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EXISTENCE OF POSITIVE PERIODIC SOLUTIONS OF HIGHER-ORDER FUNCTIONAL DIFFERENCE EQUATIONS

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Abstract. Based on the fixed-point theorem in a cone and some analysis skill, a new sufficient condition is obtained for the existence of positive periodic solutions for a class of higher-order functional difference equations. An example is used to illustrate the applicability of the main result.

Keywords: positive periodic solution; existence of positive periodic solution; fixed-point theorem; difference equation

MSC 2010: 34K13, 39A70

1. Introduction

The existence of periodic solutions of functional differential equations has been studied extensively. Many authors [1], [9] have argued that the discrete time models governed by difference equations sometimes are more appropriate than the continuous ones, for example, the predator-prey system having nonoverlapping generations. With help of differential equations with piecewise constant arguments, Fan and Wang [3] proposed a discrete analogue of the continuous time predator-prey system and gave some new sufficient conditions for the existence of a positive periodic solution of the discrete system. Discrete time models can also provide efficient computational models of continuous models for numerical simulations. It is well known that, compared to the continuous time systems, the dynamics of the discrete time systems are more difficult to deal with. It is highly nontrivial to attack the existence of positive periodic solutions of a discrete time system which is governed

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by a higher-order functional difference equation. Raffoul [7] studied the existence of positive periodic solutions for functional difference equations. By using Krasnoselskii's fixed-point theorem and the upper and lower solutions method, Zhu and Li [10] found some sets of positive values λ guaranteeing that there exist positive periodic solutions to the higher-dimensional functional difference equation of the form

$$x(n+1) = A(n)x(n) + \lambda h(n) f(x(n-\tau(n))), \quad n \in \mathbb{Z},$$

where $A(n) = \operatorname{diag}[a_1(n), a_2(n), \ldots, a_m(n)]$, $h(n) = \operatorname{diag}[h_1(n), h_2(n), \ldots, h_m(n)]$, $a_j, h_j\colon \mathbb{Z} \to \mathbb{R}^+$, $\tau\colon \mathbb{Z} \to \mathbb{Z}$ are T-periodic, $j = 1, 2, \ldots, m$, $T \geqslant 1$, $\lambda > 0$, $x\colon \mathbb{Z} \to \mathbb{R}^m$, $f\colon \mathbb{R}^m_+ \to \mathbb{R}^m_+$, where $\mathbb{R}^m_+ = \{(x_1, \ldots, x_m)^T \in \mathbb{R}^m, x_j \geqslant 0, j = 1, 2, \ldots, m\}$, $\mathbb{R}^+ = \{x \in \mathbb{R}, x > 0\}$. By using a well-known fixed-point index theorem, Li and Lu [5] studied the existence, multiplicity and nonexistence of positive periodic solutions to higher-dimensional nonlinear functional difference equations. Ma and Ma [6] investigated the existence of sign-changing periodic solutions of second order difference equations. In 2010, Wang and Chen [8] have studied the existence of positive periodic solutions for the general higher-order functional difference equation

$$x(n+m+k) - ax(n+m) - bx(n+k) + abx(n) = f(n, x(n-\tau(n))), \tag{1.1}$$

where $a \neq 1$, $b \neq 1$ are positive constants, $\tau\colon \mathbb{Z} \to \mathbb{Z}$ and $\tau(n+\omega) = \tau(n)$, $f(n+\omega, u) = f(n, u)$ for any $u \in \mathbb{R}$, $\omega, m, k \in \mathbb{N}$, where $\mathbb{N}$ denotes the set of positive integers. Based on the fixed-point theorem in a cone [8], some new sufficient conditions on the existence of positive periodic solutions to the higher-order functional difference equation (1.1) are obtained. However, the main results in [8] require that a and b should be positive constants. In this article, we consider the higher-order functional difference equation

$$\begin{aligned} x(n+m+k) - a(n+m)x(n+m) - b(n)x(n+k) + a(n)b(n)x(n) \\ + f(n, x(n-\tau(n))) = 0, \end{aligned} \tag{1.2}$$

where $a, b\colon \mathbb{Z} \to \mathbb{R}_+$ with $a(n) \neq 1$, $b(n) \neq 1$ and $a(n+\omega) = a(n)$, $b(n+\omega) = b(n)$, $\tau\colon \mathbb{Z} \to \mathbb{Z}$ and $\tau(n+\omega) = \tau(n)$, $f(n+\omega, u) = f(n, u)$ for any $u \in \mathbb{R}$, $k, \omega, m \in \mathbb{N}$.

