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# AN ITERATIVE ALGORITHM FOR COMPUTING THE CYCLE MEAN OF A TOEPLITZ MATRIX IN SPECIAL FORM

Peter Szabó

The paper presents an iterative algorithm for computing the maximum cycle mean (or eigenvalue) of $n \times n$ triangular Toeplitz matrix in max-plus algebra. The problem is solved by an iterative algorithm which is applied to special cycles. These cycles of triangular Toeplitz matrices are characterized by sub-partitions of $n-1$.

*Keywords:* max-plus algebra, eigenvalue, sub-partition of an integer, Toeplitz matrix

*Classification:* 90C27, 15B05, 15A80

## 1. INTRODUCTION

The class of Toeplitz matrices is much studied and still important within mathematics as well as in a wide range of applications (see [4, 6, 7]). Nevertheless, relatively little is known about their spectral properties. The aim of this work is to propose an efficient algorithm to find a real solution $\lambda$, $x_1, \dots, x_n \in \mathbb{R}$ to the system of equations

$$\max\{t_{i-1} + x_1,\, t_{i-2} + x_2, \dots,\, t_0 + x_i,\, x_{i+1},\, \dots,\, x_n\} = \lambda + x_i \tag{1}$$

for $i = 1, 2, \dots, n$. It will be assumed that $t_i$, for $i = 0, 1, \dots, n-1$ are non-negative real values. The system of equations (1) can be written in the form

$$A \otimes x = \lambda \otimes x$$

where $A = (a_{kj})$ is a triangular Toeplitz matrix, $a_{kj} = t_{k-j}$ for $k \geqq j$, $a_{kj} = 0$ for $k < j$ and $(\oplus, \otimes) = (max, +)$ are operations of the max-plus algebra. For a general $n \times n$ real matrix $A = (a_{ij})$ there exist standard $O(n^3)$ algorithms (see [5]) to find $\lambda,\ x_1, \dots, x_n$, solutions of the system

$$A \otimes x = \lambda \otimes x. \tag{2}$$

The proposed iterative algorithm solves the problem (1) in time $O(n^3)$ and uses special, combinatorial properties of triangular Toeplitz matrices. The algorithm is applied to special cycles which are characterized by sub-partitions of $n-1$. We show that using such cycles (sub-partitions), the values $\lambda,\ x_1, \dots, x_n$ of system (1) can be computed.

## 2. COMPUTING THE EIGENVALUE IN MAX-PLUS ALGEBRA.

In general, max-plus algebra is understood as an algebraic structure $(\overline{\mathbb{R}}, \max, +)$, where $\mathbb{R}$ is the set of real numbers, $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty\}$ and $a \oplus b = \max\{a, b\}$, $a \otimes b = a + b$ for all $a, b \in \overline{\mathbb{R}}$. Formally the operations $(\oplus, \otimes)$ can be extended to matrices and vectors in the same way as in linear algebra. The eigenvalue-eigenvector problem (2) (shortly: eigenproblem) was one of the first problems studied in max-plus algebra. Here we only discuss the case when $A$ does not contain $-\infty$, where for every matrix there is exactly one eigenvalue.

