M}_2\end{array}\right], \tag{76}$$

$$\mathsf{im}(\mathbf{M}_o^{-1}\mathbf{GKJM}_1 - \mathbf{M}_o^{-1}\mathbf{GKG}^T\mathbf{M}_2) \subseteq \mathsf{im}\left[\begin{array}{cc}\mathbf{M}_b & \mathbf{M}_2\end{array}\right]. \tag{77}$$

Part b)

In this part of the appendix the proof of the necessary conditions of eqs. (75), (76), and (77) to obtain subspace $\mathsf{im}(\mathbf{V})$ as a controlled invariant one. To show that it is useful to distinguish three possible cases depending on $\mathsf{ker}(\mathbf{\Gamma}_{uc}^T)$.

Case 1:

$\mathsf{ker}(\mathbf{\Gamma}_{uc})$ is $\mathbf{M}_o^{-1}\mathbf{GKG}^T$ invariant.
This is the easiest case, in fact if we take $\mathsf{im}(\mathbf{M}_b) = \mathsf{ker}(\mathbf{\Gamma}_{uc}^T)$ and $\mathbf{M}_2 = \mathbf{0}$, so eqs. (75) and (76) are satisfied automatically, eq. (77) is satisfied for $\mathbf{M}_1 = \mathbf{0}$.

Case 2:

$\mathsf{ker}(\mathbf{\Gamma}_{uc}^T) \not\supseteq \mathbf{M}_o^{-1}\mathbf{GKG}^T\mathsf{ker}(\mathbf{\Gamma}_{uc}^T)$ and $\mathsf{ker}(\mathbf{\Gamma}_{uc}^T) \cap \mathbf{M}_o^{-1}\mathbf{GKG}^T\mathsf{ker}(\mathbf{\Gamma}_{uc}^T) \neq \mathbf{0}$.

In this case eq. (76) is verified if the following two relations hold:
$\mathsf{im}(\mathbf{M}_2) = \mathsf{ker}(\mathbf{\Gamma}_{uc}^T)$;
$\mathbf{M}_b : \mathsf{im}(\mathbf{M}_o^{-1}\mathbf{GKG}^T\mathbf{M}_b) = \mathsf{ker}(\mathbf{\Gamma}_{uc}^T) \cap \mathbf{M}_o^{-1}\mathbf{GKG}^T\mathsf{ker}(\mathbf{\Gamma}_{uc}^T)$.
Now eq. (75) is trivially verified, while eq. (77) is verified if:

$$\mathbf{M}_o^{-1}\mathbf{GKG}^T\mathsf{ker}(\mathbf{\Gamma}_{uc}^T) \subseteq \left[\begin{array}{cc}\mathsf{im}(\mathbf{M}_o^{-1}\mathbf{GKJS}_q) & \mathsf{ker}(\mathbf{\Gamma}_{uc}^T)\end{array}\right]. \tag{78}$$

It will be demonstrated that this condition is always verified.

Case 3:

The last case to be analyzed is the following:

$$\mathsf{ker}(\mathbf{\Gamma}_{uc}^T) \cap \mathbf{M}_o^{-1}\mathbf{GKG}^T\mathsf{ker}(\mathbf{\Gamma}_{uc}^T) = \mathbf{0}. \tag{79}$$

Under this condition eq. (76) is satisfied only with $\mathbf{M}_b = \mathbf{0}$; for eq. (75) will be enough to choose $\mathsf{im}\mathbf{M}_2 = \mathsf{ker}(\mathbf{\Gamma}_{uc}^T)$. This implies the same conditions of the second case and thus to complete the demonstration it must be verified the following relation:

$$\mathbf{M}_o^{-1}\mathbf{GKG}^T\mathsf{ker}(\mathbf{\Gamma}_{uc}^T) \subseteq \left[\begin{array}{cc}\mathsf{im}(\mathbf{M}_o^{-1}\mathbf{GKJS}_q) & \mathsf{ker}(\mathbf{\Gamma}_{uc}^T)\end{array}\right]. \tag{80}$$

The following lemma shows how condition (80) is verified.