The purpose of this article is to consider the existence of a positive periodic solution of the higher-order functional difference equation (1.2). We will remove the constrains on a and b in [8]. We will replace constants a and b in [8] with functions $a(n)$ and $b(n)$. Based on a fixed point theorem in a cone, a new sufficient condition is established for the existence of positive periodic solutions for higher-order functional difference equations.

2. Some preliminaries

Let X be the set of all real ω periodic sequences. Then X is a Banach space with the maximum norm $\|x\| = \max\limits_{n\in[0,\omega-1]} |x(n)|$.

Lemma 1 ([2], [4]). *Let X be a Banach space and K a cone in X. Suppose Ω_1 and Ω_2 are open subsets of X such that $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$ and suppose that*

$$\Phi\colon\ K \cap (\overline{\Omega}_2 \setminus \Omega_1) \to K$$

is a completely continuous operator such that

(i) $\|\Phi u\| \leqslant \|u\|$ *for* $u \in K \cap \partial\Omega_1$ *and there exists* $\psi \in K\setminus\{0\}$ *such that* $x \neq \Phi x + \lambda\psi$ *for* $x \in K \cap \partial\Omega_2$ *and* $\lambda > 0$; *or*

(ii) $\|\Phi u\| \leqslant \|u\|$ *for* $u \in K \cap \partial\Omega_2$ *and there exists* $\psi \in K\setminus\{0\}$ *such that* $x \neq \Phi x + \lambda\psi$ *for* $x \in K \cap \partial\Omega_1$ *and* $\lambda > 0$.

Then Φ has a fixed-point in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

Let $d \in \mathbb{N}$. Consider the equation

$$x(n+d) = cx(n) + \gamma(n), \tag{2.1}$$

where $\gamma \in X$. Set (d,ω) as the greatest common divisor of d and ω, $p = \omega/(d,\omega)$.

Lemma 2 ([8]). *Assume that $0 < c \neq 1$. Then (2.1) has a unique periodic solution*

$$x(n) = [c^{-p} - 1]^{-1} \sum_{i=1}^{p} c^{-i}\gamma(n + (i-1)d).$$

Assume that $x \in X$ is a solution of (1.2) and let $y(n) = x(n+k) - a(n)x(n)$, $\bar{a} = \max\limits_{1\leqslant n\leqslant\omega} a(n)$, $\underline{a} = \min\limits_{1\leqslant n\leqslant\omega} a(n)$. Then (1.2) can be rewritten as

$$\begin{cases} x(n+k) = \underline{a}x(n) + y(n) + [a(n) - \underline{a}]x(n), \\ y(n+m) = b(n)y(n) - f(n, x(n-\tau(n))). \end{cases} \tag{2.2}$$

Let $h = \omega/(k,\omega)$, $l = \omega/(m,\omega)$. Since $x \in X$ is a solution of (1.2), hence $y \in X$. From Lemma 2 we have

$$x(n) = [\underline{a}^{-h} - 1]^{-1} \sum_{i=1}^{h} \underline{a}^{-i}\{y(n + (i-1)k) + [a(n + (i-1)k) - \underline{a}]x(n + (i-1)k)\}.$$