We begin with the discussion of a special digraph $D_A$ and the basic concept of the cycle mean. Let $\mathbb{R}^{n\times n}$ denotes the set of real $n \times n$ matrices. The associated digraph $D_A = (V, E)$ of a real matrix $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ is defined as a complete weighted digraph with the node set $V = N = \{1, \ldots, n\}$ and with the weights $w(i, j) = a_{ji}$ for every $(i,\ j) \in E = N \times N$. The set $E$ is called the edge set of $D_A$ and $(i,\ j) \in E$ is called a directed edge. We say that the edge $(i,\ j) \in E$ is joining vertices $i$ and $j$. In general, the path $p = \langle i_1, \ldots, i_k\rangle$ in a graph $G = (V, E)$ is a sequence of vertices $\{i_1, \ldots, i_k\} \subseteq V$ and edges $(i_{j-1}, i_j) \in E$ for $j = 2, \ldots, k$. Vertex $i_1$ is called the start vertex and vertex $i_k$ the end vertex. The path $s = \left\langle i_j, \ldots, i_l\right\rangle$ is a sub-path of $p$ if $1 \leq j$ and $l \leq k$. The paths will also be marked as $p = \left\langle p(1), p(2), \ldots, p(l + 1)\right\rangle$, where $p(i)$ are vertices for $i = 1, \ldots, l + 1$. If $p$ contains no vertices and no edges then the path $p$ is called empty. Let $p = \langle i_1, \ldots, i_k\rangle$ be a path. The number $k - 1$ is denoted as $|p|$ and called the length of $p$. The value $w(p) = a_{i_1 i_2} + \ldots + a_{i_{k-1} i_k}$ is termed the weight of $p$. If start vertex and end vertex is the same $(i_1 = i_k)$ then path $p$ is called a cycle. The cycle $p$ is termed an elementary cycle if, moreover, $i_j \neq i_l$ for $j, l = 1, \ldots, k-1$, $j \neq l$. The cycle $p$ is a loop if it contains only the vertex $i_1$ and edge $(i_1, i_1)$. If $\sigma$ is an elementary cycle then the value $\frac{w(\sigma)}{|\sigma|}$ is called the cycle mean of $\sigma$. A cycle with the maximum cycle mean is termed the critical cycle. The basic result of max-plus algebra [2] states that the maximum cycle mean in $D_A$ is equal to the unique eigenvalue of $A$.

**Theorem 2.1.** For every matrix $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ there is a unique value of $\lambda = \lambda(A)$ (called the eigenvalue of $A$) to which there is a vector $x \in \mathbb{R}^n$ satisfying (2). The unique eigenvalue is the maximum cycle mean in $D_A$ that is

$$\lambda(A) = \max_{\sigma} \frac{w(\sigma)}{|\sigma|}$$

where $\sigma = \left\langle i_1, \ldots, i_k\right\rangle$ denotes an elementary cycle in $D_A$. The maximization is taken over elementary cycles of all lengths in $D_A$, including loops.

In general, a matrix $A \in \overline{\mathbb{R}}^{n\times n}$ with $-\infty$ has several eigenvalues and the value $\lambda(A)$ from Theorem 2.1 is the greatest eigenvalue of $A$. A summary of concepts, methods, applications and combinatorial character of max-plus algebra can be found in [3] or [1]. One of the first publications to deal with max-plus algebra is [9].

## 3. GRAPHS, CYCLES AND INTEGER PARTITIONS

The class of $n \times n$ triangular Toeplitz matrices is defined as

$$T_n(t) = \begin{pmatrix} t_0 & 0 & 0 & \dots & 0 \\ t_1 & t_0 & 0 & & 0 \\ t_2 & t_1 & t_0 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ t_{n-1} & & \dots & t_1 & t_0 \end{pmatrix}$$

where $t = (t_0, t_1, \dots, t_{n-1})^T$, $t_i \in \mathbb{R}_0^+ = \langle 0, \infty)$ for $i = 0, \dots, n-1$. With every matrix $A \in T_n(t)$, a directed acyclic graph (DAG) $G_t = (N, E_t)$ can be associated, where $N = \{1, \dots, n\}$ are the vertices and $E_t = \{(i,j) | i < j;\, i,\, j = 1, \dots, n\}$ are the edges of graph $G_t$ with weight function $w_G(i,j) = a_{ji} = t_{j-i}$ for all $(i,j) \in E_t$. If $D_A$ is the associated digraph of matrix $A$ then $G_t$ is a sub-graph of $D_A$. A characterization of cycles of triangular Toeplitz matrices are presented in [8]. We recall briefly the main results of this paper.

**Definition 3.1.** Let $A \in T_n(t)$. Cycle $c_p$ in $D_A = (N, E)$ is called a *triangular Toeplitz cycle* if it can be decomposed as $c_p = p \cup e$, where $p = \langle\, p(1), \dots, p(l+1) \,\rangle$ is a path in $G_t$ and $e = (p(l+1),\, p(1)) \in E$.

**Lemma 3.2.** Let $A \in T_n(t)$ then for every cycle $c'$ from $D_A$ there is a triangular Toeplitz cycle $c_p = p \cup e$ such that $w(p) = w(c_p)$ and $\frac{w(c)}{|c|} \geq \frac{w(c')}{|c'|}$.

Hence, it follows that it is sufficient to consider only the triangular Toeplitz cycles for the computation of the eigenvalue of $A \in T_n(t)$.