Lemma 3. If $\mathbf{S}_q \neq \mathbf{0}$, and considering the following subspace:

$$\left[\begin{array}{cc}\mathsf{im}(\mathbf{M}_o^{-1}\mathbf{GKJS}_q) & \mathsf{ker}(\mathbf{\Gamma}_{uc}^T)\end{array}\right], \tag{81}$$

then it is to show that a basis matrix of subspace in (81) is a basis for the subspaces in $\Re^d$, where d is the dimension of the physical space.

Proof. $\mathbf{S}_q = min\mathcal{I}(\mathbf{M}_h^{-1}\mathbf{J}^T\mathbf{KJ}, \mathbf{M}_h^{-1}\mathbf{J}^T\mathbf{KG}^T)$ and the $\mathbf{M}_h^{-1}$ is positive definite:

$$\mathsf{im}(\mathbf{M}_o^{-1}\mathbf{GKJ}) \supseteq \mathsf{im}(\mathbf{M}_o^{-1}\mathbf{GKJS}_q) \supseteq \mathsf{im}(\mathbf{M}_o^{-1}\mathbf{GKJM}_h^{-1}\mathbf{J}^T\mathbf{KG}^T) = \mathsf{im}(\mathbf{M}_o^{-1}\mathbf{GKJ}), \tag{82}$$

this implies that

$$\mathsf{im}(\mathbf{M}_o^{-1}\mathbf{GKJS}_q) = \mathsf{im}(\mathbf{M}_o^{-1}\mathbf{GKJ}). \tag{83}$$

Now it is easy to prove that

$$\Re^d \supseteq \left[\ \mathsf{im}(\mathbf{M}_o^{-1}\mathbf{GKJ}) \quad \mathsf{ker}(\mathbf{\Gamma}_{uc}^T)\ \right] \supseteq \left[\ \mathsf{im}(\mathbf{M}_o^{-1}\mathbf{GKJ\Gamma}_{qc}) \quad \mathsf{ker}(\mathbf{\Gamma}_{uc}^T)\ \right], \tag{84}$$

$$\left[\ \mathsf{im}(\mathbf{M}_o^{-1}\mathbf{GKG}^T\mathbf{\Gamma}_{uc}) \quad \mathsf{ker}(\mathbf{\Gamma}_{uc}^T)\ \right] = \Re^d, \tag{85}$$

and

$$\mathsf{rank}(\mathbf{M}_o^{-1}\mathbf{GKG}^T\mathbf{\Gamma}_{uc}) = \mathsf{rank}(\mathbf{\Gamma}_{uc}), \tag{86}$$

because $\mathbf{M}_o^{-1}\mathbf{GKG}^T$ has null space equal to zero. □

Appendix III: Proof of Theorem 2

In order to proof the theorem shown in (2) the following two controlled invariant subspaces, $\mathsf{im}(\mathbf{V})$ e $\mathsf{im}(\mathbf{B}_{uc})$, calculated in appendix II, (68) and (58) respectively are considered. Now we will show that these two subspaces respect the following condition.

Projection Condition

Given the system represented as $(\mathbf{A}, \mathbf{B}, \mathbf{C})$. $\mathcal{F}^i$ *is a subspace of functional controllability of the output with respect to the i-derivative, if*

$$\mathcal{F}^i \subseteq \mathcal{F}^i \cup \mathcal{Z}_{i-1} + \mathcal{C}, \tag{87}$$

where $\mathcal{Z}^{i-1}$ is defined in the following way: $\mathcal{Z}_o = \mathcal{B}$ and $\mathcal{Z}_j = \mathcal{B} + \mathbf{A}(\mathcal{Z}_{j-1} \cup \mathcal{F}^i \cup \mathcal{C})$. Further details in [1].