From the second equation in (2.2) we have

$$\prod_{s=1}^{l-1} b(n+sm)y(n+m) = \prod_{s=0}^{l-1} b(n+sm)y(n) - \prod_{s=1}^{l-1} b(n+sm)f(n,x(n-\tau(n))),$$

$$\begin{aligned}\prod_{s=2}^{l-1} b(n+sm)y(n+2m) &= \prod_{s=1}^{l-1} b(n+sm)y(n+m)\\ &\quad - \prod_{s=2}^{l-1} b(n+sm)f(n+m,x(n+m-\tau(n+m))),\\ &\quad\cdots\\ y(n+lm) &= b(n+(l-1)m)y(n+(l-1)m)\\ &\quad - f(n+(l-1)m,x(n+(l-1)m-\tau(n+(l-1)m))).\end{aligned}$$

Summing the above equations yields

$$\begin{aligned}y(n+lm) = {}& \prod_{s=0}^{l-1} b(n+sm)y(n)\\ & - \sum_{j=0}^{l-2}\prod_{s=j+1}^{l-1} b(n+sm)f(n+jm,x(n+jm-\tau(n+jm)))\\ & - f(n+(l-1)m,x(n+(l-1)m-\tau(n+(l-1)m))).\end{aligned}$$

For convenience, denote $1 = \prod_{s=l}^{l-1} b(n+sm)$. Noting that $y(n+lm) = y(n)$, we obtain

$$y(n) = \sum_{j=0}^{l-1} \frac{\prod_{s=j+1}^{l-1} b(n+sm)}{\prod_{s=0}^{l-1} b(n+sm)-1} f(n+jm,x(n+jm-\tau(n+jm))). \tag{2.3}$$

Let $\bar{b} = \max\limits_{1\leqslant n\leqslant \omega} b(n)$, $\underline{b} = \min\limits_{1\leqslant n\leqslant\omega} b(n)$. We introduce the following condition:

(H) $0 < a(n) < 1$, $\underline{b} > 1$, $h = l = \omega$, and $f\colon \mathbb{R}\times(0,+\infty)\to[0,+\infty)$ is continuous.

Let

$$B = \prod_{s=0}^{l-1} b(s) = \prod_{s=0}^{\omega-1} b(s), \quad d = \min\left\{\frac{1}{B-1}, 1\right\},$$

$$G(n,j) = \frac{\prod_{s=j+1}^{l-1} b(n+sm)}{\prod_{s=0}^{l-1} b(n+sm)-1}.$$

Then

$$\frac{1}{B-1} \leqslant G(n,j) \leqslant \frac{B}{B-1} \tag{2.4}$$

and

$$y(n)=\sum_{j=0}^{l-1}G(n,j)f(n+jm,x(n+jm-\tau(n+jm))). \tag{2.5}$$

If $f(n,x(n-\tau(n)))\geqslant 0$ and $\underline{b}>1$, then $y(n)\geqslant 0$.

Define the operator T by

$$\begin{aligned}(Tx)(n)=\frac{\underline{a}^h}{1-\underline{a}^h}\sum_{i=1}^{h}\underline{a}^{-i}\sum_{j=0}^{l-1}G(n+(i-1)k,j)f\big(n+(i-1)k+jm,\\ x(n+(i-1)k+jm-\tau(n+(i-1)k+jm))\big)\\ +\frac{\underline{a}^h}{1-\underline{a}^h}\sum_{i=1}^{h}\underline{a}^{-i}[a(n+(i-1)k)-\underline{a}]x(n+(i-1)k).\end{aligned}$$

Define the cone by

$$K=\{x\in X, x(n)\geqslant\delta\|x\|\},$$

where

$$\delta=\frac{\underline{a}^h(B-1)d}{B\omega}.$$

Lemma 3. *Assume that* (H) *holds and* $0<r_1<r_2$. *Then* $T\colon\overline{K}_{r_2}\setminus K_{r_1}\to K$ *is completely continuous, where* $K_r=\{x\in K\colon\ \|x\|<r\}$ *and* $\overline{K}_r=\{x\in K\colon\ \|x\|\leqslant r\}$.