If $m = \sum_{k=1}^{l} i_k \leq n-1$ and $l > 1$ then the sequence of positive integers $i_1, \dots, i_l$ is termed a *sub-partition on the integer* $n-1$ of size $l$. Also to be noted, that if $i_1, \dots, i_l$ is a sub-partition on $n-1$ then the order of the terms in the sum $\sum_{k=1}^{l} i_k$ is not significant. Let us assume that $A \in T_n(t)$ then we say that a path $p$ in $G_t$ is given by sub-partition $i_1, \dots, i_l$ if (3) is fulfilled. We show that the paths in $G_t$ given by an arbitrary permutation of set $\{i_1, \dots, i_l\}$ have the same weight. The next result of [8] describes the basic characteristics of paths in $G_t$.

**Lemma 3.3.** Let $A \in T_n(t)$. The sequence of positive integers $i_1, \dots, i_l$ is a sub-partition on number $n-1$ if and only if there is a path in graph $G_t$ such that

$$p = \langle 1, i_1+1, i_1+i_2+1, \dots, i_1+\dots+i_l+1 \rangle = \langle p(1), p(2), \dots, p(l+1) \rangle \,. \tag{3}$$

**Lemma 3.4.** Let $A \in T_n(t)$, and $p = \langle p(1), \dots, p(l+1) \rangle$ be a path in $G_t$ given by sub-partition $i_1, \dots, i_l$. Let $\pi : \{i_1, \dots, i_l\} \to \{i_1, \dots, i_l\}$ be a permutation of the set $\{i_1, \dots, i_l\}$ and the path $p_\pi$ be given by sub-partition $\pi(i_1), \dots, \pi(i_l)$. Then $w(p) = w(p_\pi) = t_{i_1} + \ldots + t_{i_l}$ and $p(l+1) = p_\pi(l+1)$.

Proof. It follows from (3) that $p(1) = 1$, $p(j) = 1 + i_1 + \ldots + i_{j-1}$ for $j = 2, \dots, l+1$. Suppose that $A \in T_n(t)$ then the weight of edge $(p(j),\, p(j+1))$ is equal to $w(p(j), p(j+1)) = a_{p(j+1)p(j)} = t_{p(j+1)p(j)} = t_{p(j+1)-p(j)} = t_{i_j}$ for $j = 1, \dots, l$. Therefore, the weight of path $p$ equals $w(p) = t_{i_1} + \ldots + t_{i_l}$ and the path $p_\pi$ given by sub-partition $\pi(i_1), \dots, \pi(i_l)$ equals $w(p_\pi) = t_{\pi(i_1)} + \ldots + t_{\pi(i_l)}$. Thus, for each permutation

$\pi : \{i_1, \dots, i_l\} \to \{i_1, \dots, i_l\}$ we have $w(p) = t_{i_1} + t_{i_2} + \dots + t_{i_l} = t_{\pi(i_1)} + t_{\pi(i_2)} + \dots + t_{\pi(i_l)} = w(p_\pi)$ and $p(l+1) = 1 + i_1 + \dots + i_l = 1 + \pi(i_1) + \dots + \pi(i_l) = p_\pi(l+1)$. □

Figure 1 shows a graph $G_t$, where $t = (t_0, t_1, t_2, t_3, t_4)$, $n - 1 = 4$. The path $p = \langle 1, 2, 3, 5 \rangle$ in $G_t$ corresponds to a sub-partition $1, 1, 2$ of 4 and the path $p_\pi = \langle 1, 2, 4, 5 \rangle$ corresponds to a sub-partition $1, 2, 1$ and vice versa. The weight of path $p$ equals $w(p) = t_1 + t_1 + t_2 = t_1 + t_2 + t_1 = w(p_\pi)$, $l = 3$ and $p(4) = p_\pi(4) = 5$.

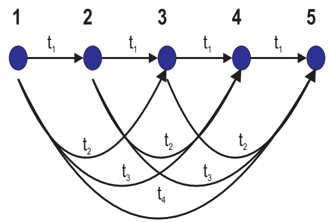


**Fig. 1.** Graph $G_t$.

## 4. AN ESTIMATION FUNCTION AND ITS FEATURES

In this chapter we define a specific function. The features of function will serve to determine the wanted eigenvalue. It will be assumed that the triangular Toeplitz matrix $A$ given by vector $t(z) = (z, t_1, \dots, t_{n-1})^T$ where $t_i \in \mathbb{R}_0^+$ are fixed numbers for $i = 1, \dots, n-1$ and $z \in \mathbb{R}_0^+$ is a variable. Note that it follows from the definition of graph $G_t$ that $G_{t(z)} = G_t$ for all $z \in \mathbb{R}_0^+$.