$$\mathsf{ker}(\mathbf{E}_{ti}) = \mathsf{im}\begin{bmatrix} \mathsf{ker}(\mathbf{Q}_k) & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}_{q\times q} \\ \mathbf{0} & \mathbf{I}_{u\times u} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathsf{ker}(\mathbf{Q}_\beta) & \mathbf{0} & -\alpha\mathbf{I}_{q\times q} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}_{u\times u} & \mathbf{0} \end{bmatrix}, \tag{88}$$

where $\mathbf{I}_{q\times q}$ and $\mathbf{I}_{u\times u}$ are the identity matrices with dimensions $q\times q$ and $u\times u$, respectively. In appendix II it is demonstrated that

$$\mathsf{ker}(\mathbf{Q}) = \mathsf{ker}(\mathbf{Q}_k) = \mathsf{ker}(\mathbf{Q}_\beta) = \mathsf{im}(\mathbf{\Gamma}_r) + \mathsf{im}(\mathbf{\Gamma}_{qc}), \tag{89}$$

then it follows that

$$\mathsf{ker}(\mathbf{E}_{ti}) = \mathsf{im}\begin{bmatrix} \mathbf{\Gamma}_r & \mathbf{0} & \mathbf{\Gamma}_{qc} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}_{q\times q} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}_{u\times u} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{\Gamma}_r & \mathbf{0} & \mathbf{\Gamma}_{qc} & \mathbf{0} & \mathbf{0} & -\alpha\mathbf{I}_{q\times q} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}_{u\times u} & \mathbf{0} \end{bmatrix}, \tag{90}$$

while

$$\ker(\mathbf{E}_{uc}) = \mathrm{im}\begin{bmatrix} \mathbf{I}_{q\times q} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \ker(\mathbf{\Gamma}_{uc}^T) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_{q\times q} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}_{u\times u} \end{bmatrix}. \tag{91}$$

It is possible to see that controlled invariant subspace $\mathrm{im}(\mathbf{V})$ defined in (68) satisfies the *Projection Condition* stated in (87). This means that the subspace $\mathrm{im}(\mathbf{V})$ is a functional controllability subspace with respect to first order derivative. In fact:

$$\mathrm{im}(\mathbf{V}) \subseteq \ker(\mathbf{E}_{ti}) + \mathrm{im}(\mathbf{V}) \cap \mathrm{im}(\mathbf{B}_\tau) = \mathcal{L}, \tag{92}$$

where subspace $\mathcal{L}$ is equal to

$$\mathcal{L} = \mathrm{im}\left[\begin{array}{cc|ccccccc} \mathbf{0} & \mathbf{0} & \mathbf{\Gamma}_r & \mathbf{0} & \mathbf{\Gamma}_{qc} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}_{q\times q} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}_{u\times u} & \mathbf{0} & \mathbf{0} \\ \mathbf{\Gamma}_h & \mathbf{S}_q\mathbf{Z} & \mathbf{0} & \mathbf{\Gamma}_r & \mathbf{0} & \mathbf{\Gamma}_{qc} & \mathbf{0} & \mathbf{0} & -\alpha\mathbf{I}_{q\times q} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}_{u\times u} & \mathbf{0} \end{array}\right]. \tag{93}$$

It is easy to show that

$$\mathrm{im}\begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{I}_{q\times q} \\ \mathbf{0} \end{bmatrix} \subseteq \mathcal{L}. \tag{94}$$

In fact, in Appendix II it was demonstrated that

$$\ker(\mathbf{Q}) \cap \mathrm{im}\begin{bmatrix} \mathbf{\Gamma}_h & \mathbf{S}_q\mathbf{Z} \end{bmatrix} = \mathbf{0}, \tag{95}$$

$$\ker(\mathbf{Q}) = \mathrm{im}(\mathbf{\Gamma}_r) + \mathrm{im}(\mathbf{\Gamma}_{qc}), \tag{96}$$

and likewise

$$\mathrm{rank}\begin{bmatrix} \mathbf{\Gamma}_h & \mathbf{S}_q\mathbf{Z} \end{bmatrix} = q - r - c, \tag{97}$$

and

$$\mathrm{rank}(\mathbf{Q}) = \mathrm{rank}(\mathbf{\Gamma}_r) + \mathrm{rank}(\mathbf{\Gamma}_{qc}). \tag{98}$$