Proof. Since $0<a(n)<1$, hence $0<\underline{a}<1$. Noting that $\underline{b}>1$ and $f(n,x(n-\tau(n)))\geqslant 0$, we have $y(n)\geqslant 0$. So $(Tx)(n)\geqslant 0$ on $[0,\omega-1]$. Since $\tau(n+\omega)=\tau(n)$ and $f(n+\omega,u)=f(n,u)$ for any $u>0$, $(Tx)(n+\omega)=(Tx)(n)$ for $x\in X$. Since $h=l=\omega/(m,\omega)=\omega$, we get

$$\sum_{j=0}^{l-1}f(n+jm,x(n+jm-\tau(n+jm)))=\sum_{j=0}^{\omega-1}f(j,x(j-\tau(j))), \tag{2.6}$$

$$\sum_{i=1}^{h}f(n+(i-1)k,x(n+(i-1)k-\tau(n+(i-1)k)))=\sum_{i=1}^{\omega}f(i,x(i-\tau(i))), \tag{2.7}$$

and

$$\sum_{i=1}^{h}[a(n+(i-1)k)-\underline{a}]x(n+(i-1)k)=\sum_{i=1}^{\omega}[a(i)-\underline{a}]x(i). \tag{2.8}$$

For any $x \in \overline{K}_{r_2} \setminus K_{r_1}$, from (2.6)–(2.8), we have

$$
\begin{aligned}
(Tx)(n) \leqslant{} & \frac{\underline{a}^h}{1-\underline{a}^h}\underline{a}^{-h}\sum_{i=1}^{h}\sum_{j=0}^{l-1}\frac{B}{B-1}\{f(n+(i-1)k+jm,\\
& x(n+(i-1)k+jm-\tau(n+(i-1)k+jm)))\}\\
& +\frac{\underline{a}^h}{1-\underline{a}^h}\underline{a}^{-h}\sum_{i=1}^{h}[a(n+(i-1)k)-\underline{a}]x(n+(i-1)k)\\
={} & \frac{\underline{a}^h}{1-\underline{a}^h}\underline{a}^{-h}\sum_{j=0}^{l-1}\frac{B}{B-1}\Bigg\{\sum_{i=1}^{h}f(n+(i-1)k+jm,\\
& x(n+(i-1)k+jm-\tau(n+(i-1)k+jm)))\Bigg\}\\
& +\frac{\underline{a}^h}{1-\underline{a}^h}\underline{a}^{-h}\sum_{i=1}^{\omega}(a(i)-\underline{a})x(i)\\
={} & \frac{1}{1-\underline{a}^h}\left(\sum_{j=0}^{l-1}\frac{B}{B-1}\right)\sum_{i=1}^{\omega}f(i,x(i-\tau(i)))\\
& +\frac{1}{1-\underline{a}^h}\sum_{i=1}^{\omega}(a(i)-\underline{a})x(i)\\
={} & \frac{1}{1-\underline{a}^h}\frac{\omega B}{B-1}\sum_{i=1}^{\omega}f(i,x(i-\tau(i)))\\
& +\frac{1}{1-\underline{a}^h}\sum_{i=1}^{\omega}(a(i)-\underline{a})x(i)\\
\leqslant{} & \frac{1}{1-\underline{a}^h}\frac{\omega B}{B-1}\sum_{i=1}^{\omega}\{f(i,x(i-\tau(i)))+(a(i)-\underline{a})x(i)\}.
\end{aligned}
$$

So

$$
\|Tx\| \leqslant \frac{\omega}{1-\underline{a}^h}\frac{B}{B-1}\sum_{i=1}^{\omega}\{f(i,x(i-\tau(i)))+(a(i)-\underline{a})x(i)\}. \tag{2.9}
$$