**Definition 4.1.** Let $A \in T_n(t(z))$ be a triangular Toeplitz square matrix given by the vector $t(z) = (z, t_1, \dots, t_{n-1})$. The vector $x(z) = (x_1(z), \dots, x_n(z))$ is called the *sub-eigenvector* of $A$ corresponding to the value $z \in \mathbb{R}_0^+$ if it is defined by the formula:

1. $x_1(z) = 0$
2. $x_i(z) = \max\{x_{i-1}(z), \max_{j=1,\dots,i-1}\{t_{i-j} + x_j(z) - z\}\}$ for $i = 2, \dots, n$.

The sub-eigenvector $x(z)$ may become an eigenvector of the matrix $A$ due to the following Lemma.

**Lemma 4.2.** Let $A \in T_n(t(z))$, $z \in \mathbb{R}_0^+$ and $x(z)$ be a sub-eigenvector of $A$. Then $A \otimes x(z) = z \otimes x(z)$ if and only if $z \geq x_n(z)$.

P r o o f . Suppose that $z \geq x_n(z)$. Let us denote $[A \otimes x(z)]_i$ the $i$th element of the vector $[A \otimes x(z)]$. It follows from Definition 4.1 that $0 = x_1(z) \leq \dots \leq x_n(z)$, therefore $[A \otimes x(z)]_1 = \max\{z + x_1(z), x_2(z), \dots, x_n(z)\} = \max\{z, x_n(z)\} = z$. For all $i > 1$ we have $x_i(z) \geq \max_{j=1,\dots,i-1}\{t_{i-j} + x_j(z)\} - z$ and by a simple computation $[A \otimes x(z)]_i = \max\{t_{i-1} + x_1(z), t_{i-2} + x_2(z), \dots, t_1 + x_{i-1}(z), z + x_i(z), x_{i+1}(z), \dots, x_n(z)\}$

=max$\{x_i(z) + z,\, x_n(z)\} = x_i(z) + z$ is obtained. Hence, $A \otimes x(z) = z \otimes x(z)$. Let us assume that $A \otimes x(z) = z \otimes x(z)$ and $x(z)$ is a sub-eigenvector of $A$. The relation $z \geq x_n(z)$ is obtained after insertion of known data $[A \otimes x(z)]_1 = \max\{z + x_1(z),\, x_2(z), \dots,\, x_n(z)\} = \max\{z,\, x_n(z)\} = z$. □

**Lemma 4.3.** Let $A \in T_n(t(z))$, $z \in \mathbb{R}_0^+$ and $x(z)$ be a sub-eigenvector of A. Then $x(z) = 0$ if and only if $z \geq \max_{j=1,\dots,n-1} t_j$.

Proof. Let $A \in T_n(t(z))$. Let us assume that $z \geq \max_{j=1,\dots,n-1} t_j$. By a simple computation it follows that $x_i(z) = 0$ for all $i = 1, \dots, n$ (shortly: $x(z) = 0$) and $A \otimes x(z) = z \otimes x(z)$. In this case $z$ is the eigenvalue and $x(z) = 0$ is the eigenvector. From the assumption $x(z) = 0$, it follows that $z \geq \max_{j=1,\dots,n-1} t_j$. □

Let $A \in T_n(t(z))$ be a triangular Toeplitz matrix where $t(z) = (z, t_1, \dots, t_{n-1})$. In the next, it will be assumed that $z < \max_{j=1,\dots,n-1} t_j$, i.e. $x(z) \neq 0$. Otherwise, according to Lemma 4.3 $z = \lambda(A)$ and $x(z) = 0$. Let us focus on the real function $y_A(z) = x_n(z) - z$.

**Definition 4.4.** Let $x(z) = (x_1(z), \dots, x_n(z))$ be a sub-eigenvector of a matrix $A \in T_n(t(z))$. The expression

$$y_A(z) = x_n(z) - z$$

is termed *an estimation function of eigenvalue* $\lambda(A)$.