This yields

$$\mathrm{rank}(\mathbf{\Gamma}_r) + \mathrm{rank}(\mathbf{\Gamma}_{qc}) + \mathrm{rank}(\mathbf{\Gamma}_h) + \mathrm{rank}(\mathbf{S}_q\mathbf{Z}) = \mathrm{rank}(\mathbf{I}_{q\times q}). \tag{99}$$

About subspace $\mathcal{R}_{\ker(\mathbf{E}_{uc})}$ with numerical simulations it is possible to show that this subspace is not a functional controllability subspace with the first and the second order derivative.

It is easy to show that this is a functional controllability subspace with respect to the third order derivative. This means that there exists a controlled invariant subspace included in $\ker(\mathbf{E}_{uc})$ for which the *Projection Property* (87) holds. Subspace $\mathrm{im}(\mathbf{B}_{uc})$ defined in (58) is considered:

$$\mathcal{Z}_0 = \mathrm{im}(\mathbf{B}_\tau), \tag{100}$$

now $\mathsf{im}(\mathbf{B}_{uc})$ is not included in

$$\mathsf{ker}(\mathbf{E}_{uc}) = \mathsf{im}\begin{bmatrix} \mathbf{I}_{q\times q} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathsf{ker}(\mathbf{\Gamma}_{uc}^{T}) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_{q\times q} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}_{u\times u} \end{bmatrix}, \tag{101}$$

it is easy to show that the element $\mathbf{M}_o^{-1}\mathbf{GBJ\Gamma}_{qc}$ in general is not included in $\mathsf{ker}(\mathbf{\Gamma}_{uc}^{T})$. This means that subspace $\mathsf{im}(\mathbf{B}_{uc})$ is not a functional controllability subspace with respect to the second order derivative. The next step:

$$\mathcal{Z}_1 = \mathsf{im}(\mathbf{B}_{\tau}) + \mathbf{A}(\mathcal{Z}_0 \cap \mathsf{im}(\mathbf{B}_{uc}) \cap \mathsf{ker}(\mathbf{E}_{uc})), \tag{102}$$

where

$$\mathcal{Z}_0 \cap \mathsf{im}(\mathbf{B}_{uc}) \cap \mathsf{ker}(\mathbf{E}_{uc}) = \mathsf{im}\begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{\Gamma}_{qc} \\ \mathbf{0} \end{bmatrix}, \tag{103}$$

thus

$$\mathcal{Z}_1 = \mathsf{im}\begin{bmatrix} \mathbf{0} & \mathbf{\Gamma}_{qc} \\ \mathbf{0} & \mathbf{0} \\ \mathbf{M}_h^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_o^{-1}\mathbf{GBJ\Gamma}_{qc} \end{bmatrix}, \tag{104}$$

we can see again that $\mathsf{im}(\mathbf{B}_{uc})$ is not included in $\mathcal{Z}_1 \cap \mathcal{R}_{B_{uc}} + \mathsf{ker}(\mathbf{E}_{uc})$. We can verify that subspace $\mathsf{im}(\mathbf{B}_{uc})$ is a functional controllability subspace with respect to the third order derivative.