At the same time,

$$
\begin{aligned}
(Tx)(n) \geqslant{} & \frac{\underline{a}^h}{1-\underline{a}^h}\underline{a}^{-1}\sum_{i=1}^{h}\sum_{j=0}^{l-1}G(n+(i-1)k,j)\{f(n+(i-1)k+jm,\\
& x(n+(i-1)k+jm-\tau(n+(i-1)k+jm)))\}\\
& +\frac{\underline{a}^h}{1-\underline{a}^h}\underline{a}^{-1}\sum_{i=1}^{h}[a(n+(i-1)k)-\underline{a}]x(n+(i-1)k)
\end{aligned}
$$

$$
\begin{aligned}
&\geqslant \frac{\underline{a}^h}{1-\underline{a}^h}\underline{a}^{-1}\sum_{j=0}^{l-1}\frac{1}{B-1}\bigg\{\sum_{i=1}^{h} f(n+(i-1)k+jm,\\
&\qquad x(n+(i-1)k+jm-\tau(n+(i-1)k+jm)))\bigg\}\\
&\qquad + \frac{\underline{a}^h}{1-\underline{a}^h}\underline{a}^{-1}\sum_{i=1}^{\omega}(a(i)-\underline{a})x(i)\\
&\geqslant \frac{\underline{a}^h}{1-\underline{a}^h}\sum_{j=0}^{l-1}\frac{1}{B-1}\bigg\{\sum_{i=1}^{\omega} f(i,x(i-\tau(i)))\bigg\}\\
&\qquad + \frac{\underline{a}^h}{1-\underline{a}^h}\sum_{i=1}^{\omega}(a(i)-\underline{a})x(i)\\
&= \frac{\underline{a}^h}{1-\underline{a}^h}\bigg(\sum_{j=0}^{\omega-1}\frac{1}{B-1}\bigg)\bigg\{\sum_{i=1}^{\omega} f(i,x(i-\tau(i)))\bigg\}\\
&\qquad + \frac{\underline{a}^h}{1-\underline{a}^h}\sum_{i=1}^{\omega}(a(i)-\underline{a})x(i)\\
&= \frac{\underline{a}^h}{1-\underline{a}^h}\Big(\frac{\omega}{B-1}\Big)\bigg\{\sum_{i=1}^{\omega} f(i,x(i-\tau(i)))\bigg\}\\
&\qquad + \frac{\underline{a}^h}{1-\underline{a}^h}\sum_{i=1}^{\omega}(a(i)-\underline{a})x(i)\\
&\geqslant \frac{\underline{a}^h}{1-\underline{a}^h}\frac{1}{B-1}\sum_{i=1}^{\omega} f(i,x(i-\tau(i)))\\
&\qquad + \frac{\underline{a}^h}{1-\underline{a}^h}\sum_{i=1}^{\omega}(a(i)-\underline{a})x(i).
\end{aligned}
$$

Then

$$
(Tx)(n) \geqslant \frac{\underline{a}^h d}{1-\underline{a}^h}\bigg[\sum_{i=1}^{\omega} f(i,x(i-\tau(i))) + (a(i)-\underline{a})x(i)\bigg], \tag{2.10}
$$

where $d=\min\{1/(B-1),1\}$.

Combining (2.9) and (2.10), we have

$$
(Tx)(n) \geqslant \delta\|Tx\|. \tag{2.11}
$$

Thus $T\colon \overline{K}_{r_2}\setminus K_{r_1}\to K$ is well defined. Since X is a finite-dimensional Banach space, one can easily show that T is completely continuous. This completes the proof. □

We can easily obtain the following result.

Lemma 4. *The fixed-point of T in K is a positive periodic solution of* (1.2).

3. The main result

Let

$$\varphi(s) = \max\{f(n,u), n \in [0,\omega-1], u \in [\delta s, s]\},$$
$$\psi(s) = \min\{f(n,u)/u, n \in [0,\omega-1], u \in [\delta s, s]\}.$$

Theorem 3.1. *Assume that* (H) *holds and there exist two positive constants α and β with $\alpha \neq \beta$ such that*

$$\varphi(\alpha) \leqslant (1-\bar{a})\frac{B-1}{\omega B}\alpha, \quad \psi(\beta) \geqslant (1-\underline{a})(B-1). \tag{3.1}$$

Then (1.2) *has at least one positive ω-periodic solution x with* $\min\{\alpha,\beta\} \leqslant \|x\| \leqslant \max\{\alpha,\beta\}$.