**Theorem 4.5.** Let $x(z) = (x_1(z), \dots, x_n(z))$ be a sub-eigenvector of a matrix $A \in T_n(t(z))$. For each $z \in \langle 0, \max_{j=1,\dots,n-1} t_j \rangle$ there is a path $p$ in $G_t$ such that

$$y_A(z) = x_n(z) - z = w(p) - (|p| + 1)z$$

and $n$ is the end vertex of $p$.

Proof. Let $z$ be an arbitrary element of the interval $\langle 0, \max_{j=1,\dots,n-1} t_j \rangle$ and $x(z)$ be a sub-eigenvector of $A$. We shall show first that there is a path $p$ in graph $G_t$ such as

$$y_A(z) = x_n(z) - z = w(p) - (|p| + 1)\, z. \tag{4}$$

From the assumption $z \in \langle 0, \max_{j=1,\dots,n-1} t_j \rangle$ and from Lemma 4.3 it follows that the sub-eigenvector $x(z) \neq 0$ and $x_n(z) > 0$. It follows from the definition of $x(z)$ that the vector components are non-decreasing, non-negative and $x_n(z) \geq t_{n-k} + x_k(z) - z$ for all $k = 1, \dots, n-1$.

We will first prove that the set $M_n(z) = \{l;\, x_n(z) = t_{n-l} + x_l(z) - z\}$ is non empty. If we assume that $x_n(z) > t_{n-k} + x_k(z) - z$ for all $k = 1, \dots, n-1$ then $x_n(z) = x_{n-1}(z)$ by Definition 4.1. The condition $x_n(z) > 0$ implies that there is an index $j$ such that $x_n(z) = x_{n-1}(z) = \cdots = x_{n-j}(z)$ and $x_{n-j}(z) = t_{n-j-l} + x_l(z) - z > 0$ for some $l$, moreover $n - j - l \geq 1$. Therefore, we obtain $x_n(z) = x_{n-j}(z) = t_{n-j-l} + x_l(z) - z \leq t_{n-(j+l)} + x_{j+l}(z) - z$, where $j + l \leq n - 1$, which is a contradiction.

Let $l_1 \in M_n(z)$ be an arbitrary index and let $p$ be an empty path in $G_t$. We add vertices $l_1$, $n$ and the edge $(l_1, n)$ to the path $p$. The value $y_A(z)$ can be written as follows: $y_A(z) = x_n(z) - z = t_{n-l_1} + x_{l_1}(z) - 2z$. If $x_{l_1}(z) = 0$ then $y_A(z) = t_{n-l_1} - 2z = w(p) - (|p| + 1)z$. If $x_{l_1}(z) > 0$ then $M_{l_1}(z) = \{j;\ x_{l_1}(z) = t_{l_1-j} + x_j(z) - z\}$ is non empty. Let $l_2 \in M_{l_1}(z)$ be an arbitrary index $(l_2 < l_1)$. We add the vertex $l_2$ and the edge $(l_2, l_1)$ to the path $p$. If $x_{l_2}(z) = 0$ then $y_A(z) = t_{n-l_1} + t_{l_1-l_2} - 3z = w(p) - (|p|+1)z$. While $x_{l_k}(z) > 0$ this procedure is repeated. If the condition $x_{l_j}(z) = 0$ is met, the procedure is finished. Such a component $x_{l_j}(z)$ of $x(z)$ exists because $x_1(z) = 0$ and $x_1(z) \leq \ldots \leq x_n(z)$. Finally, we obtain $y_A(z) = t_{n-l_1} + t_{l_1-l_2} + \ldots + t_{l_{j-1}-l_j} - (j+1)z = w(p) - (|p| + 1)z$, where $p = \langle l_j, \ldots, l_1, n\rangle$ is a path in graph $G_t$. □

Note, if for $z \in \left\langle 0, \max_{j=1,\ldots,n-1} t_j\right)$ there is a path $p$ from $G_t$ such that $y_A(z) = x_n(z) - z = w(p) - (|p| + 1)z$, so there exists such a path $p*$ of minimum length, i. e.