$$\mathcal{Z}_2 = \mathsf{im}(\mathbf{B}_{\tau}) + \mathbf{A}\left(\mathcal{Z}_1 \cap \mathsf{im}(\mathbf{B}_{uc}) \cap \mathsf{ker}(\mathbf{E}_{uc})\right), \tag{105}$$

where

$$\mathcal{Z}_1 \cap \mathsf{im}(\mathbf{B}_{uc}) \cap \mathsf{ker}(\mathbf{E}_{uc}) = \mathsf{im}\begin{bmatrix} \mathbf{0} & \mathbf{\Gamma}_{qc} \\ \mathbf{0} & \mathbf{0} \\ \mathbf{\Gamma}_{qc} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_o^{-1}\mathbf{GBJ\Gamma}_{qc} \end{bmatrix}, \tag{106}$$

thus

$$\mathcal{Z}_2 = \mathsf{im}\begin{bmatrix} \mathbf{0} & \mathbf{\Gamma}_{qc} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{M}_o^{-1}\mathbf{GKJ\Gamma}_{qc} \\ \mathbf{M}_h^{-1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_o^{-1}\mathbf{GBJ\Gamma}_{qc} & \mathbf{L} \end{bmatrix}, \tag{107}$$

and

$$\mathbf{L} = \mathbf{M}_o^{-1}\mathbf{GKJ\Gamma}_{qc} - \mathbf{M}_o^{-1}\mathbf{GBG}^{T}\mathbf{M}_o^{-1}\mathbf{GBJ\Gamma}_{qc}. \tag{108}$$

It follows that

$$\mathsf{im}(\mathbf{B}_{uc}) \cap \mathcal{Z}_2 = \mathsf{im}\begin{bmatrix} \mathbf{0} & \mathbf{\Gamma}_{qc} & \mathbf{0} & \mathbf{\Gamma}_{qc} \\ \mathbf{0} & \mathbf{0} & \mathbf{M}_o^{-1}\mathbf{GKJ\Gamma}_{qc} & \mathbf{0} \\ \mathbf{\Gamma}_{qc} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_o^{-1}\mathbf{GBJ\Gamma}_{qc} & \mathbf{L} & \mathbf{0} \end{bmatrix}, \tag{109}$$

and that

$$\mathsf{im}(\mathbf{B}_{uc}) \subseteq \mathsf{im}(\mathbf{B}_{uc}) \cap \mathcal{Z}_2 + \mathsf{ker}(\mathbf{E}_{uc}). \tag{110}$$

To show that it is enough to notice that

$$\mathsf{im}(\mathbf{M}_o^{-1}\mathbf{G}\mathbf{K}\mathbf{J}\mathbf{\Gamma}_{qc}) + \mathsf{ker}(\mathbf{\Gamma}_{uc}^T) = \mathsf{im}(\mathbf{I}_{u\times u}). \tag{111}$$

In fact, considering that

$$\mathbf{M}_o^{-1}\mathbf{G}\mathbf{K}\mathbf{J}\mathbf{\Gamma}_{qc} = \mathbf{M}_o^{-1}\mathbf{G}\mathbf{K}\mathbf{G}^T\mathbf{\Gamma}_{uc}, \tag{112}$$

and that

$$\mathsf{rank}(\mathbf{M}_o^{-1}\mathbf{G}\mathbf{K}\mathbf{G}^T\mathbf{\Gamma}_{uc}) = \mathsf{rank}(\mathbf{\Gamma}_{uc}). \tag{113}$$

Relation (113) follows from

$$\mathsf{rank}(\mathbf{M}_o^{-1}\mathbf{G}\mathbf{K}\mathbf{G}^T\mathbf{\Gamma}_{uc}) = \mathsf{rank}(\mathbf{\Gamma}_{uc}), \tag{114}$$

because of matrix $\mathbf{M}_o^{-1}\mathbf{G}\mathbf{K}\mathbf{G}^T$ having the null subspace equal to zero. To complete the demonstration, it might be worthwhile to remember that the following property holds: $\mathsf{im}(\mathbf{\Gamma}_{uc}) + \mathsf{ker}(\mathbf{\Gamma}_{uc}^T) = \mathsf{im}(\mathbf{\Gamma}_{uc}) + (\mathsf{im}(\mathbf{\Gamma}_{uc}))^\perp = \mathsf{im}(\mathbf{I}_{u\times u})$. □

(Received January 14, 2011)

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