Proof. Without loss of generality, we assume that (H) holds and $\alpha < \beta$. Obviously, $0 < \bar{a} < 1$, $0 < \underline{a} < 1$. We claim:

(i) $\|Tx\| \leqslant \|x\|$, $x \in \partial K_\alpha$,
(ii) $x \neq Tx + \lambda \cdot 1$ $\forall x \in \partial K_\beta$, $1 \in K$ and $\lambda > 0$.

From (3.1) we have that

$$f(n,x) \leqslant (1-\bar{a})\frac{B-1}{\omega B}\alpha, \quad 0 \leqslant n \leqslant \omega-1,\ \delta\alpha \leqslant x \leqslant \alpha, \tag{3.2}$$
$$f(n,x) \geqslant (1-\underline{a})(B-1)x, \quad 0 \leqslant n \leqslant \omega-1,\ \delta\beta \leqslant x \leqslant \beta. \tag{3.3}$$

In order to prove (i), let $x \in \partial K_\alpha$. Then $\|x\| = \alpha$ and $\delta\alpha \leqslant x(n) \leqslant \alpha$ for $0 \leqslant n \leqslant \omega-1$. So

$$\begin{aligned}
(Tx)(n) &\leqslant \frac{\underline{a}^h}{1-\underline{a}^h}\sum_{i=1}^{h}\underline{a}^{-i}\sum_{j=0}^{l-1}G(n+(i-1)k,j)\Big\{(1-\bar{a})\frac{B-1}{\omega B}\alpha\Big\} \\
&\quad + \frac{\underline{a}^h}{1-\underline{a}^h}\sum_{i=1}^{h}\underline{a}^{-i}[\bar{a}-\underline{a}]\|x\| \\
&\leqslant \frac{\underline{a}^h}{1-\underline{a}^h}\sum_{i=1}^{h}\underline{a}^{-i}\sum_{j=0}^{l-1}\frac{B}{B-1}\Big\{(1-\bar{a})\frac{B-1}{\omega B}\alpha\Big\} \\
&\quad + \frac{\underline{a}^h}{1-\underline{a}^h}\sum_{i=1}^{h}\underline{a}^{-i}[\bar{a}-\underline{a}]\|x\|
\end{aligned}$$

$$= \frac{\underline{a}^h}{1-\underline{a}^h}\sum_{i=1}^{h}\underline{a}^{-i}\Big(\sum_{j=0}^{l-1}\frac{B}{B-1}\Big\{(1-\bar{a})\frac{B-1}{\omega B}\alpha\Big\}\Big)$$

$$+ \frac{\underline{a}^h}{1-\underline{a}^h}\sum_{i=1}^{h}\underline{a}^{-i}[\bar{a}-\underline{a}]\|x\|$$

$$= \frac{\underline{a}^h}{1-\underline{a}^h}\sum_{i=1}^{h}\underline{a}^{-i}\Big(\sum_{j=0}^{\omega-1}\Big\{(1-\bar{a})\frac{1}{\omega}\alpha\Big\}\Big)$$

$$+ \frac{\underline{a}^h}{1-\underline{a}^h}\sum_{i=1}^{h}\underline{a}^{-i}[\bar{a}-\underline{a}]\|x\|$$

$$= \frac{\underline{a}^h}{1-\underline{a}^h}\sum_{i=1}^{h}\underline{a}^{-i}\{(1-\bar{a})\alpha\} + \frac{\underline{a}^h}{1-\underline{a}^h}\sum_{i=1}^{h}\underline{a}^{-i}[\bar{a}-\underline{a}]\alpha$$

$$= \frac{\underline{a}^h}{1-\underline{a}^h}\sum_{i=1}^{h}\underline{a}^{-i}[1-\underline{a}]\alpha = \alpha.$$