$$|p*| = \min\left\{|p|;\ y_A(z) = x_n(z) - z = w(p) - (|p| + 1)z\right\}.$$

We show how to construct such a path in time $O(n^2)$. Each element $l_1 \in M_n(z)$ from the proof of Theorem 4.5 defines a class of paths in $G_t$. This class of paths is characterized by integers $n - l_1$, $l_1 - l_2$, …, $l_{j-1} - l_j$ or by directed edges with weights $t_{n-l_1}$, $t_{l_1-l_2}$, …, $t_{l_{j-1}-l_j}$, which define the path $p_{l_1} = \left\langle l_j, \ldots, l_1, n\right\rangle$. We denote $m_i(z) = \min M_i(z) = \min\{l;\ x_i(z) = t_{i-l} + x_l(z) - z\}$ for $i = 1, \ldots, n$ and we define $m_j(z) = 0$ when $M_j(z) = \varnothing$ for some $j$. The $l_i$ values are computed as $l_i = m_{l_{i-1}}(z)$ for $i = 1, \ldots, j$. The complexity of the computation of integers $n - l_1, l_1 - l_2, l_2 - l_3, \ldots, l_{j-1} - l_j$ (or path $p_{l_1}$ ) is $O(j) \leq O(n)$. The computation and the assignment of a path $p_i^*$ is performed for each element $i \in M_n(z)$. Now just assign $|p^*| = \min\{|p_i^*|;\ y_A(z) = x_n(z) - z = w(p_i^*) - (|p_i^*| + 1)z\}$. The overall complexity of the procedure is $O(n^2)$, because $|M_n(z)| \leq n$. We will refer to the procedure of creation the path $p^*$ as a *path assignment procedure.* So the next claim is proved.

**Lemma 4.6.** For each $z \in \left\langle 0, \max_{j=1,\ldots,n-1} t_j\right)$ the path assignment procedure finds all paths $p$ in $G_t$ such that $y_A(z) = w(p) - (|p| + 1)z$ in time $O(n^2)$.

Now, we can define an equivalence relation of paths in $G_t$. Two paths $p_1$, $p_2$ are said to be equivalent if and only if $w(p_1) = w(p_2)$ and $|p_1| = |p_2|$. If a path $p$ belongs to the same class of equivalence then this class is marked as $[p]$.

**Theorem 4.7.** Let $x(z) = (x_1(z), \ldots, x_n(z))$ be a sub-eigenvector of a matrix $A \in T_n(t(z))$. The function $y_A(z) = x_n(z) - z$ is decreasing and piecewise linear on interval $\left\langle 0, \max_{j=1,\ldots,n-1} t_j\right)$ with integer slopes and moreover $y_A(z^*) = 0$ if only if $z^* = \lambda(A)$.

Proof. Let $z$ be an arbitrary element of interval $\left\langle 0, \max_{j=1,\ldots,n-1} t_j\right)$. From Theorem 4.5 it follows that there is a path in $G_t$ such that $y_A(z) = x_n(z) - z = w(p) - (|p| + 1)z$.

If there is only one equivalence class $[p*]$ such that $y_A(z) = x_n(z) - z = w(p*) - (|p*| + 1)z$ (in other words, if $[z, y_A(z)]$ is not an intersection point of two lines) then there is a small neighbourhood $(z_1, z_2)$ around $z$ where $y_A(z)$ is linear (with negative slope) and decreasing. Assume now that $y_A(z) = w(p_1) - (|p_1| + 1)z = w(p_2) - (|p_2| + 1)z$ and $|p_1| < |p_2|$. Therefore, there are two paths $p*$ and $\overline{p*}$ such that $y_A(z) = w(p*) -$

$(|p*|+1)z = w(\overline{p*}) - (|\overline{p*}|+1)z$ and $p*$ has a minimum and $\overline{p*}$ a maximum length of such paths, hence $|p*| < |\overline{p*}|$. For this reason, there is a small interval $(z_1, z\rangle$ where $y_A(z) = w(\overline{p*}) - (|\overline{p*}|+1)z$ and a small interval $\langle z, z_2)$ where $y_A(z) = w(p*) - (|p*|+1)z$. Function $y_A(z)$ on intervals $(z_1, z\rangle$ and $\langle z, z_2)$ is linear and decreasing, therefore $y_A(z)$ is a piecewise linear and decreasing on interval $\langle 0, \max_{j=1,\dots,n-1} t_j)$.