It follows that

$$\|Tx\| \leqslant \|x\|,\ x \in \partial K_\alpha. \tag{3.4}$$

Next, let $\psi = 1 \in K$ in Lemma 1. We will prove (ii) holds. If not, there exists $u_0 \in \partial K_\beta$ and $\lambda_0 > 0$ such that

$$u_0 = (Tu_0)(n) + \lambda_0. \tag{3.5}$$

Since $u_0 \in \partial K_\beta$, we have $\|u_0\| = \beta$ and $\delta\beta \leqslant u_0(n) \leqslant \beta$. Put $u_0(n) = \min\{u_0(i);\ 0 \leqslant i \leqslant \omega-1\}$ for some $n \in [0, \omega-1]$. Noting that $u_0(j) \geqslant u_0(n) > 0$ for all $j \in \mathbb{Z}$, $a(n+(i-1)k) - \underline{a} \geqslant 0$ for all $i \in \mathbb{Z}$ and $0 < \underline{a} < 1$, we have

$$\begin{aligned}
u_0(n) &= (Tu_0)(n) + \lambda_0 \\
&= \frac{\underline{a}^h}{1-\underline{a}^h}\sum_{i=1}^{h}\underline{a}^{-i}\sum_{j=0}^{l-1}G(n+(i-1)k, j)f(n+(i-1)k+jm, \\
&\qquad u_0(n+(i-1)k+jm-\tau(n+(i-1)k+jm))) \\
&\quad + \frac{\underline{a}^h}{1-\underline{a}^h}\sum_{i=1}^{h}\underline{a}^{-i}[a(n+(i-1)k)-\underline{a}]u_0(n+(i-1)k) + \lambda_0 \\
&\geqslant \frac{\underline{a}^h}{1-\underline{a}^h}\sum_{i=1}^{h}\underline{a}^{-i}\sum_{j=0}^{l-1}G(n+(i-1)k, j)\{f(n+(i-1)k+jm, \\
&\qquad u_0(n+(i-1)k+jm-\tau(n+(i-1)k+jm)))\} + \lambda_0
\end{aligned}$$

$$
\begin{aligned}
&\geqslant \frac{\underline{a}^h}{1-\underline{a}^h}\sum_{i=1}^{h}\underline{a}^{-i}\sum_{j=0}^{l-1}\frac{1}{B-1}\{f(n+(i-1)k+jm,\\
&\qquad u_0(n+(i-1)k+jm-\tau(n+(i-1)k+jm)))\}+\lambda_0\\
&\geqslant \frac{\underline{a}^h}{1-\underline{a}^h}\sum_{i=1}^{h}\underline{a}^{-i}\sum_{j=0}^{l-1}\frac{1}{B-1}(1-\underline{a})(B-1)u_0(n+(i-1)k\\
&\quad +jm-\tau(n+(i-1)k+jm))+\lambda_0\\
&= \frac{\underline{a}^h}{1-\underline{a}^h}(1-\underline{a})\sum_{i=1}^{h}\underline{a}^{-i}\sum_{j=0}^{l-1}u_0(n+(i-1)k\\
&\quad +jm-\tau(n+(i-1)k+jm))+\lambda_0\\
&\geqslant \frac{\underline{a}^h}{1-\underline{a}^h}(1-\underline{a})\sum_{i=1}^{h}\underline{a}^{-i}\sum_{j=0}^{l-1}u_0(n)+\lambda_0\\
&= \frac{\underline{a}^h}{1-\underline{a}^h}(1-\underline{a})\sum_{i=1}^{h}\underline{a}^{-i}\sum_{j=0}^{\omega-1}u_0(n)+\lambda_0\\
&= \frac{\underline{a}^h}{1-\underline{a}^h}(1-\underline{a})\sum_{i=1}^{h}\underline{a}^{-i}\omega u_0(n)+\lambda_0\\
&> \frac{\underline{a}^h}{1-\underline{a}^h}(1-\underline{a})\sum_{i=1}^{h}\underline{a}^{-i}u_0(n)+\lambda_0\\
&= \left(\frac{\underline{a}^h}{1-\underline{a}^h}(1-\underline{a})\sum_{i=1}^{h}\underline{a}^{-i}\right)u_0(n)+\lambda_0\\
&= u_0(n)+\lambda_0,
\end{aligned}
$$

which implies that $u_0(n) > u_0(n)$. This is a contradiction.

Therefore, by Lemma 1, T has a fixed-point $x \in K_\beta \setminus K_\alpha$. Furthermore, $\alpha \leqslant \|x\| \leqslant \beta$ and $x(n) \geqslant \delta\alpha$, which means that x is a positive periodic solution of (1.2). The proof is completed. □

4. Example

Now, an example is given to demonstrate our result.

Example 1. Consider the difference equation

$$
x(n+m+k)-a(n+m)x(n+m)-b(n)x(n+k)+a(n)b(n)x(n) + f(n,x(n-\tau(n)))=0, \tag{4.1}
$$

where $m=7$, $k=5$, $\omega=6$, $\tau\colon \mathbb{Z}\to\mathbb{Z}$ and $\tau(n+\omega)=\tau(n)$, $a,b\colon \mathbb{Z}\to\mathbb{R}_+$ with $a(n)=1/2+1/16\cos n\pi/3$, $b(n)=2+[1/(2\sqrt{3})]\sin n\pi/3$, $f(n,u)=(1-7/16)(1-1/2)u^9[1+1/2(-1)^n\cos\pi u/3]$.

Obviously, $a(n+\omega)=a(n+6)=a(n)$, $f(n+\omega,u)=f(n+6,u)=f(n,u)$ for any $u\in\mathbb{R}$. Further, $h=\omega/(k,\omega)=6/(5,6)=6$, $l=\omega/(m,\omega)=6/(7,6)=6$. $\bar{a}=\max\limits_{1\leqslant n\leqslant\omega}a(n)=9/16$, $\underline{a}=\min\limits_{1\leqslant n\leqslant\omega}a(n)=7/16$, $B=\prod\limits_{s=0}^{l-1}b(s)=\prod\limits_{s=0}^{\omega-1}b(s)=63^2/8^2$, $d=\min\{1/(B-1),1\}=1/(B-1)$, $\delta=\underline{a}^h(B-1)d/(B\omega)=(7/16)^6(8/63)^2 1/6$.

Let $\alpha=1/2$, then

$$\begin{aligned}\varphi(\alpha)&=\varphi\Big(\frac{1}{2}\Big)\\ &\leqslant\Big(1-\frac{7}{16}\Big)\Big(1-\frac{1}{2}\Big)\Big(\frac{1}{2}\Big)^9\Big[1+\frac{1}{2}\Big]\\ &=\Big(1-\frac{7}{16}\Big)\Big(1-\frac{1}{2}\Big)\Big(\frac{1}{2}\Big)^8\frac{3}{4}\\ &<\Big(\frac{9}{16}\times\frac{1}{2}\Big)\Big(\frac{1}{2}\Big)^7\frac{1}{2}\\ &<\Big(1-\frac{9}{16}\Big)\Big(\frac{(\frac{63}{8})^2-1}{(\frac{63}{8})^2\times 6}\Big)\frac{1}{2}.\end{aligned}$$

So $\varphi(\alpha)\leqslant(1-\bar{a})[(B-1)/(\omega B)]\alpha$.

Let $\beta=2/\delta$. If $u\in[\delta\beta,\beta]$, then $u\geqslant 2$. Furthermore,

$$\begin{aligned}\psi(\beta)&\geqslant\Big(1-\frac{7}{16}\Big)\Big(1-\frac{1}{2}\Big)\Big(\frac{2^9}{2}\Big)\Big[1-\frac{1}{2}\Big]\\ &=\Big(1-\frac{7}{16}\Big)2^6\\ &\geqslant\Big(1-\frac{7}{16}\Big)\Big(\frac{63}{8}\Big)^2.\end{aligned}$$

So $\psi(\beta)\geqslant(1-\underline{a})(B-1)$.

By Theorem 3.1, (4.1) has at least one positive 6-periodic solution.

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