Now we prove the second part of the theorem. If the condition $y_A(\overline{z}) = 0$ is met then $\overline{z} = \lambda(A)$ with regard to Lemma 4.2. Now we suppose that $\overline{z} < \max_{j=1,\dots,n-1} t_j$ and $\overline{z} = \lambda(A)$. It is necessary to prove that $y_A(\overline{z}) = x_n(\overline{z}) - \overline{z} = 0$. The condition $y_A(\overline{z}) = x_n(\overline{z}) - \overline{z} > 0$ implies that $\overline{z} \neq \lambda(A)$ by Lemma 4.2. Assume that $y_A(\overline{z}) = x_n(\overline{z}) - \overline{z} < 0$. From Lemma 4.2 it follows that for any non-critical cycle $c$ of $D_A$ the inequality $y_A(\frac{w(c)}{|c|}) > 0$ is fulfilled. The function $y_A(z)$ is piecewise linear on the interval $(\frac{w(c)}{|c|}, \overline{z}) \subseteq \langle 0, \max_{j=1,\dots,n-1} t_j)$. Therefore $y_A(z)$ is also a continuous function. Hence, there exists $z' \in (\frac{w(c)}{|c|}, \overline{z})$ such as $y_A(z') = x_n(z') - z' = 0$. The already proved sufficient condition implies that $z' = \lambda(A)$. From Theorem 2.1 it follows that $\lambda(A) = \overline{z}$ is a unique eigenvalue, but $\lambda(A) = z' \neq \overline{z}$, which contradicts with condition $y_A(\overline{z}) < 0$. □

## 5. AN ITERATIVE ALGORITHM

We propose a simple iterative algorithm to obtain the eigenvalue $\lambda(A)$ based on Theorem 4.7.

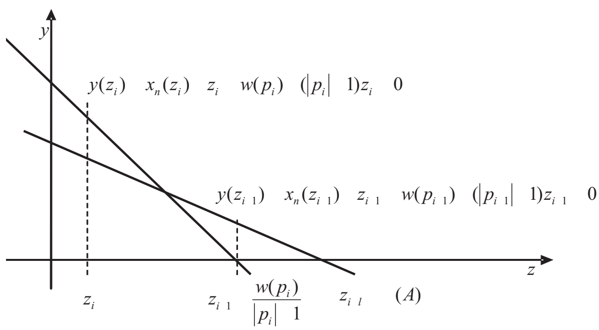


**Fig. 2.** An iterative step of the algorithm.

The Figure 2 shows an iterative step of the algorithm, where $z_i$, $z_{i+1}$ are estimates of the eigenvalue $\lambda(A)$. The algorithm solves problem (1) in $O(n^3)$ steps. Each iterative step has a complexity $O(n^2)$ (paths $p_i$ with minimum slope are created by path assignment procedure, see Lemma 4.6). The number of iterative steps does not exceed $n$, the maximum possible slope of function $y_A(z)$. The number of iterative steps depends on the initial estimate $z_0$, but on the general complexity of the iterative method it has no effect.

**Algorithm 1** An iterative algorithm

{Input: $A \in T_n(t)$, where $t = (t_0, t_1, \ldots, t_{n-1})^T$, $t_j \in \mathbb{R}_0^+$ for $j = 0 \ldots, n-1$.}
$i = 0$; $z_0 = t_0$;
**if** $y_A(z_0) \leq 0$ **then**
  {$z_0 = t_0$ is the eigenvalue, $x(z_0)$ is an eigenvector of matrix $A$ and the loop $(1,1)$ is a critical cycle.}
**end if**
**while** $y_A(z_i) > 0$ **do**
  $i = i + 1$;
  $z_i = \frac{w(p_{i-1})}{|p_{i-1}|+1}$;
**end while**
{If $y_A(z_i) = w(p_i) - (|p_i| + 1)z_i > 0$ then $i = i+1$ and $z_i = \frac{w(p_{i-1})}{|p_{i-1}|+1}$ is the next estimate of $\lambda(A)$. If $y_A(z_i) = w(p_i) - (|p_i| + 1)z_i = 0$ then $z_i$ is the eigenvalue of $A$, $x(z_i)$ is an eigenvector (see Theorem 4.7) and $c_{p_i} = p_i \cup e$ is a critical cycle. The value of $w(p_i)$ can be expressed as $t_{i_1} + \cdots + t_{i_l}$ and the indices $i_1, \ldots, i_l$ define a sub-partition of $n - 1$.}

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